Extending the Ihara-Selberg zeta function to hypergraphs

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Abstract

In the late 1960s, Ihara began work that led to the Ihara zeta function, a zeta function which is defined on a finite graph. This function is an interesting graph invariant which gives information on expansion properties of the graph. It also appears in Knot theory and has some information about colorings in graphs.

We propose two generalization of this function to hypergraphs. This will provide a framework to tie together work by Hashimoto on zeta functions of bipartite graphs and work by Feng, Li, and Solé on Ramanujan hypergraphs. We will also provide an example of viewing the generalized hypergraph zeta function as a more specialized graph zeta function, which allows us greater flexibility in distinguishing cospectral graphs.

Acknowledgements

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Contents

Introduction

In 1966 and 1968, Yasutaka Ihara wrote two papers in which he set forth the framework to define the Ihara-Selberg Zeta Function on a finite, k-regular graph [17, 18]. This function has proven to be quite fruitful, with applications relating to Ramanujan graphs, counting spanning trees on graphs, trace formulas on trees, knot theory, and spectral graph theory in general. As time has passed, there has been a lot of work done to remove the regularity condition in Ihara's original formulation as well as to simplify the factorization of the zeta function. One critical step was in 1989 when Ki-Ichiro Hashimoto showed that the reciprocal of the zeta function was related to a determinant involving an operator on the edges of a graph X [16]. Hyman Bass, in 1992, took this one step further by considering operators defined on oriented edges of X [2]. This last step provided enough flexibility to give a satisfying determinant expression of the zeta function without too many conditions being imposed upon X. For a complete picture on the current theory, we refer the interested reader to a series of articles by Harold Stark and Audrey Terras [34, 35, 36]. There has also, recently, been a generalization of this theory to digraphs by Hirobumi Mizuno and Iwao Sato [27, 28].

We hope to continue in this tradition by considering hypergraphs in place of graphs. Winnie Li [22], Cristina Ballantine [1], and others have begun a discussion of Ramanujan Hypergraphs, and it seems natural that the next step in pushing Spectral Hypergraph theory would be to define a meaningful zeta function. There is also some potential to connect with some of the theory of Buildings, for which Anton Deitmar and J. William Hoffman have begun to talk about an Ihara-Selberg Zeta Function [11]. For a good introduction to current results in Ramanujan Graphs, Spectral Graph Theory, and Ihara's Zeta Function, one might consider [30, 7, 34].

We will begin by reviewing many of the definitions and work that goes into defining and giving a determinant expression for the zeta function. Then we will look specifically at hypergraphs and form two different zeta functions. We will give expressions for both zeta functions and discuss some of the resulting properties of our zeta functions. For graphs, the path from graph to zeta function proceeds roughly in the following way:

$$\begin{tikzcd}
\text{Graph} \arrow[d, leftrightarrow] & \text{Initial determinant expression} \arrow[r] & \text{Bass's expression} \\
\text{Oriented Graph} \arrow[r, leftrightarrow] & \text{Oriented Line Graph} \arrow[u]
\end{tikzcd}$$

The key step is going from an "Oriented Line Graph" to an initial determinant expression. This will come from the Perron–Frobenius Theorem and be the same for our zeta functions as well as for Ihara's. We will see that for hypergraphs, we'll have to take a slightly more general route to get to an "oriented line graph," but the idea will be the same. Our further expressions will actually follow largely from results already in the literature once we see how they fit into our framework.

We can summarize the steps involved in the following flowchart:

$$\begin{tikzcd}
\text{Bipartite Graph} \arrow[r] \arrow[d, leftrightarrow] & \text{Hashimoto's expressions} & \\
\text{Hypergraph} \arrow[d, leftrightarrow] & \text{Initial expression} \arrow[u] \arrow[r] & \text{Bass's expression} \\
\text{Colored Oriented Graph} \arrow[r, leftrightarrow] & \text{Oriented Line Graph} \arrow[u] &
\end{tikzcd}$$

Chapter 1

The graph framework

Throughout this and the next chapter, we will try to give a clear and complete presentation of the Ihara-Selberg zeta function so that we can identify the key points and techniques used in our generalizations. We will start with some standard graph theory definitions before moving to the specifics of the zeta function. Then, we will follow the path from graph to zeta function as detailed by Motoko Kotani and Toshikazu Sunada [21]. Their strategy is to relate a graph to a strongly connected oriented graph which has a similar cycle structure; then, they take advantage of existing theorems in symbolic dynamics to give an initial determinant expression, thus relating the general graph problem to a much easier problem on oriented graphs. After they have an initial expression, they are able to reproduce Bass's expression. Though the process appears long and overly complicated, once we've seen the method once, we will be able to cut out the middle steps and jump straight to a useful expression so that in practice, computing the zeta function is quite simple. We will conclude by looking at a few consequences of the determinant expression that help illustrate why the Ihara-Selberg zeta function is a fruitful function to study.

1.1 What is a graph?

We begin by looking at some basic properties of graphs. For a good introduction to many of the ideas here, we refer to the reader to Douglas West's book [37] or to Fan Chung's book for more detailed spectral results [7].

Definition 1.1.1. *A graph $X = (V, E)$ is a set V of* vertices *and a multiset E of unordered pairs of vertices, called* edges. *If $\{u, v\} \in E$, we say that u is* adjacent *to v and write $u \sim v$. X is* finite *if $|V|$ and $|E|$ are both finite. A graph X is* simple *if there are no edges of the form $\{v, v\}$ and if there are no repeated edges. Finally, the* degree *of a vertex $v \in V$, denoted $d(v)$ is the number of vertices to which v is adjacent, counting potential multiplicities.*

Figure 1.1 is an example of a simple graph X with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}\}$. We will refer back to this graph as we follow it through the steps needed to compute the Ihara-Selberg zeta function. We will call this graph G_1 for future reference.

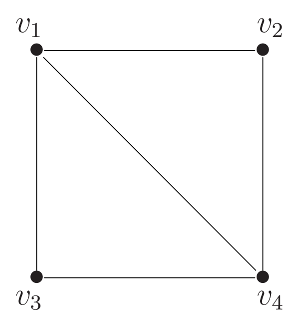

Figure 1.1: G_1: a subgraph of the complete graph on 4 vertices

A *path* on a graph is a sequence of vertices $\{v_1, v_2, \cdots, v_n\}$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i < n$. The graph is *connected* if there exists a path from x to y for any vertices $x \neq y \in X$. In general, we will only consider connected graphs as any graph that is not connected can be broken up and considered as a union of connected graphs. While we are discussing paths, we make two other definitions, which will be useful later:

Definition 1.1.2. *Let G be a connected graph, and let $x, y \in V$.*

1. *The* distance *between x and y, written $d(x, y)$ is the fewest number of edges needed to have a path from x to y.*

2. *The* diameter *of a graph, often denoted D, is the maximum distance over all pairs of vertices.*

By a function on a graph X, we mean a map from the vertices of X into the real numbers. We denote by $C(V)$ the space of functions on X. When our graph is finite, we can actually think of a function $f \in C(V)$ as a vector in $\mathbb{R}^{|V|}$ by viewing the i^{th} entry of the vector as $f(v_i)$. In fact, we can define addition of two functions $f, g \in C(V)$ and scalar multiplication by the formula

$$(f + g)(v) = f(v) + g(v), \ \ \forall\, v \in V$$

$$(cf)(v) = c(f(v)), \ \ \forall\, c \in \mathbb{R} \,\text{and}\, \forall\, v \in V.$$

Then $C(V)$ is a $|V|$-dimensional real vector space.

Remark 1.1.3. There is a standard basis for $C(V)$ given by functions $\delta_i(v_j) = 1$ if $i = j$ and 0 otherwise.

There is a very important linear operator A on $C(V)$ called the *adjacency operator*, given by:

$$(Af)(x) = \sum_{x \sim y} f(y), \tag{1.1}$$

for all $x \in V$.

This operator plays a central role in the study of spectral graph theory, as we will see shortly. First, we look at the matrix interpretation of this operator. If we label the vertices of a graph and take the basis defined in Remark 1.1.3, we can write A as a matrix, called the *adjacency matrix*, with its i, j-entry given by $a_{i,j}$:

$$a_{i,j} = \begin{cases} m & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where m is the number of times $\{v_i, v_j\}$ is listed in E.

Example 1.1.4. The adjacency matrix of G_1, the graph given in Figure 1.1, is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Remark 1.1.5. In general, the matrix defined in this way will be a symmetric $|V| \times |V|$ matrix with non-negative integer entries. We can also reverse this process: if we are given some symmetric matrix A with non-negative integer entries, we can construct a graph X with adjacency matrix A. To do this, we label the rows and columns of our matrix by $\{v_1, v_2, \ldots, v_n\}$. Then we form the graph X with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set given by: $\{v_i, v_j\} \in E$ with multiplicity $a_{i,j}$. It's important that the given matrix be symmetric because the elements of E are *unordered* pairs so that $\{v_i, v_j\} \in E$ if and only if $\{v_j, v_i\} \in E$. This idea will prove to be important later.

1.1.1 The spectrum of a graph

There are several important questions in graph theory that are related to the adjacency operator. Given a graph, we might ask how many steps you have to take before a random walk converges to the point where you are equally likely to be at any vertex on the graph. A similar question asks how efficiently information flows in a computer network. These questions are best answered by looking at the spectrum of the adjacency operator of the graph.

We first note that vertex x is adjacent to y if and only if vertex y is adjacent to x. This forces the adjacency operator A to be a *symmetric* operator. Thus, all of the eigenvalues of A must be *real* [13]. For a graph X, we denote the spectrum of A by $\mathrm{spec}(A)$ or sometimes $\mathrm{spec}(X)$. When X is a finite graph, there are only finitely many eigenvalues λ_i, which we order by size.

Proposition 1.1.6. *Let A be the adjacency matrix of a graph X, and let Δ be the maximum degree of vertices of X. Then $|\lambda| \leq \Delta$, for all $\lambda \in spec(A)$.*

Proof. Let x be an eigenvector of A with eigenvalue λ. By definition, $Ax = \lambda x$. We write $x = (x_1, \cdots, x_n)^t$ and assume, without loss of generality, that $|x_1| = \max_{1 \leq i \leq n} |x_i|$. Then, we have

$$|\lambda||x_1| = \left| \sum_{j=1}^{n} a_{1j} x_j \right| \leq |x_1| \sum_{j=1}^{n} a_{1j}$$

where the equality is given by matrix multiplication, and the inequality is because $|x_1|$ was assumed to be the maximum value and that each $a_{i,j} \geq 0$. But $\sum\limits_{j=1}^{n} a_{1j}$ is exactly $d(v_1)$, so we have

$$|x_1| \sum_{j=1}^{n} a_{1j} = |x_1| d(v_1) \leq |x_1| \Delta.$$

We note that $x_1 \neq 0$ since x is an eigenvector and not zero. Dividing through by $|x_1|$ completes the proof. $\square$

Usually, we will order the eigenvalues by size; then, we can write

$$\Delta \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|} \geq -\Delta. \tag{1.3}$$

We say that a graph X is *k-regular* if every vertex is adjacent to exactly k other vertices. In this case, k is actually an eigenvalue with eigenfunction the constant function, and all of the other eigenvalues are less than or equal to k in absolute value, by the proposition, so we can say a bit more:

$$k = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|} \geq -k. \tag{1.4}$$

The fundamental question then becomes how large is $|\lambda_i|$ when i is not 1. To see the importance of this, imagine that you are looking at a random walk on a k-regular graph. At each step, you can move to k places. To model the number of times you can visit each vertex in m steps, you apply the adjacency operator m times. As m grows large, λ_1 will begin to dominate and you will be forced into being just as likely to be at one vertex as any other. The speed at which this happens is determined by the size of the second largest eigenvalue in absolute value. To have a random walk converge quickly, it's important that $|\lambda_i|$, and particularly the second largest eigenvalue in absolute value, be as small as possible. In general, $\lambda_2 = \lambda_1$ if and only if the graph is not connected. We also have $\lambda_{|V|} = -k$ if and only if the graph is bipartite [37]. Studying this eigenvalue question leads to the celebrated Alon–Boppana Theorem, which gives us a bound on how small the second eigenvalue can be as our graphs get large [24].

Theorem 1.1.7 (Alon–Boppana)**.** *Let $\{X_m\}$ be a family of connected k-regular graphs with $|V(X_m)| \to \infty$ as $m \to \infty$. Then*

$$\liminf_{m\to\infty} \lambda_2(X_m) \geq 2\sqrt{k-1}.$$

We defer the proof of the theorem to Corollary 3.2.6 for a more general result.

This Theorem led Alexander Lubotzky, Ralph Phillips, and Peter Sarnak to make the following definition [24]:

Definition 1.1.8 (Lubotzky, Phillips, and Sarnak)**.** *A $(q+1)$-regular graph is* Ramanujan *iff for every eigenvalue λ of the adjacency operator A such that $|\lambda| \neq q+1$, we have:*

$$|\lambda| \leq 2\sqrt{q}.$$

The terminology "Ramanujan" comes from a deep connection to the Ramanujan–Petersson conjecture for congruence subgroups which was proved by Deligne. One method of constructing Ramanujan graphs involves exploiting the continuous structures to build graphs with the appropriate spectra.

For a long time, it was an open problem to construct an infinite family of such graphs. Lubotzky, Phillips, and Sarnak as well as Moshe Morgenstern were finally able to construct such families using number theoretic and other methods [24, 29]. The known constructions produce k-regular Ramanujan graphs with k a prime power or $k-1$ prime. It is still an open question to construct infinite families of Ramanujan graphs for other values of k. For current results in Ramanujan graphs, we recommend a survey by Murty [30]. We will see that the Ihara-Selberg zeta function will have a great deal to say about whether a given k-regular graph is Ramanujan or not.

Chapter 2

The Ihara-Selberg zeta function

Our focus of this chapter will be the definition and determinant expressions of the Ihara-Selberg zeta function. We will take a historical approach, beginning with the main definition. From there, we will outline the graph constructions that let us change the problem of factoring the Ihara-Selberg zeta function of a graph into a problem of factoring a zeta function of an oriented graph, which is much easier. From our initial expression, we will jump to the expression given by Hyman Bass. This will enable us to talk about a "Riemann Hypothesis" for the zeta function and study some of the properties associated with it.

We will see that for k-regular graphs, the Ihara-Selberg zeta function satisfies many of the properties that number theorists look for in a reasonable zeta function: Euler product expansion, functional equations, and a "Riemann hypothesis" (that is sometimes true and sometimes not). We will also be able to see some properties of a graph that are determined by the zeta function. For the most part, this chapter is a survey of known results until the end when we discuss the coefficients of the zeta function.

2.1 The definition

Before we can define the Ihara-Selberg zeta function of a graph, we must decide on our notion of a "prime." Then we will define the zeta function as a product over our primes, copying the general form of the Euler product expansion of the Riemann zeta function. Our "primes" will be a special family of cycles in the graph which have no backtracking or tails. We need to make several definitions. Throughout this section, we will assume, unless specified otherwise, that X is a finite, connected graph.

A *closed path* in X is a sequence $c = (v_1, e_1, v_2, e_2, \cdots, v_k, e_k, v_1)$ such that $v_i \in e_{i-1}, e_i$ for $i \in \mathbb{Z}/k\mathbb{Z}$. Note that this implies that $v_1 \in e_k$ so that this path really is "closed." We say that c has *backtracking* if there is a subsequence of c of the form $(v_i, e_i, v_{i+1}, e_i, v_i)$. Intuitively, at some point this means we leave a vertex via an edge e and then take that same edge directly back to the vertex. If X has multiple edges, it's possible to return directly to the previous vertex without "backtracking" so long as you use a different edge or a loop. We emphasize that backtracking requires using the same edge twice in sucession. By c^r, we mean the closed path obtained by going r times around c. We say that c is *tail-less* if c^2 does not have backtracking. We will say that c is a *closed geodesic* if c has no backtracking and is tail-less. Finally, a closed geodesic$c$ is *primitive* if it is not b^r for some other geodesic b and integer $r \geq 2$. The *length* of a primitive geodesic c, denoted $|c|$, is the number of edges in the sequence.

We define an equivalence relation on primitive geodesics. We say that two primitive geodesics c and b are equivalent if one is a cyclic permutation of the other. See Figure 2.1 for an example of two equivalent, primitive geodesics. We call a representative of $[c]$ a *prime cycle* and the equivalence class a *prime cycle class*.

We gather all of the pertinent definitions together for future use:

Path Criteria 1. *Let X be a graph. Then,*

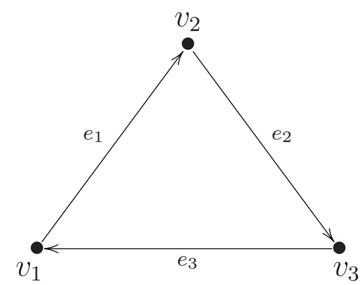

Figure 2.1: The primitive geodesic $\{v_1, e_1, v_2, e_2, v_3, e_3, v_1\}$ is equivalent to $\{v_2, e_2, v_3, e_3, v_1, e_1, v_2\}$.

1. *A* closed geodesic *c is a closed path with no backtracking or tails.*

2. *A closed geodesic is* primitive *if it is not b^r for some other geodesic b and integer $r \geq 2$.*

3. *A* prime cycle *c is a representative of the equivalence class $[c]$ of primitive closed geodesics, identified by cyclic permutation.*

We now have everything we need to define the Ihara-Selberg zeta function:

Definition 2.1.1. *Let P be the set of all prime cycle classes of a finite, connected graph X. Then the* Ihara-Selberg zeta function *$Z_X(u)$ is given, for sufficiently small $u \in \mathbb{C}$, by*

$$Z_X(u) = \prod_{\mathfrak{p} \in P} \left(1 - u^{|\mathfrak{p}|}\right)^{-1}. \tag{2.1}$$

Remark 2.1.2. Unless our graph has just two or fewer prime cycle classes, we have an infinite number of prime cycles. We see this by considering two primitive cycles α and β, with β not given by taking α in the opposite direction, which intersect. We shift them so that they both start at the same vertex v and assume, without loss of generality, that β contains an edge that α does not; then, the cycle $\alpha^n\beta$ will be primitive for any $n \in \mathbb{Z}^+$ since any edge that is in β but not α will only appear once. This means the zeta function is, in general, an infinite product.

A priori, $Z_X(u)$ is quite difficult to express nicely for computational purposes. We will do it in two stages. First, we will write $Z_X(u)$ to realize it as the determinant of some linear operators. Then, with some linear algebra, we can produce a much nicer expression. In practice, we will be able to jump directly to the final, explicit expression and skip all of the intermediate steps. We will go through the steps to get the initial determinant expression in detail since this is the process we will need when we change our view to hypergraphs. The strategy is to construct from X a new directed graph X_L which is strongly connected and has the same "cycle structure" as X. Then we can make use of the Perron–Frobenius operator and an easy lemma of Rufus Bowen and O.E. Lanford III [6] to produce our initial expression. For a more detailed look at this strategy, we note that this is Kotani and Sunada's method [21].

2.2 From connected graph to oriented graph

In this section, we will show how to start with a graph X and produce an oriented graph X_L^o which has the same cycle structure as X. We will see that X_L^o satisfies the conditions of the Perron–Frobenius theorem, so we will be able to produce an initial expression of the Ihara-Selberg zeta function using that framework.

Since we will be working with oriented graphs, we need a few extra tools to help us. An oriented edge $e = \{x, y\}$ is an ordered pair of vertices x, $y \in V$. We say that x is the *origin* of e, denoted by $o(e)$, and y is the *terminus* of e, denoted by $t(e)$. We also have the *inverse edge* $\bar{e}$ given by switching the origin and terminus. In a general oriented graph, we may not have the inverse edge in the given edge set; however, the oriented graphs we consider will be constructed to always have the inverse edge.

Let X be a finite connected graph. We label the edges of X: $E = \{e_1, e_2, \cdots, e_n\}$. By an *orientation* on X, we mean a choice of direction for each edge $e_i \in E$; i.e., given an edge, we specify a origin vertex and a terminus vertex. Our first step will

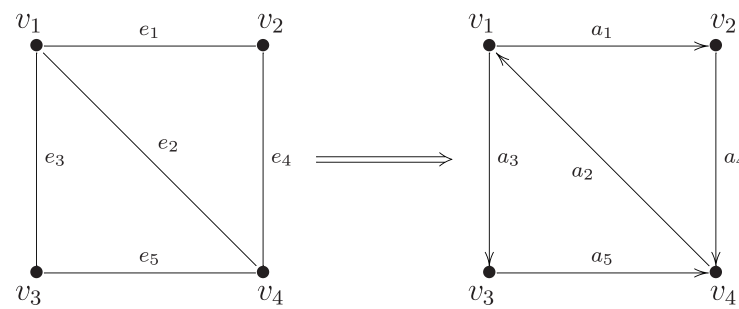

Figure 2.2: Orienting the graph G_1.

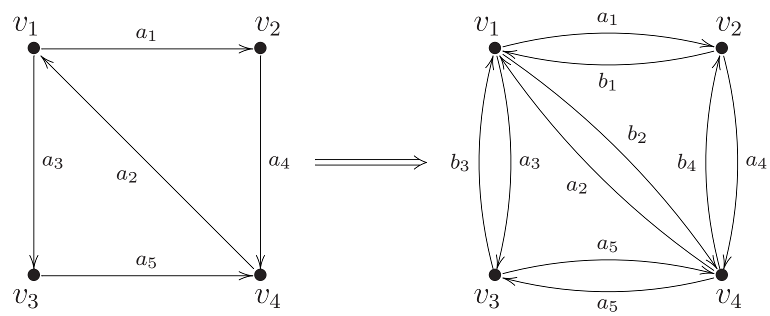

Figure 2.3: Completing the orientation on G_1.

be to orient the graph X. To maintain clarity, we relabel the edges $\{a_1, \cdots, a_n\}$. In general, there are many different possible orientations you can choose. We will see in the end that the final expression is independent of orientation, so there is no problem if you change orientations. See Figure 2.2 for one way to orient G_1.

Once we've oriented our graph X, our next step is to add in the opposing orientation. We will let b_i be the oriented edge which is opposite of a_i. In the event that a_i is an oriented loop, we let b_i be an additional oriented loop from the same vertex. In this case, b_i is the inverse edge of a_i and vice versa. We denote the graph formed in this manner by X_o. Then X_o is a graph with $2|E|$ oriented edges. We complete the orientation started for G_1 in Figure 2.3.

Finally, we construct the *oriented line graph* $X_L^o = (V_L, E_L^o)$ associated with our

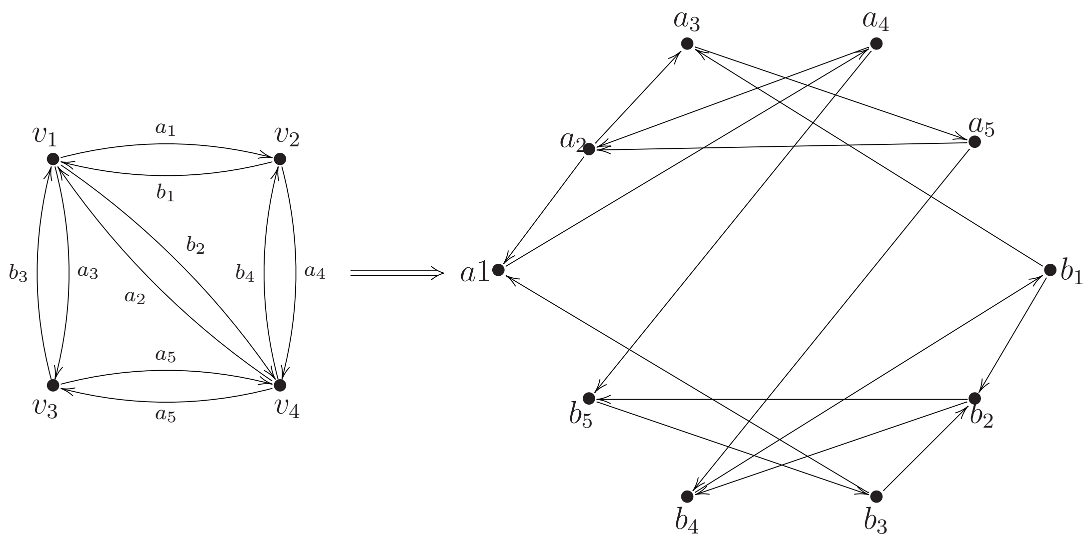

Figure 2.4: Construction of an oriented line graph of G_1.

choice of X_o by

$$V_L = E(X_o),$$
$$E_L^o = \{(e_i, e_j) \in E(X_o) \times E(X_o);\, \bar{e}_i \neq e_j,\, t(e_i) = o(e_j)\}.$$

We can think of E_L as the set of paths of length 2 without backtracking in X_o. This graph is also oriented, with incidence map (o, t) induced from the identity map on $E(X_o) \times E(X_o)$. Intuitively, we are building a graph which has vertices given by "legal" moves you could make to get a geodesic. Ruling out backtracking is why we explicitly disallow $\{e, \bar{e}\}$ in forming the edge set. If we had chosen a different orientation when we formed X_o, we would get the same oriented line graph but with a different labeling. See Figure 2.4 for the resulting graph X_L^o that corresponds to our choice of orientation for G_1.

We will leave this construction for a moment and take a short diversion into the Perron–Frobenius framework. Once we have developed the appropriate tools, we will return to the construction of an oriented line graph, showing that it satisfies several

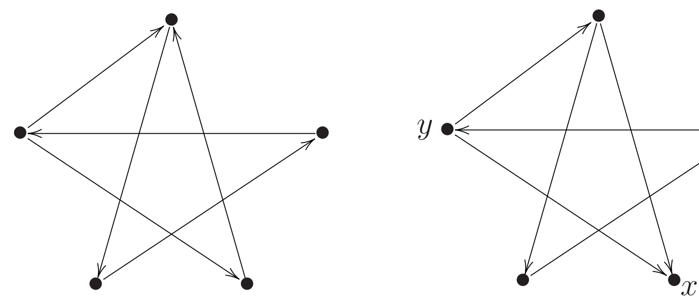

Figure 2.5: The oriented graph on the left is strongly connected; whereas, the graph on the right is not strongly connected since there is no admissible path from x to y.

nice conditions, and then rewrite the zeta function as a determinant expression.

2.3 The Perron–Frobenius framework

We will try to keep this section completely self-contained until the end, when we apply the tools described here to oriented line graphs. It should be noted that we are still following the method laid out by Kotani and Sunada [21]. We assume throughout that $X_o = (V_o, E_o)$ is a finite oriented graph. We repeat many of the path and cycle definitions from before to see how they fit into this context.

An *admissible path* in X_o is a sequence $c = (e_1, \cdots, e_k)$ where $e_i \in E_o$ and $t(e_i) = o(e_{i+1})$ for every i. We let $o(c) = o(e_1)$ and $t(c) = t(e_k)$. Then, the admissible path c is *closed* if $t(c) = o(c)$. The oriented graph X_o is *strongly connected* if, for any $x, y \in V_o$, there exists an admissible path c with $o(c) = x$ and $t(c) = y$. It's helpful to think of this as an analogue for *connected* in the unoriented case. We give in Figure 2.5 an example of an oriented graph which is strongly connected and one which is not.

For $m \geq 1 \in \mathbb{Z}$, we let N_m be the number of admissible closed paths of length m in X_o. Then, we define the *zeta function* of X_o by

$$Z^o_{X_o}(u) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m\right). \tag{2.2}$$

The motivation for this definition comes from the shape of the Weil zeta functions of projective algebraic varieties over finite fields [19, 20].

Rationality of $Z^o_{X_o}(u)$ will follow from the *Perron–Frobenius operator* $T : C(V_o) \mapsto C(V_o)$ given by

$$(Tf)(x) = \sum_{e \in E_0(x)} f(t(e)),$$

where $E_0(x) = \{e \in E_o \mid o(e) = x\}$ is the set of all oriented edges with x as their origin vertex. We think of T as an oriented version of the adjacency operator on non-oriented graphs. With this in mind, we see that taking a power of T has a nice representation given by

$$(T^n f)(v) = \sum_{\substack{\text{ad. paths } c \\ |c|=n;\ o(c)=v}} f(t(c)). \tag{2.3}$$

This operator is the focus of much study, but we will take only a few of the results about it for our uses. By the Perron–Frobenius theorem [14], we have

Lemma 2.3.1. *Let X_o be a finite, strongly connected, oriented graph with Perron–Frobenius operator T. We denote by $(T1)$, the Perron–Frobenius operator applied to the constant function 1. Then,*

1. *T has at least one positive eigenvalue. The maximal positive eigenvalue, called the* Perron–Frobenius root *and denoted α, is simple and has a positive-valued eigenfunction.*

2. *$|\lambda| \leq \alpha$ for any eigenvalue λ of T.*

3. *$\min_{v \in V_o}(T1)(v) \leq \alpha \leq \max_{v \in V_o}(T1)(v)$.*

4. *If $Tf = \lambda f, f \geq 0, f \neq 0$, then $\lambda = \alpha$ and $f > 0$.*

We refer the reader to [14] for a proof.

When X_o is not just a cycle, we have the following lemma from Rufus Bowen and O. E. Lanford III [6] which gives us the determinant expression of $Z^o_{X_o}(u)$ that we are looking for.

Lemma 2.3.2 (Bowen and Lanford)**.** *Let X_o be a finite, strongly connected, oriented graph which is not just a cycle, and let T be the associated Perron–Frobenius operator with Perron–Frobenius root α. Then,*

1. *The power series $\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m$ converges absolutely when $|u| < \alpha^{-1}$.*
2. *$Z^o_{X_o}(u) = \det(I - uT)^{-1}$. In particular, $Z^o_{X_o}(u)$ is a rational function of u and has a simple pole at $u = \alpha^{-1}$.*

Proof. 1. Let δ_v denote the indicator function on the set $\{v\}$. Then by (2.3)

$$(T^m \delta_v)(w) = \#\{c \,|\, \text{admissible paths with } o(c) = w,\, t(c) = v,\, |c| = m\}.$$

We can sum the above quantity over all of the functions δ_v for $v \in V_o$ to recover the trace of a power of T in the following way:

$$\operatorname{tr} T^m = \sum_{v \in V_o} (T^m \delta_v)(v) = N_m.$$

However, we also know, from elementary linear algebra, that the trace relates the operator to its eigenvalues by

$$\operatorname{tr} T^m = N_m = \sum_{\lambda_i \in \operatorname{spec}(T)} \lambda_i^m.$$

Recalling that α is the largest eigenvalue in absolute value lets us conclude that $\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m$ converges absolutely when $|u| < \alpha^{-1}$.

2. Let $\{\lambda_1, \cdots, \lambda_{|V_o|}\}$ be the spectrum of T. We first recall the definition of $Z^o_{X_o}(u)$

and then rewrite the series in terms of the eigenvalues since $\mathrm{tr}T^m = N_m$:

$$Z^o_{X_o}(u) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m\right) = \exp\left(\sum_{m=1}^{\infty} \sum_{i=1}^{|V_o|} \frac{1}{m} \lambda_i^m u^m\right).$$

Now we use the series expansion $-\log(1-x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k$ with $x = \lambda_i u$:

$$\begin{aligned} Z^o_{X_o}(u) = \exp\left(\sum_{m=1}^{\infty} \sum_{i=1}^{|V_o|} \frac{1}{m} \lambda_i^m u^m\right) &= \prod_{i=1}^{|V_o|} \exp(-\log(1-\lambda_i u)) \\ = \prod_{i=1}^{|V_o|} \frac{1}{1-\lambda_i u} &= \det(I - uT)^{-1}. \end{aligned}$$

The final equality follows because I and T commute, allowing us to simultaneously diagonalize them. □

In the context of finite, strongly connected, oriented graphs, this lemma gives us exactly what we need. The zeta function must equal the reciprocal of a finite degree polynomial with constant term 1. In particular, it is a rational function with only finitely many poles. Our job now is to connect this zeta function with the Ihara-Selberg zeta function defined in Definition 2.1.1. To do this, we must do two things: rewrite the oriented graph zeta function so that it looks like a product instead of a sum. Then, we must show how the cycle structure of a finite, connected graph X relates to the cycle structure of an oriented line graph X^o_L constructed in the previous section. We will begin by establishing the Euler product expansion.

We let P be the set of prime cycles, defined by imposing the same equivalence relation as before on closed, primitive admissible paths, of a finite, strongly connected, oriented graph X_o. We denote by M_k the number of prime cycles of length k. The following lemma relates M_k and N_m:

Lemma 2.3.3 (Sunada)**.** *Let X_o be a finite, strongly connected, oriented graph. Let M_k denote the number of admissible prime cycles of length k and N_m the number of* *closed admissible paths of length m. Then,*

$$\sum_{k|m} kM_k = N_m,$$

where k runs over all divisors of m.

Proof. We first consider a representative c of a prime cycle of length k. If k divides m, then we get a closed path of length m by going $\frac{m}{k}$ times around c; i.e. by taking $c^{\frac{m}{k}}$. Recalling the equivalence relation, we see that there are exactly k closed paths of length m that arise from $[c]$. Hence, each prime cycle of length k yields k closed paths of length m whenever $k|m$. This establishes:

$$\sum_{k|m} kM_k \leq N_m.$$

The other direction of the inequality follows because we are looking at *directed* graphs. Since there is no possibility of backtracking or tails, the only consideration is that a closed path is the k-multiple of some other closed path. Suppose b is a closed path of length m, and $b = c^k$ for $k \in \mathbb{N}$ with c a *primitive* closed path – in the sense that c is not a non-trivial k-multiple of some other closed path. Then c is a representative of a prime cycle, and $k|c| = |b| = m$, so $k|m$. Hence, b is counted in $\sum_{k|m} kM_k$, which gives:

$$\sum_{k|m} kM_k \geq N_m.$$

□

We have the following Euler product expansion for the zeta function of X_o [21]:

Theorem 2.3.4 (Sunada)**.** *Let X_o be a finite, strongly connected, oriented graph, and* *let P be the set of admissible prime cycles of X_o. Then,*

$$Z^o_{X_o}(u) = \prod_{\mathfrak{p}\in P} \left(1 - u^{|\mathfrak{p}|}\right)^{-1}.$$

Proof. We begin with Definition 2.2 and consider $\log Z^o_{X_o}(u)$:

$$\log Z^o_{X_o}(u) = \sum_{m=1}^{\infty} \frac{1}{m} N_m u^m.$$

We invoke Lemma 2.3.3 to break the sum up:

$$\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{l|m} l M_l u^m.$$

Now we make the substitution $kl = m$:

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{l|m} l M_l u^m = \sum_{l,k=1}^{\infty} \frac{1}{k} M_l u^{kl}.$$

We split the sum out again:

$$\sum_{l,k=1}^{\infty} \frac{1}{k} M_l u^{kl} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{\infty} M_l u^{kl}.$$

The number of prime cycles of length l is exactly M_l:

$$\sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{\infty} M_l u^{kl} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{\infty} \sum_{\substack{\mathfrak{p}\in P \\ |\mathfrak{p}|=l}} u^{kl}.$$

We now switch the summation order and group the outer sum into a sum over the prime cycles:

$$\sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{\infty} \sum_{\substack{\mathfrak{p}\in P \\ |\mathfrak{p}|=l}} u^{kl} = \sum_{\mathfrak{p}\in P} \sum_{k=1}^{\infty} \frac{1}{k} u^{k|\mathfrak{p}|}.$$

Since u is small, and $|\mathfrak{p}| \geq 2$, we can use the series expansion $\log \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{1}{k} x^k$:

$$\sum_{\mathfrak{p} \in P} \sum_{k=1}^{\infty} \frac{1}{k} u^{k|\mathfrak{p}|} = \sum_{\mathfrak{p} \in P} \log \frac{1}{1 - u^{|\mathfrak{p}|}}.$$

Thus $Z^o_{X_o}(u) = \prod_{\mathfrak{p} \in P} \left(1 - u^{|\mathfrak{p}|}\right)^{-1}$. $\square$

The Euler product expansion has exactly the same form as the definition for the Ihara-Selberg zeta function. For a finite connected graph X, if we construct an oriented line graph X^o_L and can show that prime cycles in X are in one-to-one correspondence with prime cycles in X^o_L of the same length, we will be able to conclude that $Z_X(u) = Z^o_{X^o_L}(u) = \det(I - uT)^{-1}$ where T is the Perron–Frobenius operator associated with X^o_L.

2.3.1 Linking the Ihara-Selberg zeta function to oriented line graphs

We can now explicitly connect the Ihara-Selberg zeta function to the Perron–Frobenius operator of an associated line graph. We let $X = (V, E)$ be a finite, connected graph and X^o_L an oriented line graph constructed from X.

Lemma 2.3.5 (Sunada)**.** *There is a one-to-one correspondence between admissible paths of length k in X^o_L and geodesics of length k in X.*

Proof. Let $c = (v_1, e_1, \cdots, v_k, e_k, v_{k+1})$ be a geodesic of length k in X. This corresponds to the path $c_o = ((\{v_1, v_2\}, \{v_2, v_3\}), (\{v_2, v_3\}, \{v_3, v_4\}), \cdots, (\{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}))$ in X^o_L. Thus, c_o is an admissible path because the oriented graph X_o was constructed to have all ordered pairs $\{v_i, v_j\}$ (provided $\{v_i, v_j\}$ is an edge in X) as oriented edges, so we see that $(\{v_i, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}) \in E^o_L$ by construction. The length of c_o is exactly k, the same length as c.

All edges in E_L^o are of the form $(\{v_i, v_j\}, \{v_j, v_k\})$ whenever $\{v_i, v_j\}$ and $\{v_j, v_k\} \in E$, by construction, so we can represent an arbitrary admissible path as $c_o = ((\{v_1, v_2\}, \{v_2, v_3\}), (\{v_2, v_3\}, \{v_3, v_4\}), \cdots, (\{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}))$. For $(\{v_i, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}) \in E_L^o$, we must have $\{v_i, v_{i+1}\}$ and $\{v_{i+1}, v_{i+2}\} \in E$. Then c_o corresponds to the geodesic $c = (v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, \cdots, v_k, \{v_k, v_{k+1}\}, v_{k+1})$ in X. The only problem is that c may have backtracking, but this can only occur if $\{v_i, v_{i+1}\} = \{v_{i+1}, v_{i+2}\}$, as unordered pairs, for some i. This would mean that $v_i = v_{i+2}$, but this would force us to have the oriented edge $\{v_i, v_{i+1}\}$ and its inverse $\{v_{i+1}, v_i\}$ adjacent to each other in X_L^o. This was explicitly disallowed by the construction of X_L^o, so c cannot have backtracking and is thus a geodesic. Thus, c has the same length as c_o. So we see that there is a one-to-one correspondence between geodesics of length k in X and admissible paths of length k in X_L^o. $\square$

In particular, let us look at cycles in X and X_L^o:

Corollary 2.3.6 (Sunada)**.** *Let X be a finite, connected graph and X_L^o an oriented line graph associated with X. Then,*

1. *There is a one-to-one correspondence between admissible prime cycles in X_L^o and prime cycles in X of the same length.*

2. $Z_X(u) = Z^o_{X_L^o}(u) = \det(I - uT)^{-1}$, *where T is the Perron–Frobenius operator on X_L^o.*

Proof. 1. The assertion follows directly from the previous lemma by considering cycles instead of paths and noting that the prime classes come from the same equivalence relation on both sides.

2. The equality $Z_X(u) = Z^o_{X_L^o}(u)$ follows from Definition 2.1.1, Theorem 2.3.4, and part 1 above. The second part of the equality $Z^o_{X_L^o}(u) = \det(I - uT)^{-1}$, where T is the Perron–Frobenius operator, follows from Lemma 2.3.2. $\square$

This corollary is exactly what we were looking for. Given a finite graph X, we can now realize the Ihara-Selberg zeta function, which is defined as a possibly infinite product, as a rational function which we know how to calculate. As an example, we compute the Ihara-Selberg zeta function of G_1.

Example 2.3.7. We wish to compute the Ihara-Selberg zeta function for the graph G_1, shown in Figure 1.1. We've already written down an oriented line graph associated with G_1 in Figure 2.4. From that, we compute

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Computing $\det(I - uT)^{-1}$ then gives us

$$Z_{G_1}(u) = \frac{1}{1 - 4u^3 - 2u^4 + 4u^6 + 4u^7 + u^8 - 4u^{10}}.$$

The main issue now is that it takes several steps to produce the Perron–Frobenius operator that shows up in the determinant expression. However, by writing the zeta function as a determinant, we've given ourselves a starting point to use linear algebra to factor it further. Hyman Bass provided, in 1992, a further determinant expression which is much more useful [2].

2.4 Bass's expression and consequences

We will give the main result of this section without proof but will refer the interested reader to Bass's paper [2] or to Kotani and Sunada's paper [21]. Bass uses non-commutative determinants to derive the main result; while, Kotani and Sunada use linear algebra and vector space decomposition. Before we state Bass's theorem, we need to define one more operator on $C(V)$. Throughout this section, our graphs will be finite and connected.

We first define a function $q : V \mapsto \mathbb{Z}$ by $q(v) = d(v) - 1$, where $d(v)$ is the *degree* of vertex v as defined in Definition 1.1.1. Then we define $Q : C(V) \mapsto C(V)$ by $(Qf)(v) = q(v)f(v)$. Taking the standard basis, we see that Q is represented by a diagonal matrix with $q(v_i)$ on the diagonals.

Bass gave the following expression of $Z_X(u)$, even in the case when X is not a regular graph [2]. For regular graphs, Ihara gave this same formulation [17]:

Theorem 2.4.1 (Bass). *Let X be a finite, connected graph with adjacency operator A and operator Q as defined above. Let I be the identity operator on $C(V)$. Then,*

$$Z_X(u) = (1 - u^2)^{\chi(X)} \det(I - uA + u^2Q)^{-1}$$

where $\chi(X) = |V| - |E|$ is the Euler Number of the graph X.

In general, $\chi(X) \leq 0$ with equality if and only if X is just a cycle. Thus, we are actually seeing an expression that looks like the reciprocal of a polynomial. From a historical perspective, we should note that Ihara proved the above result for k-regular graphs X via a combinatorial argument [17]. Ihara's result can be associated with degree $p + 1$ graphs associated with a co-compact discrete subgroup of the p-adic linear group $SL_2(\mathbb{Q}_p)$. His proof relied heavily on the combinatorial nature of the underlying Hecke algebra, which we cannot generalize to include non-regular graphs.

This problem explains the 26-year gap between Ihara's paper and Bass's general expression.

With this expression in hand, we can safely forget about the constructions necessary to use the Perron–Frobenius operator. We've written the Ihara-Selberg zeta function as a product of $(1-u^2)$ to an easily computable power and as a determinant of operators which are easy to compute from X. This formulation is easier to deal with since all of the operators are defined on X, allowing us to forget about the oriented line graph construction if we are only interested in the zeta function.

Now that we have a satisfying expression for the zeta function, we can begin to ask some more structural questions. What does the Ihara-Selberg zeta function tell us about the graph? How many graphs have the same Ihara-Selberg zeta function? How are those graphs related, if at all? How are the poles of the Ihara-Selberg zeta function distributed for random graphs? We first show a connection between the distribution of poles of the Ihara-Selberg zeta function and Ramanujan graphs. Then, in the next section we will look at some thoughts on which graphs can have the same zeta function.

The *Riemann hypothesis* for the Riemann zeta function says that if $0 < \operatorname{Re} s < 1$ and $\zeta(s) = 0$, then $\operatorname{Re}(s) = \frac{1}{2}$. Thus, all of the non-trivial zeroes conjecturally lie on the line $\frac{1}{2} + it$. We can make a similar definition for the Ihara-Selberg zeta function:

Definition 2.4.2 (Ihara)**.** *Suppose X is a finite connected $(q+1)$-regular graph. Then $Z_X(u)$ satisfies the* Riemann hypothesis *iff for*

$$\mathrm{Re(s)} \in (0,1),\ \ \mathrm{Z_X^{-1}(q^{-s})} = 0 \Longrightarrow \mathrm{Re(s)} = \frac{1}{2}.$$

While the Riemann hypothesis is still unknown for Riemann's zeta function, in our case, we have a complete solution as a corollary of Bass's work:

Corollary 2.4.3 (Ihara)**.** *Assume X is a finite, connected $(q+1)$-regular graph. Then,*

$Z_X(u)$ satisfies the Riemann hypothesis if and only if X is a Ramanujan graph.

We will break the proof up into several lemmas. This is the same spirit of proof as found in [21]. We assume throughout that X is a $(q+1)$-regular graph.

Lemma 2.4.4. *Suppose $\lambda \in \mathbb{R}$, $q > 0$, and $|\lambda| \leq 2\sqrt{q}$. Let $f(u) = qu^2 - \lambda u + 1$. Then the roots of $f(u)$ lie in the complex plane on the circle $|u| = q^{-\frac{1}{2}}$.*

Proof. By the quadratic formula, $f(u)$ has roots at

$$u = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q}.$$

Since $|\lambda| \leq 2\sqrt{q}$, the discriminant is non-positive, so the roots are $\frac{\lambda \pm i\sqrt{-\lambda^2+4q}}{2q}$. They lie on the circle given by

$$|u|^2 = \frac{\lambda^2}{4q^2} + \frac{-\lambda^2 + 4q}{4q^2} = q^{-1}.$$

Hence, u lies on the circle $|u| = q^{-\frac{1}{2}}$. □

Lemma 2.4.5. *Suppose $\lambda \in \mathbb{R}$, $q > 0$, and $q+1 > |\lambda| > 2\sqrt{q}$. Let $f(u) = qu^2 - \lambda u + 1$. Then $f(u)$ has a root on the real axis of the complex plane in the interval $[q^{-\frac{1}{2}}, 1)$ or the interval $(-1, -q^{-\frac{1}{2}}]$.*

Proof. The quadratic formula gives us roots at

$$u = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q}.$$

If $q+1 > \lambda > 2\sqrt{q}$, the root with the plus sign is in $[q^{-\frac{1}{2}}, 1)$; while if $-(q+1) < \lambda < -2\sqrt{q}$, the root with the negative sign is in $(-1, -q^{-\frac{1}{2}}]$. □

We state one more quick lemma before we look at the structure of the Ihara-Selberg zeta function.

Lemma 2.4.6. *Let $q \in \mathbb{Z}^+$. For $u \in \mathbb{C}$, the identification $u = q^{-s}$ sends points on the circle $|u| = q^{-\frac{1}{2}}$ to points on the line $s = \frac{1}{2} + it$ where t ranges over the real numbers.*

We now need a lemma to show us how to find the poles of the Ihara-Selberg zeta function of a $(q+1)$-regular graph.

Lemma 2.4.7. *Suppose X is a $(q+1)$-regular graph. Then we can write $Z_X(u)$ as*

$$Z_X(u) = (1-u^2)^{\chi(X)} \det(I - uA + qu^2 I)^{-1} = (1-u^2)^{\chi(X)} \prod_{\lambda_i \in spec\ A} \left(1 - \lambda_i u + qu^2\right)^{-1}.$$

Proof. We use Theorem 2.4.1 for the first equality. The matrix A is symmetric, so there exists some matrix Q such that QAQ^{-1} is diagonal. Since determinants are invariant under conjugation, we can write

$$\begin{aligned}
\det(I - uA + qu^2 I) &= \det(Q(I - uA + qu^2 I)Q^{-1}) \\
&= \det(QIQ^{-1} - uQAQ^{-1} + qu^2 QIQ^{-1}) \\
&= \det(I - uQAQ^{-1} + qu^2 I) \\
&= \prod_{\lambda_i \in \text{spec } A} \left(1 - \lambda_i u + qu^2\right).
\end{aligned}$$

The last equality follows because we're actually taking the determinant of a diagonal matrix with entries on the diagonal of the form $1 - \lambda_i u + qu^2$. ☐

Lemma 2.4.7 combined with the previous three lemmas serve to tell us exactly where the poles of a $(q+1)$-regular graph's zeta functions lie. We now complete the proof to Corollary 2.4.3.

Proof. Suppose that X is a $(q+1)$-regular Ramanujan graph. We consider the expression $Z_X(u) = (1-u^2)^{\chi(X)} \det(I - uA + qu^2 I)^{-1}$ and ask where poles might appear.

The term $(1-u^2)^{\chi(X)}$ only contributes poles at $u = \pm 1$. Rewriting u as q^{-s}, we see that Re(s) $= 0$, which is outside of $(0, 1)$ for these poles.

Thus, we need only consider the poles that come from the determinant expression. By Lemma 2.4.7, we can actually reduce this to considering zeroes of the polynomial $f(u) = qu^2 - \lambda_i u + 1$ for $\lambda_i \in \text{Spec } A$. There are two cases to consider. First, since X is $(q+1)$-regular, at least one eigenvalue satisfies $\lambda = q + 1$. In this case, we see that $f(u)$ has zeroes at 1 and at $\frac{1}{q}$, corresponding to Re(s) $= 0$ and 1. These are outside the interval Re(s) $\in (0, 1)$. Since the graph is Ramanujan, all other eigenvalues satisfy $|\lambda| \leq 2\sqrt{q}$. Applying Lemma 2.4.4 followed by Lemma 2.4.6 puts all of the zeroes on Re(s) $= \frac{1}{2}$ as desired.

Thus, if X is Ramanujan, any pole of $Z_X(q^{-s})$ with Re(s) $\in (0, 1)$ must actually satisfy Re(s) $= \frac{1}{2}$. To prove the other direction, we note that if X is not Ramanujan, it has an eigenvalue λ in the interval $q + 1 > |\lambda| > 2\sqrt{q}$. A similar analysis as before and an application of Lemma 2.4.5 gives us a pole in the wrong location. Thus, the Riemann Hypothesis cannot hold for a non-Ramanujan $(q+1)$-regular graph. $\square$

We've actually said quite a bit about the poles. Another way to rephrase the Riemann hypothesis is as follows: a $(q+1)$-regular graph's Ihara-Selberg zeta function $Z_X(u)$ satisfies the Riemann hypothesis if its real poles are of the form $|u| = 1$ or q^{-1} or possibly with $|u| = q^{-\frac{1}{2}}$ in the event that $\lambda = \pm 2\sqrt{q}$ is an eigenvalue. The pole $u = q^{-1}$ must be simple. The complex poles are all forced to be on the circle of radius $q^{-\frac{1}{2}}$ and centered at 0, so we need only check the real poles to determine if a graph is Ramanujan or not.

Now that we know about the poles of the zeta function, we look at any symmetries that may arise in the values. We have functional equations whenever X is $(q+1)$-regular. We state several now that were found by Stark and Terras [34], but we will postpone the proof until later.

Corollary 2.4.8 (Bass; Stark and Terras)**.** *Let X be a $(q+1)$-regular graph with*

$n = |V|$. *Then, among others, we have the following functional equations for* $Z_X(u)$*:*

1. $\Lambda_X(u) := (1-u^2)^{\frac{n}{2}-\chi(X)}(1-q^2u^2)^{\frac{n}{2}} Z_X(u) = (-1)^n \Lambda_X(\frac{1}{qu})$.

2. $\xi_X(u) := (1+u)^{-\chi(X)}(1-u)^{-\chi(X)+n}(1-qu)^n Z_X(u) = \xi_X(\frac{1}{qu})$.

3. $\Xi_X(u) := (1-u^2)^{-\chi(X)}(1+qu^2)^n Z_X(u) = \Xi_X(\frac{1}{qu})$.

We refer the reader to a proof of Corollary 5.4.3 for an example.

Taking a step back for a moment, we see that, when X is k-regular, we have all of the properties that number theorists look for in a zeta function:

1. $Z_X(u)$ satisfies an Euler product expansion.

2. $Z_X(u)$ has a Riemann hypothesis which is sometimes true and sometimes not, but we know explicitly when it is.

3. $Z_X(u)$ satisfies functional equations.

From a number theorist's point of view, these will be the of the properties that we look for in generalizing the Ihara-Selberg zeta function to hypergraphs. In the next section, we will take a look at how well the Ihara-Selberg zeta function serves as a graph invariant.

2.5 The Ihara-Selberg zeta function as graph invariant

In this section, we want to focus on a more graph theoretic question: if two graphs X and Y have $Z_X(u) = Z_Y(u)$, what, if anything, can we say about the relationship between the graphs X and Y? It will turn out that the answer is quite a mixed bag, but we will outline the results we know about here. These sorts of questions have received a great deal of attention lately and some preliminary results can be found in [8, 10, 33]. From Corollary 2.3.6, it is immediate that X and Y have the same zeta function if and only if the spectra of their T operators is the same. The Perron–Frobenius operator T is defined on the oriented line graph, meaning we can't directly study the graphs from this statement. We hope to be able to address the actual graph structure that is determined by having the same zeta function. First, we need a new definition:

Definition 2.5.1 (Harary, King, Mowshowitz, and Read)**.** *Two graphs X and Y are* cospectral *if their adjacency operators have the same spectrum; i.e., if spec*$(X) =$ *spec*(Y)*.*

Suppose X and Y are both k-regular graphs; then, it turns out that having the same Ihara-Selberg zeta function is equivalent to X and Y being cospectral. This should not be surprising in light of Lemma 2.4.7 but is worth pointing out. In fact, this result seems to have first been found by Aubi Mellein [26].

Theorem 2.5.2 (Mellein, 2001)**.** *Suppose X and Y are k-regular graphs. Then $Z_X(u) = Z_Y(u)$ if and only if X and Y are cospectral.*

We take a moment to recall the *binomial coefficient* notation from combinatorics. For $r, s \in \mathbb{Z}_{\geq 0}$, we define

$$\binom{r}{s} = \frac{r!}{(r-s)!s!} \tag{2.4}$$

which is the number of ways to choose an ordered subset of size s from a set of r elements. We also adopt that convention that if $s < 0$ or if $r < s$, $\binom{r}{s} = 0$. The following relation is the rule of formation of Pascal's triangle and will be useful to us:

Lemma 2.5.3 (Pascal's formula)**.** *Let $r, s \in \mathbb{Z}$, then*

$$\binom{r}{s} + \binom{r}{s-1} = \binom{r+1}{s}$$

.

Before proving the main result, we establish the following Lemma:

Lemma 2.5.4. *Let q be a positive integer and let $\{\lambda_1, \cdots, \lambda_n\}$ be a set of real numbers. Then*

$$\prod_{\lambda_i}(1 - \lambda_i u + qu^2) = 1 + c_1 u + c_2 u^2 + \cdots + c_{2n} u^{2n}.$$

If we let, for $j \leq n$,

$$d_j = (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_j}, \tag{2.5}$$

then, for $j \leq n$, we can explicitly write down the c_j's in terms of the d_j's. For future calculations, we adopt the convention that $d_0 = 1$ and $d_{-m} = 0$ for $m \in \mathbb{N}$. We have

$$c_j = \sum_{l=0}^{[\frac{j}{2}]} \binom{n - j + 2l}{l} q^l d_{j-2l}. \tag{2.6}$$

Proof. We consider the coefficients found by multiplying out the polynomial $\prod_{\lambda_i}(1 - \lambda_i u + qu^2)$. The coefficient of u^j is found from choosing l copies of qu^2, where l runs up to $\frac{j}{2}$, choosing $j - 2l$ factors $-\lambda_i u$, and copies of 1. For a given l, the contribution to c_j of the $j-2l$ linear factors is d_{j-2l}, and the contribution of the l quadratic factors is $q^l \binom{n-j+2l}{l}$. The lemma follows. □

With this lemma in hand, we now prove Theorem 2.5.2:

Proof. This proof is similar to that given by Mellein [26]; although, some of the details are different. Suppose G and H are k-regular graphs. We will first show, by closely examining Theorem 2.4.1, that if they are cospectral, they have the same Ihara-Selberg zeta functions.

Since G and H are cospectral, they must have the same number of vertices. Since they are both k-regular, they must also have the same number of edges; hence, they have the same Euler Number χ. To show that $Z_G(u) = Z_H(u)$, we need only show that the parts which arise from the determinant expression are the same.

So we consider the expression $\det(I - uA + u^2Q)$. When our graph is k-regular, Q is a diagonal matrix with $q = k - 1$ in every entry on the diagonal. This lets us rewrite:

$$\det(I - uA + u^2Q) = \det(I - uA + qu^2I).$$

Now since the identity matrix commutes with the adjacency matrix, we can simultaneously diagonalize A and I, allowing us to rewrite the determinant expression as a product over the eigenvalues of the adjacency matrix:

$$\det(I - uA + qu^2I) = \prod_{\lambda_i \in \mathrm{spec} A} (1 - \lambda_i u + qu^2).$$

Hence when a graph is k-regular, the determinant expression is completely determined by the number of vertices in the graph, the eigenvalues of the adjacency operator, and k. This gives us $Z_G(u) = Z_H(u)$.

We now suppose that G and H are both k-regular with $Z_G(u) = Z_H(u)$. As before, G and H must have the same Euler Number (the number of eigenvalues of the adjacency operator is the number of vertices, so they have the same number of vertices and thus edges), so their determinant expressions must be the same. We will show that knowing $\det(I - uA(G) + qu^2)$ is enough to determine exactly the spectrum of G.

As before, we have

$$\det(I - uA + qu^2I) = \prod_{\lambda_i \in \mathrm{spec} A} (1 - \lambda_i u + qu^2);$$

however, Lemma 2.5.4 tells us explicitly what the coefficients must be in terms of the eigenvalues. This means we have all of the numbers c_j up to $j = n$. We recursively recover the numbers d_j by noting that

$$d_1 = c_1$$

and

$$d_s = c_s - \sum_{k=1}^{[\frac{s}{2}]} \binom{n-s+2k}{k} q^k d_{s-2k}.$$

Now we notice that

$$(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_n) = x^n + d_1x^{n-1} + d_2x^{n-2} + \cdots + d_n.$$

Since we can explicitly compute the d_j's, we can actually write down the characteristic polynomial of $A(G)$. Its roots form the spectrum of $A(G)$. Thus if two k-regular graphs have the same Ihara-Selberg zeta function, they must be cospectral. □

We should point out that this result actually follows from a theorem of Gregory Quenell [32]. He focused on the set of closed paths of varying lengths and was able to prove a similar statement. Before stating his result, we need a few definitions.

The universal cover of a k-regular graph G is the infinite k-regular tree, which we denote X_k. We let $\mathrm{Aut}(X_k)$ be the group of automorphisms of X_k. Then the graph G can be viewed as the quotient of X_k by a subgroup H of $\mathrm{Aut}(X_k)$ that acts freely on the vertices of X_k. We write $G = H \setminus X_k$; then the vertices of G are the orbits Hx of vertices in X_k, and Hx is adjacent to Hy if and only if each element of Hx is adjacent to some element of Hy in X_k. With this framework in mind, we state Quenell's theorem:

Theorem 2.5.5 (Quenell). *Let $H \setminus X_k$ be an n-vertex, simple k-regular graph. For each integer $r \geq 1$, let*

$$P_r = \sum_{[h_i]_H \subset [t_r]_{\mathrm{Aut}(X_k)}} L(C_H(h_i))$$

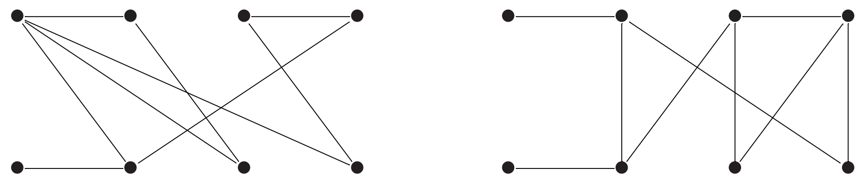

Figure 2.6: Two cospectral graphs with different Ihara-Selberg zeta functions.

where $[t_r]_{\mathrm{Aut}(X_k)}$ *is the* $\mathrm{Aut}(X_k)$*-conjugacy class containing all length-r translations in* H *and* $L(C_H(h_i))$ *denotes the length of a generator of the centralizer* $C_H(h_i)$ *of* h_i *in* H*.*

Then the spectrum of $H \setminus X_k$ *determines and is determined by the sequence* $P_1, P_2, \cdots, P_n$*.*

Quenell's proof relies heavily on covering space theory and spherical functions on X_k. He then connects this theorem to the Ihara-Selberg zeta function with the following remark:

Remark 2.5.6 (Quenell)**.** Let G be a k-regular graph. Then $u\frac{d}{du}\log Z_G(u)$ is a generating function for the numbers P_r.

From this remark and the fact that $Z_G(0) = 1$, we can deduce Theorem 2.5.2. In fact, his result is stronger because he only needs the first n coefficients of the logarithmic derivative; whereas, we have required all of them.

We're also interested in the more general case when X may not necessarily be a regular graph. Unfortunately, we do not know of a nice formulation at the moment to determine if two graphs have the same Ihara-Selberg zeta function or what it means for two graphs to have the same Ihara-Selberg zeta function. We offer two examples to illustrate the difficulties. In both cases, the graphs we used came from Willem Haemers and Edward Spence's article [15].

Example 2.5.7. Figure 2.6 gives an example of two graphs which are cospectral but have different Ihara-Selberg zeta functions. If we denote by H_1 the leftmost graph

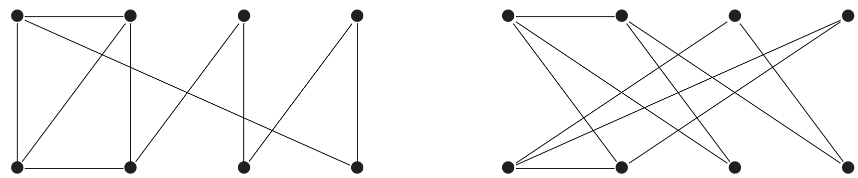

Figure 2.7: Two graphs with same laplacian spectrum but with different Ihara-Selberg zeta functions.

and H_2 the rightmost, we have:

$$Z_{H_2}^{-1}(u) = -(-1+u)^2(1+u)(1+u+u^2)(1+u+u^2+u^3+u^4)$$
$$\times(-1+u-u^2+2u^3-2u^4+3u^5-3u^6+3u^7),$$

and

$$Z_{H_1}^{-1}(u) = -(-1+u)^2(1+u)(1+u+2u^2+u^3+2u^4)(-1+u^3+u^5+u^6+2u^7).$$

Example 2.5.8. Often, people will study the *combinatorial laplacian* $\Delta = I + Q - A$ instead of the adjacency operator. When two k-regular graphs are cospectral, they also have the same laplacian spectrum. When two graphs are not k-regular, it's possible to have the same laplacian spectrum and not be cospectral. We might expect that this is a better operator to study given the operators that appear in the determinant expression of Bass's factorization; however, we have the same problem as the previous example when we consider this operator. Figure 2.7 is an example of two graphs which have the same laplacian spectrum but have different Ihara-Selberg zeta functions. We let H_1 be the left graph and H_2 the right; then

$$Z_{H_1}^{-1} = (-1+u)^3(1+u)^2(1+u+2u^2)(1+u+2u^2+2u^3+2u^4+2u^5+2u^6)$$
$$\times(-1+u+2u^3-2u^4+2u^5-2u^6+4u^7),$$

and

$$Z_{H_2}^{-1}(u) = (-1+u)^3(1+u)^2(-1+u-2u^2+4u^3-3u^4+5u^5-2u^6+4u^7)$$
$$\times (1+2u+3u^2+2u^3+3u^4+4u^5+7u^6+6u^7+4u^8).$$

In her Ph.D. dissertation, Debra Czarneski extended Mellein's result to cover biregular bipartite graphs (bipartite graphs where the degree of each vertex in an independent set is the same as the degree of the other vertices in that set) [10]. Thus, we can conclude that if two graphs X and Y are both k-regular, $Z_X(u) = Z_Y(u)$ if and only if X and Y are cospectral. The conclusion is identical if X and Y are both (d, r)-biregular, bipartite graphs. There doesn't seem to be much progress on non-regular cases. The k-regular case can use Bass's determinant expression as a tool; while, the biregular bipartite case uses Hashimoto's expression, stated in Theorem 5.3.6 in an identical way. Lacking this sort of tool, we are forced to look at the Perron–Frobenius operator, which is more delicate.

2.5.1 The coefficients of the Ihara-Selberg zeta function

Setting aside the larger question of exactly what conditions will force non-isomorphic graphs to have the same zeta function, we look at a more specific question: what properties of a graph are encoded in the Ihara-Selberg zeta function? Yaim Cooper has several preliminary results in this direction [8], but we would like to add a new result here. This is a generalization of the proof Norman Biggs gives to show that the adjacency operator determines the number of triangles in a graph [5].

Theorem 2.5.9. *Let X be a graph with no loops or multiple edges and write*

$$Z_X(u)^{-1} = 1 + c_1u + c_2u^2 + c_3u^3 + \cdots + c_mu^m.$$

Then the coefficient c_3 is the negative of twice the number of triangles in X.

Proof. We first note that the number of triangles in X is one half the number of oriented triangles in X_L^o. This comes from the correspondence established in Lemma 2.3.5 and the observation that one triangle gives rise to two prime cycle classes because direction of travel matters.

We construct X_L^o and then look at the Perron–Frobenius operator T. We consider the characteristic polynomial

$$\chi(T;x) = \det(xI - T) = x^n + d_1x^{n-1} + d_2x^{n-2} + d_3x^{n-3} + \cdots + d_n.$$

Then $-d_3$ is the sum of the principal minors of T which have 3 rows and columns. We consider all such possible minors. We first make a note about the edge relations that may arise. Suppose e_1 and e_2 are oriented edges such that $t(e_1) = o(e_2)$. Then, by construction, $t(e_2) \neq o(e_1)$. This means that if the i, j-entry of T is 1, the j, i-entry must be 0. Moreover the diagonal entries are zero since $t(e_i) \neq o(e_i)$ (there are no loops) as well. Up to reordering the edges, the only non-trivial principal minors with three rows and columns that may arise are:

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Of these three, the determinants of the last two are trivial, and the determinant of the first is 1. However, the first principal minor exactly corresponds to the triangle given by $\{e_1, e_2, e_3\}$ where $t(e_1) = o(e_2)$, $t(e_2) = o(e_3)$, and $t(e_3) = o(e_1)$. One such minor will appear for each oriented triangle in X_L^o.

The coefficient d_3 is the same as the coefficient c_3 given by sending x to $1/\lambda$ and then rewriting $1/\lambda = u$ in the expression $\det(I - uT)$. $\square$

We feel that these sorts of questions can be quite important as they let us take full advantage of the number theory toolkit to try to solve problems in graph theory. Hopefully, we will be able to give exact results to determine when two graphs have the same Ihara-Selberg zeta function in the future. As Quenell's theorem shows, this has an immediate link to paths on graphs and can then be pushed towards other questions of that sort.

We will now leave the graph setting and turn towards hypergraphs. We will define two different zeta functions that will generalize the Ihara-Selberg zeta function on a graph. In both cases, they will give the Ihara-Selberg zeta function if we actually start with a graph. We will show many of the same results as in the graph case but as applies particularly to hypergraphs. We will also show that our second zeta function is a non-trivial generalization and will be able to produce rational functions that graphs cannot.

Chapter 3

Hypergraphs

Before looking at the zeta functions, we review some definitions and results about hypergraphs. For the most part, results cited in this section are taken from papers by Keqin Feng and Wen-Ch'ing Winnie Li as well as Li and Patrick Solé [12, 23]. Two excellent books that review hypergraph theory are written by Claude Berge [3, 4].

3.1 What is a hypergraph?

Definition 3.1.1 (Berge)**.** *A hypergraph $\mathbb{H}$ is a set of hypervertices $V(\mathbb{H})$ and a multiset of hyperedges $E(\mathbb{H})$ such that each hyperedge is a multiset consisting of elements of $V(\mathbb{H})$ and the union of all the hyperedges is $V(\mathbb{H})$. We note that a hypervertex may be repeated in the same hyperedge. We also allow for hyperedges to repeat. A hypervertex v is incident to a hyperedge e if $v \in e$. We call the cardinality of a hyperedge e the* order *of the hyperedge and denote it $|e|$. We require that $|e| > 1$ for each hyperedge e. We restrict our attention to the case where the number of hypervertices, the number of hyperedges, and the order of each hyperedge are all finite.*

Bipartite graphs play an important role in the study of hypergraphs, and we define them here. A *bipartite graph B* is a graph whose vertex set V can be partitioned into

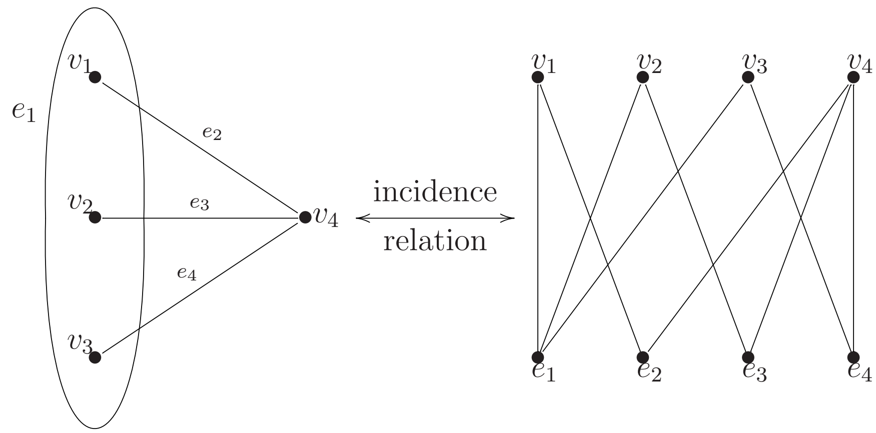

Figure 3.1: A hypergraph with $V = \{v_1, v_2, v_3, v_4\}$ and hyperedges given by $e_1 = \{v_1, v_2, v_3\}, e_2 = \{v_1, v_4\}, e_3 = \{v_2, v_4\}$, and $e_4 = \{v_3, v_4\}$. On the right is the associated bipartite graph formed by the incident relation. We see that the vertex e_1 has degree 3, indicating that e_1 is a hyperedge of order 3.

two subsets V_1 and V_2 such that every edge of B connects a vertex from V_1 to one from V_2. The incidence relation allows us to associate with $\mathbb{H}$ a bipartite graph B in the following way: the vertices of B are $V(\mathbb{H})$ and $E(\mathbb{H})$. Vertices $v \in V(\mathbb{H})$ and $e \in E(\mathbb{H})$ are adjacent in B if v is incident to e in $\mathbb{H}$. This associated bipartite graph is a very important tool in the general study of hypergraphs and will be quite useful to us later. See Figure 3.1 for an example of a hypergraph and its associated bipartite graph.

Notation 1. *Given a hypergraph $\mathbb{H}$, we will denote by $B_{\mathbb{H}}$ its associated bipartite graph.*

For now, we can use the associated bipartite graph to generalize to hypergraphs the adjacency matrix. We associate to $\mathbb{H}$ an adjacency matrix $A(\mathbb{H})$ whose rows and columns are parameterized by $V(\mathbb{H})$. Then the ij-entry of $A(\mathbb{H})$ is the number of paths in B from v_i to v_j of length 2 with no backtracking. This last condition does allow a diagonal entry to be nonzero if a vertex appears more than once in some hyperedge since there will be a multiple edge in the bipartite graph.

Remark 3.1.2. Of course, we might also define the ij-entry to be the number of hyperedges which contain both v_i and v_j with appropriate compensation for v_i or v_j appearing more than once in a hyperedge. This second definition is equivalent to the first since every path of length 2 with no backtracking from v_i to v_j goes through a vertex in $B_{\mathbb{H}}$ given by a hyperedge. Each path corresponds to v_i and v_j being in the hyperedge in the middle of the path. The backtracking condition serves to keep us from saying v_i is adjacent to v_i when v_i appears only once in a hyperedge.

3.2 The spectrum of a hypergraph

As with graphs, we will be interested in studying the spectrum of the adjacency operator on hypergraphs. To do this, it's sometimes useful to actually change from a problem on hypergraphs into a problem on graphs. For an arbitrary hypergraph $\mathbb{H}$, the adjacency matrix will be a symmetric $|V(\mathbb{H})| \times |V(\mathbb{H})|$ matrix with non-negative integer entries. By Remark 1.1.5, we can construct a graph $G_{\mathbb{H}}$ with $V(G_{\mathbb{H}})$ labeled the same as $V(\mathbb{H})$ and with $A(G_{\mathbb{H}}) = A(\mathbb{H})$. This construction is actually very intuitive: for each hyperedge e, we construct an $|e|$-clique—a clique on a set of vertices is formed when every pair of vertices in the set is an edge—on the vertices of e by adding an edge joining v and w for each pair of hypervertices $v, w \in e$. We give an example of this in Figure 3.2.

Notation 2. *Given a hypergraph $\mathbb{H}$, we denote by $G_{\mathbb{H}}$ the graph with $V = V(\mathbb{H})$ and the same adjacency matrix as $\mathbb{H}$.*

Once we realize that studying the adjacency operator on a hypergraph is the same as studying it on a graph, we can get a lot of immediate results. We know immediately that the eigenvalues must be real and that they are bounded by the maximum degree of vertices considered in $G_{\mathbb{H}}$. Given a hypergraph $\mathbb{H}$ and hypervertex v, we would

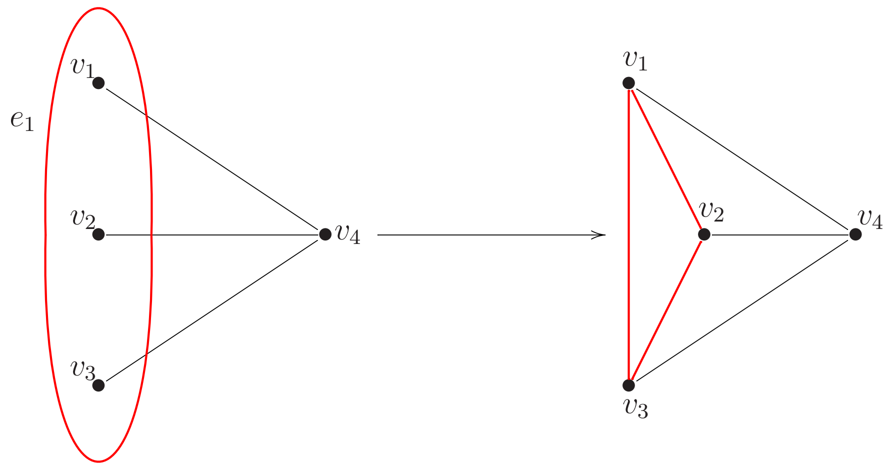

Figure 3.2: Going from a hypergraph to a graph on the same vertex set and with the same adjacency matrix

like to know what the *degree* of v is when considered as a vertex of $G_{\mathbb{H}}$. We compute this explicitly as $d(v) = \sum_{v \in e_i} (|e_i| - 1)$. We let Δ be the maximum degree of vertices in $G_{\mathbb{H}}$; then, as before, we have

$$\Delta \geq \lambda_1 \geq \cdots \geq \lambda_{|V(\mathbb{H})|} \geq -\Delta.$$

Before we look at the gap between the largest eigenvalue and the second largest, we need the proper notions of "connected" and "regular." In the spectrum, λ_1 will be explicit as it was with regular graphs, and we will be able to look for the next largest eigenvalue as before.

Definition 3.2.1. *A hypergraph* $\mathbb{H}$ *is* connected *if* $G_{\mathbb{H}}$ *is connected as a graph.*

Definition 3.2.2 (Li and Solé)**.** *A hypergraph* $\mathbb{H}$ *is* (d, r)*-regular if:*

1. *Every hypervertex is incident to exactly* d *hyperedges, and*

2. *Every hyperedge contains exactly* r *hypervertices.*

Suppose $\mathbb{H}$ is a (d, r)-regular hypergraph; then, $G_{\mathbb{H}}$ actually has a very special structure. Every vertex of $G_{\mathbb{H}}$ is an element of *exactly* d r-cliques. In this case, the degree of each vertex in $G_{\mathbb{H}}$ is $k = d(r-1)$, and $\lambda_1 = d(r-1)$. By viewing (d, r)-regular hypergraphs as a special class of k regular graphs, Feng and Li establish the following result on graphs which gives us an Alon–Boppana-type corollary for the second eigenvalue of regular hypergraphs [12]:

Theorem 3.2.3 (Feng and Li)**.** *Let G be a k-regular graph. Suppose that there is a constant g such that for any pair of adjacent vertices in G, there are at least g vertices of G adjacent to both vertices. If the diameter of G is $\geq 2l + 2 \geq 4$ for some $l \in \mathbb{Z}$, then*

$$\lambda_2(G) > g + 2\sqrt{q} - \frac{2\sqrt{q} - 1}{l},$$

where $q = k - g - 1$.

We will give a complete proof as it illustrates some important techniques that Spectral Graph Theory has taken from Spectral Geometry. The proof is a modification of a proof by Alon Nilli for Theorem 1.1.7 [31].

Definition 3.2.4 (Feng and Li)**.** *Let G be a graph. The* combinatorial laplacian $\mathcal{L}$ *is a linear operator on $C(V)$ given by:*

$$(\mathcal{L}f)(v) = \sum_{\substack{u \\ v \sim u}} (f(v) - f(u)). \tag{3.1}$$

If G is a k-regular graph, $\mathcal{L}$ can be written $\mathcal{L} = kI - A$. Then $\mathcal{L}$ is symmetric and has eigenvalues given by $\eta_i = k - \lambda_i$ where the λ_i's are the eigenvalues of A.

There is a natural inner product that we associate with $C(V)$, given by

$$\langle f, g \rangle = \sum_{v \in V} f(v)g(v).$$

Proposition 3.2.5. *The laplacian operator $\mathcal{L}$ on a k-regular graph G is self-adjoint with respect to the above inner product.*

Proof. Since G is k-regular, we use the expression $\mathcal{L} = kI - A$. Let $f, g \in C(V)$. Then

$$\begin{aligned}
\langle \mathcal{L}f, g\rangle &= k\langle f, g\rangle - \langle Af, g\rangle \\
&= k \sum_{x\in V(G)} f(x)g(x) - \sum_{x\in V(G)} g(x) \sum_{\substack{y\in V(G)\\ \{x,y\}\in E(G)}} f(y) \\
&= \sum_{\{x,y\}\in E(G)} \left(f(x)g(x) - 2g(x)f(y) + f(y)g(y)\right) \\
&= \langle f, \mathcal{L}g\rangle.
\end{aligned}$$

The second to last equality comes from changing our sum from vertices to a sum over edges. We have to include both vertices in the edge, which is why we get the term $f(x)g(x)$ as well as $f(y)g(y)$. The symmetry in the third line lets us put everything back together with the laplacian operator on the other side of the inner product. $\square$

Since $\mathcal{L}$ is self-adjoint with respect to the inner product, there exists an orthogonal basis of elements of $C(V)$ which are eigenfunctions of $\mathcal{L}$ [13]. When the graph is k-regular, the constant function $f_0 \equiv 1$, called the *harmonic eigenfunction*, is an eigenfunction with eigenvalue $\eta_1 = 0$. Then, the other eigenvalues of $\mathcal{L}$ are positive (the dimension of the 0-eigenspace is 1 if G is connected) with eigenfunctions perpendicular to f_0. By the variational characterization of the eigenvalues in terms of the Rayleigh quotient of $\mathcal{L}$, we can realize the second smallest eigenvalue $\eta_2 = k - \lambda_2$ as

$$\eta_2 = \min_{\substack{f\neq 0, f\in C(V)\\ \langle f, f_0\rangle = 0}} \frac{\langle \mathcal{L}f, f\rangle}{\langle f, f\rangle}. \tag{3.2}$$

Our strategy to prove Theorem 3.2.3 will be to pick a suitable test function f to use in (3.2) to get an upper bound on η_2, which will become a lower bound on λ_2.

Proof. We first note that it's sufficient to consider only connected graphs. If G is not connected and is k-regular, we have $\lambda_2 = k$ and the theorem holds trivially. Thus let G be a connected k-regular graph with diameter $\geq 2l + 2 \geq 4$. Suppose that there is a constant integer g such that for any pair of adjacent vertices in G, there are at least g vertices of G adjacent to both vertices. By our assumption on the diameter of G, there exist two vertices u and v such that $\text{dist}(u, v) \geq 2l+2 \geq 4$. We define some neighborhoods around these vertices: for $i \geq 0$, let

$$U_i = \{x \in V(G) | \text{dist}(x, u) = i\}, \text{ and}$$
$$V_i = \{x \in V(G) | \text{dist}(x, v) = i\}.$$

By our choice of u and v, $U_0, \cdots, U_l, V_0, \cdots, V_l$ are pairwise disjoint. In addition, no vertex in $U = \cup_{i=1}^{l} U_i$ is adjacent to any vertex in $V = \cup_{i=0}^{l} V_i$. Our assumptions on G allow us to estimate the size of each of these sets. We have $|U_0| = |V_0| = 1$ and $|U_1| = |V_1| = k$. For each vertex $x \in U_i$ with $i \geq 1$, at least 1 of its k neighbors lies in U_{i-1}. Denote one such vertex by y. By our assumption, there are at least g vertices adjacent to both x and y; thus, at least g of x's neighbors lie in U_i and U_{i-1}. This leaves at most $q = k - g - 1$ neighbors of x in U_{i+1}. This means we can bound the size of U_{i+1} by that of U_i via $|U_{i+1}| \leq q|U_i|$ for $i = 1, \cdots, l-1$. Similarly we have $|V_{i+1}| \leq q|V_i|$ for $i = 1, \cdots, l-1$. We are now ready to define our test function $f \in C(V)$ to use in the Rayleigh quotient:

$$f(x) = \begin{cases} a & \text{if } x \in U_0 \cup U_1, \\ aq^{-(i-1)/2} & \text{if } x \in U_i,\ 2 \leq i \leq l, \\ -b & \text{if } x \in V_0 \cup V_1, \\ -bq^{-(i-1)/2} & \text{if } x \in V_i,\ 2 \leq i \leq l, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $q = k - g - 1$ as before, and a, b are positive real numbers chosen so that $\langle f, f_0 \rangle = 0$.

We can now begin to estimate the terms in the Rayleigh quotient. We first look at $\langle f, f \rangle$:

$$\begin{aligned}
\langle f, f \rangle &= \sum_{x \in X} f(x)^2 \\
&= \sum_{x \in U} f(x)^2 + \sum_{x \in V} f(x)^2
\end{aligned}$$

since f is 0 outside of U and V. We consider each sum separately:

$$\begin{aligned}
A_1 &= \sum_{x \in U} f(x)^2 \\
&= a^2 \left(1 + \sum_{i=1}^{l} |U_i| q^{-(i-1)} \right) \\
&\geq a^2 \left(1 + l \frac{|U_l|}{q^{l-1}} \right).
\end{aligned}$$

The last inequality follows by repeated application of the inequality $|U_{i+1}| \leq q|U_i|$ for $i = 1, \cdots, l-1$. We can also make a similar statement for $B_1 = \sum_{x \in V} f(x)^2 \geq b^2 (1 + l \frac{|V_l|}{q^{l-1}})$.

Now we estimate the numerator $\langle \mathcal{L} f, f \rangle$. Performing the same calculation as in the proof of Proposition 3.2.5, we have:

$$\begin{aligned}
\langle \mathcal{L} f, f \rangle &= k \langle f, f \rangle - \langle A f, f \rangle \\
&= k \sum_{x \in V(G)} f(x)^2 - \sum_{x \in V(G)} f(x) \sum_{\substack{y \in V(G) \\ \{y, x\} \in E(G)}} f(y) \\
&= \sum_{\{x, y\} \in E(G)} \left(f(x)^2 - 2 f(x) f(y) + f(y)^2 \right) \\
&= \sum_{\{x, y\} \in E(G)} (f(x) - f(y))^2.
\end{aligned}$$

We now use the fact that no vertex in U is adjacent to a vertex in V to split this into two sums, one over edges with an endpoint in U and one over edges with an endpoint in V. We write $\langle \mathcal{L}f, f\rangle = A_2 + B_2$ with

$$A_2 = \sum_{\substack{\{x,y\}\in E(G)\\ x\in U}} (f(x) - f(y))^2.$$

We define B_2 in the same way by replacing U with V. We now use our definition of the test function f and the fact that for each $x \in U_i$, x has at most q neighbors in U_{i+1} to approximate A_2. Since any edge incident to the vertex in U_0 is incident to a vertex in U_1, we can begin our sum at U_1:

$$\begin{aligned}
A_2 &\leq \sum_{i=1}^{l-1} |U_i| q (q^{-(i-1)/2} - q^{-i/2})^2 a^2 + |U_l| q q^{-(l-1)} a^2 \\
&= (\sqrt{q}-1)^2 (|U_1| + |U_2| q^{-1} + \cdots + |U_{l-1}| q^{-(l-2)} + |U_l| q^{-(l-1)}) a^2 \\
&\quad + a^2 (2\sqrt{q} - 1) |U_l| q^{-(l-1)} \\
&\leq (\sqrt{q}-1)^2 (A_1 - a^2) + (2\sqrt{q}-1) \frac{A_1 - a^2}{l} \\
&< \left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q}-1}{l}\right) A_1.
\end{aligned}$$

Similarly, we can estimate B_2:

$$B_2 < \left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q}-1}{l}\right) B_1.$$

We can now put everything together to estimate η_2 with the Rayleigh quotient:

$$\begin{aligned}
k - \lambda_2 = \eta_2 &\leq \frac{A_2 + B_2}{A_1 + B_1} \\
&< \left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q}-1}{l}\right) \frac{A_1 + B_1}{A_1 + B_1} \\
&= 1 + q - 2\sqrt{q} + \frac{2\sqrt{q}-1}{l}.
\end{aligned}$$

This gives us an estimate for the second eigenvalue of the adjacency operator

$$\lambda_2 > g + 2\sqrt{q} - \frac{2\sqrt{q}-1}{l},$$

as desired. $\square$

Before we return to the hypergraph setting, we make one short note about the diameter of a graph as the number of vertices grows large. If we let D be the diameter of a k-regular graph G. Then

$$|V(G)| \leq 1 + k + k(k-1) + \cdots + k(k-1)^{D-1} < 1 + k + \cdots + k^D,$$

so

$$D \geq \frac{\log |V(G)|}{\log k} - O(1).$$

Hence, as the number of vertices grows large, D will tend to infinity.

Theorem 3.2.3 gives a strong statement about graphs, so we need to connect it back to hypergraphs. Suppose $\mathbb{H}$ is a (d, r)-regular hypergraph. Then for every pair of adjacent vertices in $G_{\mathbb{H}}$, there are at least $r - 2$ vertices adjacent to both vertices since two vertices are adjacent if and only if they both lie in the same r-clique. Thus, we take $g = r - 2$, which gives $q = k - (r - 1) = (d - 1)(r - 1)$. Combining these two numbers with the statement about the diameter growing large as the number of vertices grows large, we get the corollary we were interested in:

Corollary 3.2.6 (Feng and Li)**.** *Let $\{\mathbb{H}_m\}$ be a family of connected (d, r)-regular hypergraphs with $|V(\mathbb{H}_m)| \to \infty$ as $m \to \infty$. Then*

$$\liminf_{m\to\infty} \lambda_2(\mathbb{H}_m) \geq r - 2 + 2\sqrt{q},$$

where $k = d(r-1)$ and $q = (d-1)(r-1) = k - (r-1)$.

This is the key ingredient in their definition of Ramanujan hypergraphs; however, the associated bipartite graph also has an equally important role to play in the structure of the eigenvalues of $\mathbb{H}$. We will focus on this connection in the next section.

3.3 The associated bipartite graph

We now turn our attention to the relations between $\mathbb{H}$ and its associated bipartite graph $B_{\mathbb{H}}$. $B_{\mathbb{H}}$ is a bipartite graph with vertex sets coming from $V(\mathbb{H})$ and $E(\mathbb{H})$. This suggests that, if we are given an arbitrary bipartite graph, we'd have two ways to construct a hypergraph from it. We could pick one of the independent sets to be the hypervertices and the other to characterize the incidence relation, or we could switch it and pick the other set initially as our hypervertices. With this idea, we construct the dual hypergraph $\mathbb{H}^*$ of $\mathbb{H}$. We begin with $\mathbb{H}$ and construct $B_{\mathbb{H}}$. Then, we take $B_{\mathbb{H}}$ and construct a new hypergraph with hypervertices coming from the vertex set of $B_{\mathbb{H}}$ parameterized by $E(\mathbb{H})$ and hyperedges defined from the vertex set of $B_{\mathbb{H}}$ parameterized by $V(\mathbb{H})$. Figure 3.3 gives an example of a hypergraph in addition to its associated bipartite graph and dual hypergraph.

We will be particularly interested in the special case of regular hypergraphs. Suppose $\mathbb{H}$ is (d,r)-regular, then $B_{\mathbb{H}}$ is a (d,r)-biregular bipartite graph. When we construct $\mathbb{H}^*$, we finish with a (r,d)-regular hypergraph. We see in [23] that, for regular hypergraphs, the adjacency operators of these three structures are intimately related:

$$A(B_{\mathbb{H}}) = \begin{pmatrix} 0 & M \\ {}^tM & 0 \end{pmatrix}, \tag{3.3}$$

$$A(B_{\mathbb{H}})^2 = \begin{pmatrix} M{}^tM & 0 \\ 0 & {}^tMM \end{pmatrix} = \begin{pmatrix} A(\mathbb{H}) + dI_V & 0 \\ 0 & A(\mathbb{H}^*) + rI_E \end{pmatrix},$$

where $M = M(V,E)$ is the incidence matrix of $\mathbb{H}$, and I_V and I_E are identity matrices

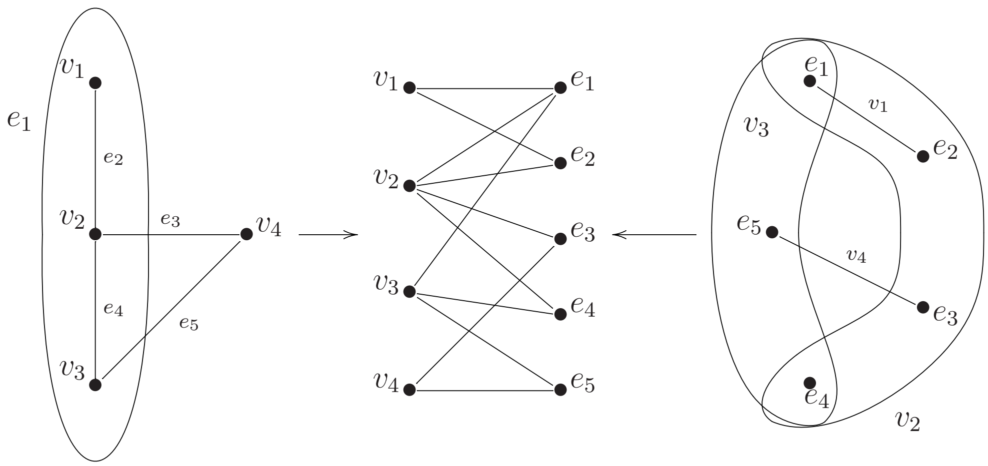

Figure 3.3: Two hypergraphs with the same associated bipartite graph. These hypergraphs are duals of each other.

with rows and columns parameterized by V and E respectively. The identity matrices appear in $A(B)^2$ since the adjacency operators of $\mathbb{H}$ and $\mathbb{H}^*$ are defined by looking at paths of length 2 in the respective structure with no backtracking. Introducing the identity matrix adds the paths with backtracking back in so that we get the full picture.

Let $P(x)$, $P^*(x)$, and $Q(x)$ denote the characteristic polynomials of $A(\mathbb{H})$, $A(\mathbb{H}^*)$, and $A(B)^2$ respectively. Then these polynomials are related as follows:

$$Q(x) = P(x-d)P^*(x-r). \tag{3.4}$$

In addition, the polynomials $P(x)$ and $P^*(x)$ satisfy the following relation, given in [9]:

$$x^{|V|}P^*(x-r) = x^{|E|}P(x-d). \tag{3.5}$$

This gives a very explicit connection between the spectrum of $\mathbb{H}$ and $\mathbb{H}^*$. Since

$Q(x)$ is the characteristic polynomial of $A(B)^2$, it has non-negative roots. By (3.4), this forces the eigenvalues of $\mathbb{H}$ and $\mathbb{H}^*$ to be at least $-d$ and $-r$ respectively. Furthermore, the eigenvalues of B are d more than the square roots of the eigenvalues of $\mathbb{H}$ and r more than the eigenvalues of $\mathbb{H}^*$. One immediate consequence of these relations is that the spectrum of a (d, r)-biregular bipartite graph is contained in the interval $[\sqrt{dr}, -\sqrt{dr}]$ since $d(r-1)$ is the largest eigenvalue of a (d, r)-regular hypergraph. In fact, $\lambda_1(B) = \sqrt{dr}$ and $\lambda_{|V(B)|} = -\sqrt{dr}$ whenever B is (d, r)-biregular.

When d and r are not equal, comparing the powers of x in both sides of (3.5) gives the *obvious* eigenvalue $-d$ of $\mathbb{H}$ with multiplicity $|V(\mathbb{H})| - |E(\mathbb{H})|$ or $-r$ of $\mathbb{H}^*$ with multiplicity $|E(\mathbb{H}) - |V(\mathbb{H})|$, depending on whether $d < r$ or $r < d$. In general, once we know the spectrum of $\mathbb{H}$, $\mathbb{H}^*$, or B, we get a great many results about the spectra of the other two structures.

As an application of (3.4), we rewrite Corollary 3.2.6 for biregular, bipartite graphs. This result tells us how far the second eigenvalue can be from $\sqrt{dr}$:

Corollary 3.3.1 (Li and Solé)**.** *Let $\{B_m\}$ be a family of connected (d, r)-biregular bipartite graphs with $|V(B_m)| \to \infty$ as $m \to \infty$. Then*

$$\liminf_{m\to\infty} \lambda_2(B) \geq \sqrt{d-1} + \sqrt{r-1}.$$

Proof. We begin by using B_m to construct a (d, r)-regular hypergraph $\mathbb{H}$ and its dual $\mathbb{H}^*$. We let $Q(x), P(x)$, and $P^*(x)$ be the characteristic polynomials of $A(B_m)$, $A(\mathbb{H})$, and $A(\mathbb{H}^*)$ respectively. Now, we investigate the critical case when $\mathbb{H}$ has second eigenvalue $\lambda_2(\mathbb{H}) = r-2+2\sqrt{(d-1)(r-1)}$. In this case, $P\left(r-2+2\sqrt{(d-1)(r-1)}\right) = 0$, which implies that $Q\left(d+r-2+2\sqrt{(d-1)(r-1)}\right) = 0$ by (3.4). We simplify this:

$$\begin{aligned} d + r - 2 + 2\sqrt{(d-1)(r-1)} &= (r-1) + (d-1) + 2\sqrt{(d-1)(r-1)} \\ &= \left(\sqrt{r-1} + \sqrt{d-1}\right)^2. \end{aligned}$$

Hence, if $\mathbb{H}$ has eigenvalue $r - 2 + 2\sqrt{(d-1)(r-1)}$, B_m must have eigenvalues $\lambda = \pm\left(\sqrt{r-1} + \sqrt{d-1}\right)$. The corollary now follows from the observation that if we look at $\alpha > r - 2 + 2\sqrt{(d-1)(r-1)}$, the corresponding root $\alpha + d$ of $Q(x)$ is larger, and thus the absolute value of the corresponding eigenvalues of B_m are larger. $\square$

We see from the computation in the above proof that the eigenvalues of biregular bipartite graphs are symmetrically arranged about 0. Zero also can occur as an eigenvalue: in particular, zeros in the spectrum correspond to the obvious eigenvalues of a hypergraph; although, zero may appear with greater multiplicity than the obvious eigenvalues do. In practice, we will often consider the squares of the eigenvalues since this gives us the same information but makes the eigenvalues a bit easier to work with.

We can now make the appropriate generalizations of *Ramanujan* hypergraph and bipartite graph. Our definition of a *Ramanujan biregular bipartite graph* is the same as that given by Hashimoto [16], and the definition for a *Ramanujan regular hypergraph* agrees with Li and Solé's [23]:

Definition 3.3.2 (Hashimoto; Li and Solé). *We make the following definitions:*

1. *Let X be a finite, connected (d, r)-biregular bipartite graph. We say X is a* Ramanujan bipartite graph *if*

$$|\lambda^2 - (d-1) - (r-1)| \leq 2\sqrt{(d-1)(r-1)}, \tag{3.6}$$

for all $\lambda \in Spec(X)$ such that $\lambda^2 \neq dr$.

2. *Let* $\mathbb{H}$ *be a finite, connected* (d, r)*-regular hypergraph. We say* $\mathbb{H}$ *is a* Ramanujan hypergraph *if*

$$|\lambda - r + 2| \leq 2\sqrt{(d-1)(r-1)}, \tag{3.7}$$

for all non-obvious *eigenvalues* $\lambda \in Spec(\mathbb{H})$ *such that* $\lambda \neq d(r-1)$.

Remark 3.3.3. This definition can be troublesome. For instance, if $d = 2$ and $r = 10$, then a Ramanujan (d, r)-regular hypergraph could have no eigenvalues of absolute value smaller than 2, which seems to defeat the goal of having small eigenvalues. We refer the reader to [25] for some explicit constructions of Ramanujan hypergraphs; although, the r considered is never larger than 4.

Remark 3.3.4. Suppose that X is a finite, connected (k, k)-biregular bipartite graph. Then X is a k-regular graph, and we see that the definition of *Ramanujan* given here corresponds to the definition given in Definition 1.1.8. Similarly, if $\mathbb{H}$ is a $(d, 2)$-regular hypergraph, $\mathbb{H}$ is just a d-regular graph, and our definition of *Ramanujan Hypergraph* agrees with the graph definition.

Proposition 3.3.5 (Li and Solé)**.** *Suppose* $\mathbb{H}$ *is a finite, connected* (d, r)*-regular hypergraph. Then* $\mathbb{H}$ *is a Ramanujan hypergraph if and only if* $B_{\mathbb{H}}$ *is a Ramanujan bipartite graph.*

Proof. This is a direct application of (3.4). □

These are the main results and definitions we will need for the spectral theory on hypergraphs. We now turn our attention to the generalizations of the Ihara-Selberg zeta function to hypergraphs.

Chapter 4

The "naive" generalization

In this chapter, we will look at what is possibly the more "naive" zeta function. We will see that it doesn't encapsulate the Ramanujan condition in a nice way; however, it will serve to identify isospectral (d, r)-regular hypergraphs. The motivation that leads to this definition is simple and has been profitable in other areas of study for hypergraphs and graphs. We will try to encapsulate the idea that a (d, r)-regular hypergraph is, in fact, a special case of a $d(r-1)$-regular graph.

The most critical part of generalizing the Ihara-Selberg zeta function is deciding what exactly we mean by a "prime cycle." For graphs, we had the notions of "backtracking", "tails", and an equivalence relation. All of these things need to be appropriately defined for hypergraphs. We will focus mainly on different generalizations of "backtracking" and see that it can lead to drastically different results. In this chapter, we take the "weakest" possible generalization of backtracking and see where it leads.

4.1 Paths and the zeta function

We let $\mathbb{H}$ be a connected hypergraph. We begin with a *closed path*: A closed path in $\mathbb{H}$ is a sequence $c = (v_1, e_1, v_2, e_2, \cdots, v_k, e_k, v_1)$ such that $v_i \in e_{i-1}, e_i$ for $i \in \mathbb{Z}/k\mathbb{Z}$.

Note that this implies that $v_1 \in e_k$ so that this path really is "closed."

We say c has *hypervertex backtracking* if there is a subsequence of c of the form $(v_i, e_i, v_{i+1}, e_i, v_i)$. Intuitively, this means that at some point we leave a hypervertex via a hyperedge e and then take that same hyperedge directly back to the hypervertex. It is permissible to return immediately to the hypervertex as long as you use a *different* hyperedge. It is also permissible to reuse the same hyperedge immediately as long as you go to a different hypervertex. This will turn out to be the weakest generalization of backtracking that we examine. We now make the same definitions that we used for graphs.

We denote by c^m the m-multiple of c formed by going around the closed path m times. Then, c is *tail-less* if c^2 does not have hypervertex backtracking. We call c a *closed geodesic* if c has no hypervertex backtracking and is tail-less. In addition, if c is not the non-trivial m-multiple of some other closed geodesic b, we say that c is a *primitive geodesic*. As before, we denote by $|c|$, the number of hyperedges in c, which we call the *length* of c.

Now that we have the definitions to give us a primitive geodesic, we need to impose an equivalence relation to obtain prime cycles. We use the same equivalence relation that worked for graphs. Namely, two primitive geodesics are equivalent if one is a cyclic permutation of the other. We call the equivalence class $[c]$ a *prime cycle*. To maintain clarity, we gather these definitions into one set of Path Criteria:

Path Criteria 2. *Let $\mathbb{H}$ be a hypergraph. Then the prime cycles on $\mathbb{H}$ satisfy Path Criteria 2 if:*

1. *A* closed geodesic *c is a closed path with no hypervertex backtracking or tails.*

2. *A closed geodesic is* primitive *if it is not b^m for some other geodesic b and integer $m \geq 2$.*

3. *A* prime cycle *c is a representative of the equivalence class $[c]$ of primitive*

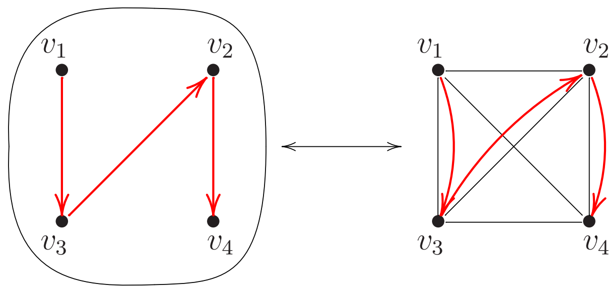

Figure 4.1: A legal path in a hyperedge of order 4 and the corresponding legal path in a 4-clique on the same vertices. The correspondence goes both ways.

geodesics, identified by cyclic permutation.

We define the generalization of the Ihara-Selberg zeta function that arises from Path Criteria 2 as follows:

Definition 4.1.1. *Let $\mathbb{H}$ be a finite, connected hypergraph. Let P be the set of prime cycles in $\mathbb{H}$ defined by Path Criteria 2. Then the* naive zeta function *of $\mathbb{H}$ is given, for sufficiently small $u \in \mathbb{C}$, by*

$$Z_{\mathbb{H}}(u) = \prod_{\mathfrak{p} \in P} \left(1 - u^{|\mathfrak{p}|}\right)^{-1}. \tag{4.1}$$

We use the word "naive" because we've imposed a very weak backtracking condition on the prime cycles. If we think about a single hyperedge, we realize that the only thing that matters is not returning to the same hypervertex via the hyperedge you just used. If we imagine replacing the hyperedge $e = \{v_1, \cdots, v_k\}$ with a clique on $\{v_1, \cdots, v_k\}$, we see that not backtracking in the hyperedge is equivalent to not backtracking – in the graph sense – in the clique. Indeed, any legal move you could make in the clique, you can make in the hyperedge. We see an example of this in Figure 4.1. This leads us to our next proposition:

Proposition 4.1.2. *Let $\mathbb{H}$ be a hypergraph and $G_{\mathbb{H}}$ the graph associated with it defined earlier. Then there is a one-to-one correspondence between prime cycles of length l*

that satisfy Path Criteria 2 in $\mathbb{H}$ and prime cycles of length l that satisfy Path Criteria 1 in $G_{\mathbb{H}}$.

Proof. Let $\mathbb{H}$ be a hypergraph and $G_{\mathbb{H}}$ be the graph formed by taking every hyperedge and replacing it with a clique on the vertices of the hyperedge. This is the graph defined above. We assume all discussion of geodesics is referring to the Path Criteria stated in the proposition.

For an edge $e \in E(G_{\mathbb{H}})$, we denote by $\hat{e}$ the hyperedge in $E(\mathbb{H})$ that gave rise to e. For an edge $\hat{e} \in E(\mathbb{H})$ with $v, w \in \hat{e}$, we denote by $\{v, w\}_{\hat{e}}$, the edge in $G_{\mathbb{H}}$ that comes from $\hat{e}$ and joins v and w.

Now suppose $\{v_1, e_1, v_2, e_2, \cdots, v_k, e_k, v_1\}$ is a primitive geodesic in $G_{\mathbb{H}}$. Then we claim that $\{v_1, \hat{e_1}, v_2, \hat{e_2}, \cdots, v_k, \hat{e_k}, v_1\}$ is a primitive geodesic in $\mathbb{H}$. We suppose that $\{v_1, \hat{e_1}, v_2, \hat{e_2}, \cdots, v_k, \hat{e_k}, v_1\}$ has hypervertex backtracking. Then there is some subsequence of the form $\{v_i, \hat{e_i}, v_{i+1}, \hat{e_i}, v_i\}$. In $G_{\mathbb{H}}$, this subsequence corresponds to $\{v_i, e_i, v_{i+1}, e_{i+1}, v_i\}$. By our construction, we have $e_i = \{v_i, v_{i+1}\}_{\hat{e_i}}$, and $e_{i+1} = \{v_{i+1}, v_i\}_{\hat{e_i}}$. This means $e_i = e_{i+1}$ and we actually have backtracking in the graph, but this is a contradiction. A similar argument rules out the possibility of a tail, so $\{v_1, \hat{e_1}, v_2, \hat{e_2}, \cdots, v_k, \hat{e_k}, v_1\}$ is a primitive geodesic in $\mathbb{H}$.

Now suppose $\{v_1, \hat{e_1}, v_2, \hat{e_2}, \cdots, v_k, \hat{e_k}, v_1\}$ is a primitive geodesic in $\mathbb{H}$. Then we claim that $\{v_1, \{v_1, v_2\}_{\hat{e_1}}, v_2, \cdots, v_k, \{v_k, v_1\}_{\hat{e_k}}, v_1\}$ is a primitive geodesic in $G_{\mathbb{H}}$ of the same length. The argument is the same as the one above, and we leave it to the reader to complete. □

This correspondence means that the naive zeta function for $\mathbb{H}$ is exactly the Ihara-Selberg zeta function of $G_{\mathbb{H}}$. In particular, this means that if $\mathbb{H}$ is actually a graph, we would get the same function as the Ihara-Selberg zeta function. We can rewrite Theorem 2.4.1 for an arbitrary hypergraph:

Theorem 4.1.3. *Let $\mathbb{H}$ be a finite, connected hypergraph with adjacency operator A.*

Let I be the identity operator on $C(V)$, and define an operator $\tilde{Q}$ on $C(V)$ by

$$(\tilde{Q}f)(v) = \left[\sum_{\substack{e \in E \\ v \in E}} (|e| - 1) - 1 \right] f(v).$$

Then,

$$Z_{\mathbb{H}}(u) = Z_{G_{\mathbb{H}}}(u) = (1 - u^2)^{\chi(\mathbb{H})} \det(I - uA + u^2\tilde{Q})^{-1}$$

where

$$\chi(\mathbb{H}) = \chi(G_{\mathbb{H}}) = |V| - |E(G_{\mathbb{H}})| = |V| - \sum_{e \in E} \binom{|e|}{2}$$

is the Euler number of $G_{\mathbb{H}}$.

Proof. We have $Z_{\mathbb{H}}(u) = Z_{G_{\mathbb{H}}}(u)$ by Proposition 4.1.2. The main expression is exactly Theorem 2.4.1, so we need only show that the Euler number and operators match the corresponding Euler number and operators on $G_{\mathbb{H}}$.

We begin with the Euler number $\chi(G_{\mathbb{H}})$. For a given hyperedge $e \in E(\mathbb{H})$, e induces exactly $\binom{|e|}{2}$ edges in $G_{\mathbb{H}}$ because we form a clique on $|e|$ vertices. This establishes that the definition given for $\chi(\mathbb{H})$ is what is needed for Bass's Theorem.

$G_{\mathbb{H}}$ was constructed explicitly to have the same adjacency operator as $\mathbb{H}$, so we need only confirm that the operator $\tilde{Q}$ does the correct thing. For a given hypervertex v in a hyperedge e, v is adjacent to the other $|e| - 1$ hypervertices in e when we pass to $G_{\mathbb{H}}$. This means that the *degree* of v is the sum of $|e| - 1$ over all hyperedges which contain v. We then realize the Q operator on $G_{\mathbb{H}}$ by subtracting one from the degree before multiplying by $f(v)$. $\square$

In the particular case that $\mathbb{H}$ is (d, r)-regular, then $G_{\mathbb{H}}$ is a $d(r-1)$-regular graph, so we can formulate a reasonable Riemann hypothesis for regular hypergraphs, using the one for graphs. In this case, $\tilde{Q}$ is a diagonal matrix with $d(r - 1) - 1$, which is the degree of each vertex in $G_{\mathbb{H}}$ minus 1, in each diagonal entry. We now give the corresponding Riemann hypothesis for $Z_{\mathbb{H}}$.

Definition 4.1.4. *Suppose* $\mathbb{H}$ *is a* (d, r)*-regular hypergraph. Let* $\tilde{q} = d(r-1) - 1$. *Then* $Z_{\mathbb{H}}(\tilde{q}^{-s})$ *satisfies the* Riemann Hypothesis *iff for*

$$\mathrm{Re(s)} \in (0,1), \;\; \mathrm{Z_X^{-1}(q^{-s})} = 0 \Longrightarrow \mathrm{Re(s)} = \frac{1}{2}.$$

Unfortunately, the relation between the Riemann hypothesis and the Ramanujan condition is not as strong as before. The key issue is that if the underlying graph $G_{\mathbb{H}}$ is Ramanujan, it does not necessarily mean that $\mathbb{H}$ is Ramanujan. The problem comes when we examine the eigenvalues which are less than zero. Let us suppose that $G_{\mathbb{H}}$ is a $d(r-1)$-regular Ramanujan graph. This means that any non-trivial eigenvalue $\lambda \in \mathrm{Spec}(G_{\mathbb{H}})$ satisfies

$$-2\sqrt{d(r-1)-1} \leq \lambda. \tag{4.2}$$

However, if we want $\mathbb{H}$ to be a Ramanujan (d, r)-regular hypergraph, we would need for λ to satisfy

$$r - 2 - 2\sqrt{(d-1)(r-1)} \leq \lambda. \tag{4.3}$$

If we are just given the information in (4.2), we cannot conclude that the condition in (4.3) is met, as can be quickly seen by taking the values $r = 5$ and $d = 4$. We should also point out that Equation (3.4) implies that

$$-d \leq \lambda;$$

however, this is still not sufficient information to give us the bound in (4.3). If we were only to require that $Z_{\mathbb{H}}$ satisfy the Riemann hypothesis, we would only know that $G_{\mathbb{H}}$ is a Ramanujan graph, which is not enough to force $\mathbb{H}$ to satisfy the appropriate eigenvalue bounds.

This does not mean that we cannot obtain any spectral information from this zeta function. If $\mathbb{H}$ is Ramanujan, we can use Theorem 4.1.3 to identify where poles can appear.

Proposition 4.1.5. *Suppose $\mathbb{H}$ is a finite, connected (d, r)-regular Ramanujan hypergraph. Let $\tilde{q} = d(r-1) - 1$, and let $i = \sqrt{-1}$. Then*

1. *$Z_{\mathbb{H}}(u)$ has poles at $u = \pm 1$ corresponding to the factor $(1-u^2)^{\chi}(G_{\mathbb{H}})$ of the zeta function.*

2. *$Z_{\mathbb{H}}(u)$ has an additional pole at $u = 1$ and a simple pole at $u = \frac{1}{\tilde{q}}$ corresponding to the eigenvalue $\lambda = d(r-1)$.*

3. *If $d < r$, $Z_{\mathbb{H}}(u)$ has poles at*

$$u = \frac{-d \pm i\sqrt{4\tilde{q} - d^2}}{2\tilde{q}}$$

   with density $1 - d/r$ corresponding to the obvious *eigenvalues of $\mathbb{H}$.*

4. *$Z_{\mathbb{H}}(u)$ has poles on the circle*

$$|u| = \frac{1}{\tilde{q}^{1/2}} \textit{with } \mathrm{Re}(u) < 0 \textit{ and } |\mathrm{Im(u)}| \geq \frac{\sqrt{4\tilde{\mathrm{q}} - (\mathrm{r} - 2 - 2\sqrt{(\mathrm{d}-1)(\mathrm{r}-1)})^2}}{2\tilde{\mathrm{q}}}$$

   corresponding to eigenvalues which satisfy

$$r - 2 - 2\sqrt{(d-1)(r-1)} \leq \lambda < 0.$$

5. *$Z_{\mathbb{H}}(u)$ has poles on the circle $|u| = \frac{1}{\tilde{q}^{1/2}}$ with $\mathrm{Re(u)} \geq 0$, corresponding to eigenvalues which satisfy*

$$0 \leq \lambda \leq 2\sqrt{\tilde{q}}.$$

6. $Z_{\mathbb{H}}(u)$ has poles on the real axis in the interval

$$[\frac{1}{\tilde{q}^{1/2}}, \frac{(r-2+2\sqrt{(d-1)(r-1)}) + \sqrt{(r-2+2\sqrt{(d-1)(r-1)})^2 - 4\tilde{q}}}{2\tilde{q}}]$$

corresponding to eigenvalues which satisfy

$$2\sqrt{\tilde{q}} \leq \lambda \leq r-2+2\sqrt{(d-1)(r-1)}.$$

Conversely if $\mathbb{H}$ is a finite, connected (d,r)-regular hypergraph, $\mathbb{H}$ is Ramanujan if the poles of $Z_{\mathbb{H}}(u)$ fall appropriately in the given regions.

Proof. Suppose $\mathbb{H}$ is a (d,r)-regular hypergraph, then all of the above conditions are a consequence of the expression

$$Z_{\mathbb{H}}(u) = (1-u^2)^{\chi(\mathbb{H})} \prod_{\lambda \in \mathrm{Spec}(\mathbb{H})} \left(1 - \lambda u + \tilde{q}u^2\right),$$

and the quadratic formula for the polynomial

$$f(u) = \tilde{q}u^2 - \lambda u + 1.$$

We make a few remarks and leave the bulk of the proof to the reader. It is spiritually identical to the proof of Corollary 2.4.3.

1. Statements 1 and 2 follow immediately from Bass's expression and by writing the zeta function as a product over eigenvalues.

2. Statement 3 considers the *obvious* eigenvalues for hypergraphs. When deciding if the poles determine the Ramanujan condition, it is important that the multiplicity of the described pole be correct.

3. Statement 4 has the condition on the imaginary part of the poles on the left

half circle to rule out eigenvalues in the range

$$-2\sqrt{\tilde{q}} \leq \lambda < r - 2 - 2\sqrt{(d-1)(r-1)}.$$

If $r = 2$, we see that this condition is trivial and we can have any pole on the left half circle, just as before with graphs.

4. Statement 5 is clear from Corollary 2.4.3.

5. Statement 6 locates the poles that occur from eigenvalues larger than the graph Ramanujan condition allows but still within the hypergraph Ramanujan condition. The left endpoint of the interval is $\frac{1}{\tilde{q}^{1/2}}$ because

$$\lambda \geq \sqrt{\lambda^2 - 4\tilde{q}},$$

so the negative branches of the square root can never push us too far to the left. $\square$

Hence, we do have a complete characterization of the Ramanujan condition in terms of the poles of the naive zeta function; however, it is not nearly as clean as the result for regular graphs. Since a (d, r)-regular hypergraph can be viewed as a special case of a regular graph, we can directly take many of the results that apply to the Ihara-Selberg zeta function of a regular graph. We summarize the main properties here:

1. $Z_{\mathbb{H}}(u)$ is defined in terms of an Euler product expansion. There is a sum expansion as before though we typically don't need it.

2. The poles of $Z_{\mathbb{H}}(u)$ determine when $\mathbb{H}$ is a Ramanujan (d, r)-regular hypergraph.

3. $Z_{\mathbb{H}}(u)$ satisfies functional equations as in Corollary 2.4.8.

4. Two (d, r)-regular hypergraphs are cospectral if and only if they have the same naive zeta function.

We haven't really managed to take advantage of the extra hypergraph structure. In the next section, we will show an additional property of the naive zeta function of a (d,r)-regular hypergraph if the hypergraph satisfies a certain coloring condition.

4.2 Strongly r-colorable hypergraphs

In this section, we will restrict our attention to a more specific class of (d,r)-regular hypergraphs. We will be able to compute explicitly some of the eigenvalues of the adjacency operator of these hypergraphs, which will allow us to pinpoint more factors of the naive zeta function.

We will be interested in *strong r-colorings* of hypergraphs, as defined by Berge [3].

Definition 4.2.1 (Berge)**.** *A* strong r-coloring *of a hypergraph $\mathbb{H}$ is an r-coloring of the hypervertices of $\mathbb{H}$ such that no two hypervertices contained in the same hyperedge have the same color. The* strong chromatic number $\gamma(\mathbb{H})$ *is the smallest integer r for which there exists a strong r-coloring.*

If $\mathbb{H}$ is a (d,r)-regular hypergraph, we must have $\gamma(\mathbb{H}) \geq r$ since every hyperedge is of order r. If we actually have $\gamma(\mathbb{H}) = r$, we can pick out an extra factor of the naive zeta function if our hypergraph has enough extra structure.

Proposition 4.2.2. *Suppose $\mathbb{H}$ is a finite, connected (d,r)-regular hypergraph with $\gamma(\mathbb{H}) = r$. Suppose, in addition, that any two hyperedges in $\mathbb{H}$ intersect in at most one hypervertex. Then $A(\mathbb{H})$ has eigenvalue $-d$ with multiplicity at least $r-1$.*

Proof. We begin by fixing a strong r-coloring of $\mathbb{H}$. Let $m = |E(\mathbb{H})|$, so there are exactly m hypervertices of each color. We denote by $v_i^{(j)}$ the i^{th} hypervertex of color

j. Now we order the hypervertices by color:

$$v_1^{(1)}, \cdots, v_m^{(1)}$$
$$v_1^{(2)}, \cdots, v_m^{(2)}$$
$$\cdots$$
$$v_1^{(r)}, \cdots, v_m^{(r)}.$$

With this ordering of the hypervertices, the adjacency operator A has block form

$$\begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{r1} & \cdots & A_{rr} \end{pmatrix},$$

where A_{ij} is an $m \times m$ block. In addition, we know a good bit about the blocks. The diagonal blocks A_{ii} must be zero matrices since the diagonal blocks record the adjacency relations of hypervertices of the same color. By the definition of a strong r-coloring, no hypervertices of the same color may be adjacent.

Now suppose that A_{ij} is one of the off-diagonal blocks; i.e., $i \neq j$. Then, we claim that each row and column of A_{ij} has exactly d 1's. Each hypervertex is in exactly d hyperedges, and each hyperedge has each color possible attached to its hypervertices. Since any two hyperedges intersect in at most one hypervertex, a particular hypervertex is adjacent to exactly d distinct hypervertices of each different color.

W now exhibit $r-1$ linearly independent eigenvectors, each having eigenvalue $-d$. For $2 \leq k \leq r$, define the vector w_k by putting a 1 in the first m entries of w_k and -1 in the k^{th} m entries. The set $\{w_k\}$ defined in this way has exactly $r-1$ elements, and it is clear that it is linearly independent. Moreover, $Aw_k = -dw_k$ for each w_k due to the structure of A described above and the usual matrix operations. $\square$

Knowing these eigenvalues allows us to factor the determinant expression of the naive zeta function further. We have the following corollary:

Corollary 4.2.3. *Suppose $\mathbb{H}$ is a finite, connected (d, r)-regular hypergraph with $\gamma(\mathbb{H}) = r$. Suppose, in addition, that any two hyperedges in $\mathbb{H}$ intersect in at most one hypervertex. Then the polynomial $(1 + du + (d(r-1)-1)u^2)^{r-1}$ divides $Z_{\mathbb{H}}^{-1}(u)$.*

Proof. Since $\mathbb{H}$ is (d, r)-regular, we have $\tilde{Q} = (d(r-1)-1)I$ where I is the identity matrix. This allows us to rewrite:

$$Z_{\mathbb{H}}^{-1}(u) = (1-u^2)^{-\chi_{\mathbb{H}}} \det(I - Au + (d(r-1)-1)Iu^2).$$

Since A and I commute, we can simultaneously diagonalize them to rewrite

$$Z_{\mathbb{H}}^{-1}(u) = (1-u^2)^{-\chi_{\mathbb{H}}} \prod_{\lambda \in \mathrm{Spec(A)}} (1 - \lambda u + (d(r-1)-1)u^2).$$

The previous proposition then gives the result by considering the terms when $\lambda = -d$. $\square$

While this is certainly a nice statement, the class of hypergraphs for which it applies is very small. In general, we haven't truly taken advantage of the freedom of a hypergraph in our definition of the naive zeta function. In the next chapter, we will revisit the backtracking definitions to take more advantage of the hypergraph structure. We will see that the theory develops in a much cleaner way to give us a very satisfying Riemann hypothesis statement as well as interesting graph applications.

Chapter 5

Generalized Ihara-Selberg zeta function

We will now look at what we feel to be a more useful generalization of the Ihara-Selberg zeta function to hypergraphs. Where the naive zeta function had difficulty identifying the Ramanujan condition for (d, r)-regular hypergraphs, we will see that this new zeta function behaves nearly identically to the Ihara-Selberg zeta function. We will also be able to view this zeta function as a product over prime cycles on a graph; however, we will be able to choose our cycles with greater flexibility than before. In this sense, we feel we have a true generalization. Taking this viewpoint, we will be able to distinguish some non-isomorphic, cospectral k-regular graphs, something the Ihara-Selberg zeta function cannot do because of Theorem 2.5.2.

We organize this chapter as follows. We begin with the path definitions that are used to define the zeta function. Then, we generalize the constructions that lead us through the Perron–Frobenius framework used to obtain an initial determinant expression of the Ihara-Selberg zeta function. Once we have an initial expression of the zeta function in terms of a Perron–Frobenius operator on some strongly connected, oriented graph, we will shift viewpoints and recall the connection between hypergraphs and bipartite graphs. This will let us draw from Hashimoto's [16] results for a more detailed expression.

5.1 The definition

As with the naive zeta function, the most critical step is to decide what the appropriate notion of backtracking is. We will take a more restrictive view this time so that we can hope to take better advantage of the hypergraph structure. We first recall the definition of a *chain* in a hypergraph, found in Berge [3]:

Definition 5.1.1. *In a hypergraph* $\mathbb{H}$, *a* chain of length l *is defined to be a sequence* $\{x_1, e_1, x_2, \cdots, e_l, x_{l+1}\}$ *such that*

1. $x_1, x_2, \cdots, x_l$ *are all distinct hypervertices of* $\mathbb{H}$,
2. $e_1, \cdots, e_l$ *are all distinct hyperedges of* $\mathbb{H}$,
3. $x_k, x_{k+1} \in e_k$ *for* $k = 1, 2, \cdots, l$.

This definition is a bit too restrictive for us since we don't mind reusing hypervertices and hyperedges so long as we wander around some in our hypergraph first. We can use the spirit of this definition to motivate our path definitions.

A *closed path* in $\mathbb{H}$ is a sequence $c = (v_1, e_1, v_2, e_2, \cdots, v_k, e_k, v_1)$ such that $v_i \in e_{i-1}, e_i$ for $i \in \mathbb{Z}/k\mathbb{Z}$. Note that this implies that $v_1 \in e_k$ so that this path really is "closed." We require that $v_i \neq v_{i+1}$ unless v_i is repeated in e_i, in which case we imagine that we are going to a "different" copy of v_i. We say c has *hyperedge backtracking* if there is a subsequence of c of the form (e_j, v_{j+1}, e_j). This simply says that we use a hyperedge twice in a row. It is permissible to return directly to a hypervertex so long as a different hyperedge is used. An example of *hyperedge backtracking* is illustrated in Figure 5.1.

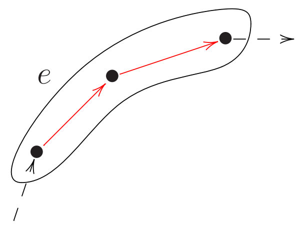

Figure 5.1: Hyperedge backtracking in a 3-edge e.

We denote by c^m the m-multiple of c formed by going around the closed path m times. Then, c is *tail-less* if c^2 does not have hyperedge backtracking. We call a closed path c a *closed relaxed chain* if c has no hyperedge backtracking and is tail-less. In addition, if c is not the non-trivial m-multiple of some other closed relaxed chain b, we say that c is a *primitive, closed, relaxed chain.* As before, we denote by $|c|$, the number of hyperedges in c, which we call the *length* of c. These definitions, of course, are entirely identical to those given for the previous zeta function. We've changed the names to maintain clarity between the two different path types.

We use the same equivalence relation as before: namely, two primitive, closed, relaxed chains are equivalent if one is a cyclic permutation of the other. We call the equivalence class $[c]$ a *prime chain.* We state these definitions as Path Criteria 3:

Path Criteria 3. *Let $\mathbb{H}$ be a hypergraph. Then the prime chains on $\mathbb{H}$ satisfy Path Criteria 3 if:*

1. *A* closed relaxed chain *c is a closed path with no hyperedge backtracking or tails.*
2. *A closed relaxed chain is* primitive *if it is not b^m for some other relaxed chain b and $m \geq 2 \in \mathbb{Z}$.*
3. *A* prime chain *c is a representative of the equivalence class $[c]$ of primitive, closed, relaxed chains, identified by cyclic permutation.*

We define the *generalized Ihara-Selberg zeta function* of a hypergraph $\mathbb{H}$ as follows:

Definition 5.1.2. *Let $\mathbb{H}$ be a finite, connected hypergraph. Let P be the set of prime chains in $\mathbb{H}$ that satisfy Path Criteria 3, then the* generalized Ihara-Selberg zeta function *of $\mathbb{H}$ is given, for sufficiently small $u \in \mathbb{C}$, by*

$$\zeta_{\mathbb{H}}(u) = \prod_{\mathfrak{p} \in P} \left(1 - u^{|\mathfrak{p}|}\right)^{-1}. \tag{5.1}$$

Remark 5.1.3. There is a very pleasing interpretation of this zeta function as it can be applied to graphs. Suppose X is a graph with one triangle. We can imagine removing the edges in the triangle and replacing them with a hyperedge of order three. Then we could form this zeta function. This would be different from the Ihara-Selberg zeta function as what we've really done is taken the product of all prime cycles in the graph X with the extra condition that you cannot use two edges of the triangle consecutively. We will elaborate on this idea later when we show that two cospectral graphs with the same Ihara-Selberg zeta function are not, in fact, isomorphic.

With the naive zeta function, it was fairly immediate that the path structure for $\mathbb{H}$ from Path Criteria 2 matched up with the path structure given by Path Criteria 1 on $G_{\mathbb{H}}$. It will turn out that the associated bipartite graph $B_{\mathbb{H}}$ will have an important role to play here; however, this is not immediately clear. As with the Ihara-Selberg zeta function, we will first begin by constructing a strongly connected, oriented graph with the same path structure as $\mathbb{H}$ as given by Path Criteria 3. Then we will be able to use the Perron–Frobenius framework to start our factorization. Our strategy will be to generalize Kotani and Sunada's method that was detailed for graphs [21].

5.2 From hypergraph to directed graph

In this section, we show how to take an arbitrary hypergraph $\mathbb{H}$ and convert it into a strongly connected, oriented graph which has the admissible prime cycles in one-to-one correspondence with the prime chains on $\mathbb{H}$. When we did this construction for graphs, the idea was to first orient the graph and then to add in the opposing orientation. We then constructed the *oriented line graph* by looking at how oriented edges fed into each other. The key to dealing with backtracking in the initial graph was to not let an oriented edge feed into its inverse edge when we were forming the oriented line graph. This neatly resolved the issue.

For hypergraphs, the construction is essentially the same, but we will have a stronger restriction on which oriented edges can feed into other oriented edges. This will allow us to disallow using the same hyperedge to go to two consecutive hypervertices even if its a different hypervertex. The key step will be to color the edges of an oriented graph and then construct an oriented line graph with the rule that you cannot use two colors in a row. This will let us generalize the backtracking condition we had on graphs in the appropriate way.

Let $\mathbb{H}$ be a finite, connected hypergraph. We label the edges of $\mathbb{H}$: $E = \{e_1, e_2, \cdots, e_m\}$ and fix m colors $\{c_1, c_2, \cdots, c_m\}$. We now construct an edge-colored graph $G\mathbb{H}_c$ as follows. The vertex set $V(G\mathbb{H}_c)$ is the set of all hypervertices $V(\mathbb{H})$. For each hyperedge $e_j \in E(\mathbb{H})$, we construct an $|e_j|$-clique in $G\mathbb{H}_c$ on the hypervertices in e_j. We then color the edges of this $|e_j|$-clique c_j. Thus if e_j is a hyperedge of order i, we have $\binom{i}{2}$ edges in $G\mathbb{H}_c$, all colored c_j. This construction should look very similar to the construction of $G_{\mathbb{H}}$. In fact, it is the same except that we've colored the edges depending upon which hyperedge they are induced from.

Once we've constructed $G\mathbb{H}_c$, we orient all of the edges. As before, we then include the opposition orientation as well, giving the inverse edges the same colors, so we obtain a graph $G\mathbb{H}_c^o$ which has twice as many colored, oriented edges as $G\mathbb{H}_c$.

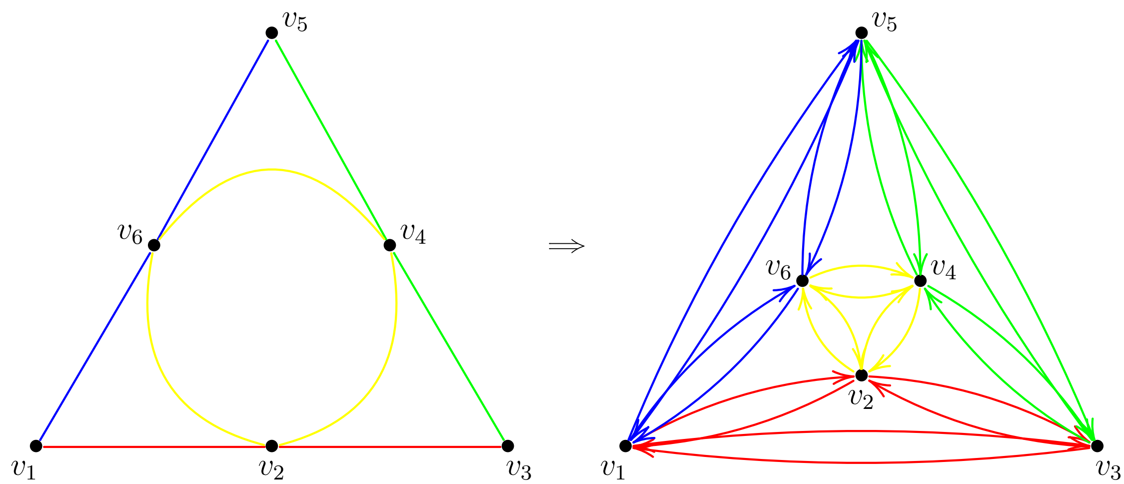

Figure 5.2: One possible construction of the graph $G\mathbb{H}_c^o$ from a hypergraph.

$G\mathbb{H}_c^o$ will serve the same role as X_o served for graphs.

Example 5.2.1. See Figure 5.2 for an example of this construction. The object on the left is a hypergraph with colors used to denote hyperedges. The hypervertices are $\{v_1, \cdots, v_6\}$, and the hyperedges are $\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_1, v_5, v_6\}$, $\{v_2, v_4, v_6\}$. The figure on the right is a construction of the edge-colored, oriented graph $G\mathbb{H}_c^o$. The hyperedge $\{v_1, v_2, v_3\}$ has become 6 red, oriented edges on the vertices $\{v_1, v_2, v_3\}$ in $G\mathbb{H}_c^o$.

Remark 5.2.2. There are many possible ways to construct a graph $G\mathbb{H}_c^o$, depending upon how we choose to orient the edges of $G\mathbb{H}_c$. We will get the same result irrespective of this choice, so in principle, it won't matter which orientation we choose when doing our constructions.

Finally, we construct the *oriented line graph* $\mathbb{H}_L^o = (V_L, E_L^o)$ associated with our choice of $G\mathbb{H}_c^o$ by

$$V_L = E(G\mathbb{H}_c^o),$$

$$E_L^o = \{(e_i, e_j) \in E(G\mathbb{H}_c^o) \times E(G\mathbb{H}_c^o); c(e_i) \neq c(e_j), t(e_i) = o(e_j)\},$$

where $c(e_i)$ is the colored assigned to the oriented edge $e_i \in E(G\mathbb{H}_c^o)$.

This condition can look slightly complicated, but all we are doing is cataloguing which oriented edges feed into oriented edges of a different color. If our hypergraph is actually just a graph, the only oriented edge with the same color is the inverse edge, so we actually recover the construction of X_L^o given earlier. We give an example in Figure 5.3 which is a bit smaller than the hypergraph in Figure 5.2 so that we can fully see what is happening.

Now that we have a construction for the oriented line graph of a hypergraph, we need to show that it is strongly connected and has the same cycle structure as the prime chains on $\mathbb{H}$. If we can do this, we will be able to write the generalized Ihara-Selberg zeta function of $\mathbb{H}$ in terms of a determinant involving the Perron–Frobenius operator T of $\mathbb{H}_L^o$ by invoking Lemma 2.3.2.

Proposition 5.2.3. *Suppose $\mathbb{H}$ is a finite, connected hypergraph such that every hypervertex is in at least two hyperedges. Further, suppose that $\mathbb{H}$ contains more than two prime chains. Then, the oriented line graph $\mathbb{H}_L^o$ constructed as above is finite and strongly connected.*

Proof. The vertices of $\mathbb{H}_L^o$ are of the form $\{v, w\}_e$ where $e \in E(\mathbb{H})$ and $v, w \in e$. This represents using the hyperedge e to go from v to w. To show that $\mathbb{H}_L^o$ is strongly connected, we must show that given any two vertices $x, y \in V(\mathbb{H}_L^o)$, there exists a path which begins at x and ends at y. We write vertex x as $\{v_1, v_2\}_{e_1}$ for some $e_1 \in E(\mathbb{H})$ and $v_1, v_2 \in e_1$. Similarly, we write y as $\{v_k, v_{k+1}\}_{e_k}$ for some $e_k \in E(\mathbb{H})$ and $v_k, v_{k+1} \in e_k$. To construct a path in $\mathbb{H}_L^o$ which begins at x and ends at y, we first find a path c in $\mathbb{H}$ of the form $c = (v_1, e_1, v_2, e_2, \cdots, e_{k-1}, v_k, e_k, v_{k+1})$ such that c has no hyperedge backtracking. If c has no hyperedge backtracking, when we follow this path through the construction of the oriented line graph, we get a path in $\mathbb{H}_L^o$ which begins at $x = \{v_1, v_2\}_{e_1}$ and ends at $y = \{v_k, v_{k+1}\}_{e_k}$, as desired. We now restrict our attention to the hypergraph $\mathbb{H}$ and show that there is always such a path c.

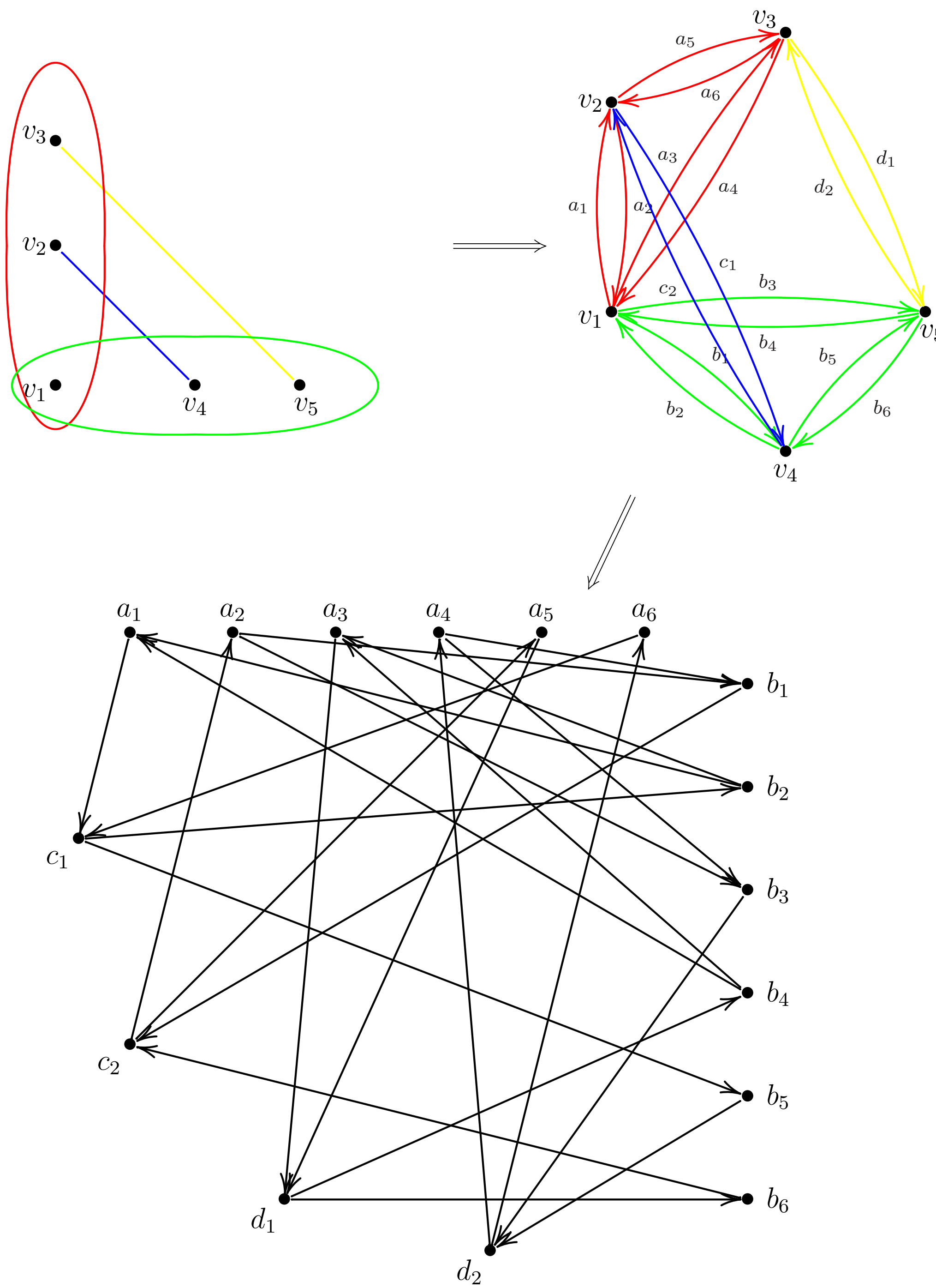

Figure 5.3: We begin with a hypergraph $\mathbb{H}$ in the top left. Then we construct one possible edge-colored oriented graph $G\mathbb{H}_c^o$. From this graph, we construct the corresponding oriented line graph. We notice that there are no edges that go from a_i to a_j; this is because they represent the red edges in $G\mathbb{H}_c^o$.

Since $\mathbb{H}$ is connected and every hypervertex is in at least 2 hyperedges, there exists a path d which begins with $(v_1, e_1, v_2, \cdots)$ and finishes at vertex v_k. We can refine d to remove any consecutive use of hyperedges. For instance, if there is a subsequence of the form $\{v_i, e_i, v_{i+1}, e_i, v_{i+2}\}$, we could replace it with $\{v_i, e_i, v_{i+1}\}$. Thus, we can assume without loss of generality that d has no hyperedge backtracking.

Now there are two cases: either the path used e_k in the last step to get to v_k or it did not. If the path did not use e_k, we can use e_k to go to v_{k+1}, and we are done.

To illustrate how the second case can occur, we refer to figure 5.4. It is impossible to begin a path with $\{v_1, e_1, v_2\}$ and end it with $\{v_4, e_3, v_3\}$ because our start gets us off in the wrong direction, and there is no room to turn around without backtracking. Hence, we need the additional hypothesis that there are more than two prime chains. We can get the desired path by leaving v_k via a hyperedge different than e_k. Then there is some chain (which may have a tail) which returns to v_k via a hyperedge different than e_k (and doesn't use hyperedge backtracking — we can refine in the same manner as before). Now we can go from v_k to v_{k+1} via e_k without hyperedge backtracking. This yields the desired path. We refer to figure 5.5 for an example. We begin with $\{v_1, e_1, v_2\}$ and wish to finish with $\{v_4, e_3, v_3\}$. To do this, we take advantage of a different chain originating from v_5 to be able to finish the path in the appropriate direction.

In essence, we need more than two prime chains to allow ourselves to "turn around" if we start in the wrong direction. Since we have the desired path in the hypergraph, as mentioned before, we get a path which starts at x and finishes at y in $\mathbb{H}_L^o$. Hence, $\mathbb{H}_L^o$ is strongly connected.

That $\mathbb{H}_L^o$ is finite is clear since $\mathbb{H}$ is finite. $\square$

Proposition 5.2.3 tells us that $\mathbb{H}_L^o$ is a candidate for the Perron–Frobenius framework we established earlier. Now we need to show that $\mathbb{H}_L^o$ is the correct model to understand the generalized Ihara-Selberg zeta function on $\mathbb{H}$.

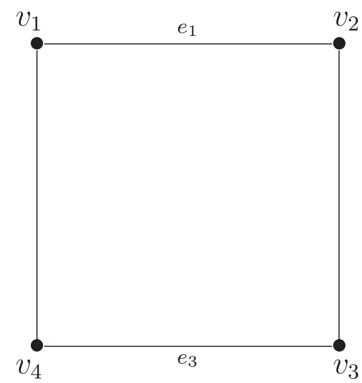

Figure 5.4: A cycle of length 4

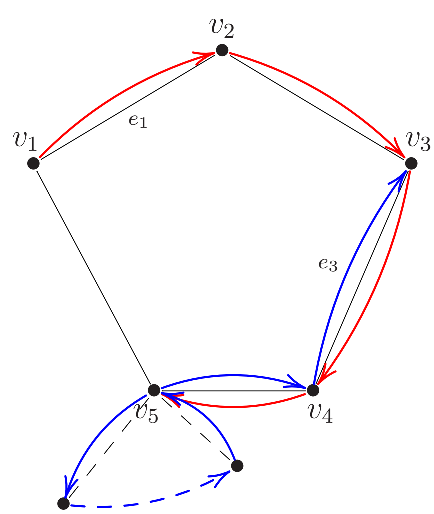

Figure 5.5: “Using a second chain to turn around”

Proposition 5.2.4. *Let $\mathbb{H}$ be a finite, connected hypergraph and let $\mathbb{H}_L^o$ be the associated oriented line graph. Then there is a one-to-one correspondence between admissible prime cycles in $\mathbb{H}_L^o$ and prime chains in $\mathbb{H}$.*

Proof. To show the cycle correspondence, we will actually show that there is a correspondence between paths in $\mathbb{H}$ with no hyperedge backtracking and admissible paths in $\mathbb{H}_L^o$. The cycle correspondence will then follow since all the relations imposed on paths are the same.

Suppose v and w are hypervertices contained in a hyperedge e. Then we denote by $\{v, w\}_e$ the oriented edge in $G\mathbb{H}_c^o$ with origin v, terminus w, and color given by the color chosen for e. We let $c = (v_1, e_1, v_2, e_2, \cdots, v_k, e_k, v_{k+1})$ be a path in $\mathbb{H}$ with no hyperedge backtracking. This corresponds to the path $c^o = (\{v_1, v_2\}_{e_1}, \{v_2, v_3\}_{e_2}, \cdots, \{v_k, v_{k+1}\}_{e_k})$ in $G\mathbb{H}_c^o$. Since there is no hyperedge backtracking, $e_i \neq e_{i+1}$ at every step, so we change colors as we follow each oriented edge. Then the corresponding path $\tilde{c} = ((\{v_1, v_2\}_{e_1}, \{v_2, v_3\}_{e_2}), (\{v_2, v_3\}_{e_2}, \{v_3, v_4\}_{e_3}), \cdots, (\{v_{k-1}, v_k\}_{e_{k-1}}, \{v_k, v_{k+1}\}_{e_k}))$ in $\mathbb{H}_L^o$ is admissible with length k.

Similarly, given an admissible path in $\mathbb{H}_L^o$, we can realize it as a path in $G\mathbb{H}_c^o$ which changes colors at every step. That means the corresponding path in $\mathbb{H}$ changes hyperedges at every step; i.e., it does not have hyperedge backtracking. The lengths, then, are the same. $\square$

Putting these two propositions together, we can invoke Lemma 2.3.2 to start our factorization:

Theorem 5.2.5. *Let $\mathbb{H}$ be a finite, connected hypergraph with more than two prime chains. Then the generalized Ihara-Selberg zeta function $\zeta_{\mathbb{H}}(u)$ satisfies*

$$\zeta_{\mathbb{H}}(u) = Z_{\mathbb{H}_L^o}(u) = \det(I - uT)^{-1},$$

where T is the Perron–Frobenius operator associated with $\mathbb{H}_L^o$.

Proof. The equality $\zeta_{\mathbb{H}}(u) = Z_{\mathbb{H}_L^o}(u)$ follows from Proposition 5.2.4.

To get the second equality, we first note that $\mathbb{H}_L^o$ is finite and strongly connected by Proposition 5.2.3. Then we can invoke Lemma 2.3.2 to realize $Z_{\mathbb{H}_L^o}(u) = \det(I - uT)^{-1}$. $\square$

This theorem gives us an initial determinant expression of $\zeta_{\mathbb{H}}(u)$. In particular, $\zeta_{\mathbb{H}}(u)$ must be a rational function and will have a simple pole at $u = \alpha^{-1}$ where α is the Perron–Frobenius root of T, defined in Lemma 2.3.1. In the next section, we will see that the associated bipartite graph $B_{\mathbb{H}}$ will have an important role to play in understanding the zeta function. This will allow us to relate our zeta function to the zeta functions Hashimoto considered on bipartite graphs [16].

5.3 Hashimoto's determinant expressions

In the previous section, we were able to realize the generalized Ihara-Selberg zeta function as a determinant of linear operators. In this section, we will see that by shifting our view to the associated bipartite graph of a hypergraph, we can do much better. Once we've established the needed relation between chains on hypergraphs and paths on bipartite graphs, we will draw very heavily from Hashimoto's work on zeta functions of bipartite graphs [16].

To motivate the relation we are looking for, we look at a simple example. In Figure 5.6, we look at the closed relaxed chain given by $c = \{v_1, e_1, v_2, e_3, v_4, e_2, v_1\}$. This corresponds to a primitive geodesic $\tilde{c} = \{v_1, e_1, v_2, e_3, v_4, e_2, v_1\}$ in the associated bipartite graph. In fact, this sort of correspondence is true in general:

Proposition 5.3.1. *Let $\mathbb{H}$ be a finite, connected hypergraph with associated bipartite graph $B_{\mathbb{H}}$. Then there is a one-to-one correspondence between prime chains of length l in $\mathbb{H}$ and prime cycles of length $2l$ in $B_{\mathbb{H}}$.*

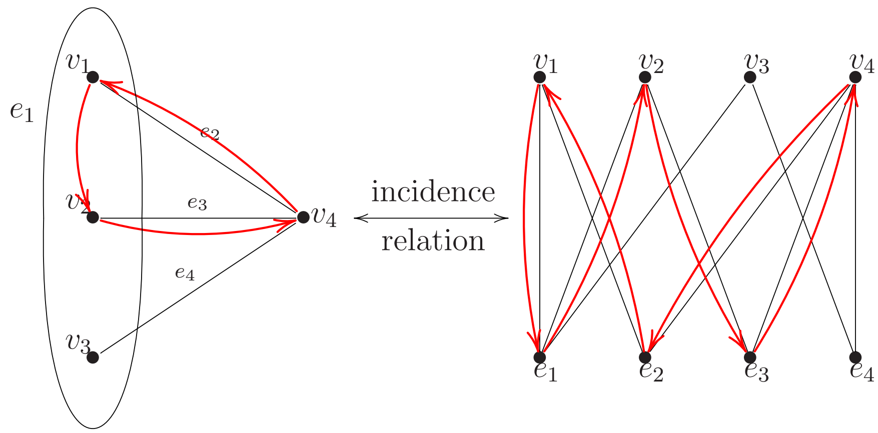

Figure 5.6: A simple example of a relaxed chain of length 3 in a hypergraph and a corresponding primitive geodesic of length 6 in its associated bipartite graph.

Proof. We will begin with a representative of a prime chain of length l in $\mathbb{H}$. Let $c = \{v_1, e_1, \cdots, v_l, e_l, v_1\}$ be a closed relaxed chain in $\mathbb{H}$. Then we claim that $\tilde{c} = \{v_1, e_1, \cdots, v_l, e_l, v_1\}$ is a primitive geodesic in $B_{\mathbb{H}}$. It is clear that $\tilde{c}$ is both closed and primitive if c is, so we need only check that $\tilde{c}$ has no backtracking or tails.

Let's look at what hyperedge backtracking in the hypergraph means. We say that c has hyperedge backtracking if we use the same hyperedge twice in a row. On the bipartite graph side, this means we leave a vertex in the set from $E(\mathbb{H})$, go to a vertex in the set $V(\mathbb{H})$ and then backtrack to the vertex in $E(\mathbb{H})$. Still on the bipartite side, the only other way to backtrack is to go from a vertex in $V(\mathbb{H})$ to a vertex in $E(\mathbb{H})$ and directly back. Thus, we would have the following sequence in the hypergraph: (v_i, e_i, v_i). This type of sequence is expressly disallowed unless v_i is repeated more than once in e_i. If this happens, there is a multiple edge in $B_{\mathbb{H}}$ representing this, which means we can actually return to the first vertex without backtracking. Putting all of this together, we see that no hyperedge backtracking in $\mathbb{H}$ is equivalent to no backtracking on the corresponding path in $B_{\mathbb{H}}$. Once we know that backtracking isn't an issue, having no tails also follows immediately since backtracking in $\tilde{c}^2$ would correspond to hyperedge backtracking in c^2. Thus, each prime chain of length l in $\mathbb{H}$ corresponds to a primitive geodesic of length $2l$ in $B_{\mathbb{H}}$.

We now look at prime cycles in $B_{\mathbb{H}}$ and show that they correspond to prime chains in $\mathbb{H}$. Without loss of generality, we can assume that the first entry in a representative of a prime cycle in $B_{\mathbb{H}}$ is a vertex parameterized by the set $V(\mathbb{H})$. If it is not, we simply shift the cycle one slot in either direction, and we will have an appropriate representative because the graph is bipartite. Suppose the representative looks like $\tilde{c} = \{v_1, e_1, \cdots, v_l, e_l, v_1\}$; then we have the following closed relaxed chain in $\mathbb{H}$: $c = \{v_1, e_1, \cdots, v_l, e_l, v_1\}$. This is a closed relaxed chain by the same reasons as above since $\tilde{c}$ is a primitive geodesic. Also, $|\tilde{c}| = 2l = 2|c|$, so we see that given a prime cycle in $B_{\mathbb{H}}$, it corresponds to a prime chain of half the length in $\mathbb{H}$. □

This correspondence allows us to relate the generalized Ihara-Selberg zeta function of a hypergraph to the Ihara-Selberg zeta function of its associated bipartite graph.

Theorem 5.3.2. *Let $\mathbb{H}$ be a finite, connected hypergraph such that every hypervertex is in at least two hyperedges. Then,*

$$\zeta_{\mathbb{H}}(u) = Z_{B_{\mathbb{H}}}(\sqrt{u}).$$

Proof. Let $P_{\mathbb{H}}$ be the set of prime chains on $\mathbb{H}$ and $P_{B_{\mathbb{H}}}$ the set of prime cycles on $B_{\mathbb{H}}$. Then we rely on the previous proposition to write:

$$\begin{aligned}\zeta_{\mathbb{H}}(u) &= \prod_{\mathfrak{p}\in P_{\mathbb{H}}} \left(1 - u^{|\mathfrak{p}|}\right)^{-1} = \prod_{\mathfrak{p}\in P_{\mathbb{H}}} \left(1 - u^{2|\mathfrak{p}|/2}\right)^{-1} \\ &= \prod_{\ell\in P_{B_{\mathbb{H}}}} \left(1 - u^{|\ell|/2}\right)^{-1} = Z_B(\sqrt{u}).\end{aligned}$$

□

As an immediate consequence of this relation, we see that, for an arbitrary hypergraph $\mathbb{H}$ which satisfies the conditions of Theorem 5.3.2 and its dual hypergraph $\mathbb{H}^*$, we must have

$$\zeta_{\mathbb{H}}(u) = \zeta_{\mathbb{H}^*}(u)$$

since they have the same associated bipartite graph $B_{\mathbb{H}}$. In addition, we can rewrite Theorem 2.4.1 to give us a form of $\zeta_{\mathbb{H}}(u)$ which is more amenable to computation:

Corollary 5.3.3. *Let $\mathbb{H}$ be a finite, connected hypergraph such that every hypervertex is in at least two hyperedges. Let $A_{B_{\mathbb{H}}}$ be the adjacency operator on $B_{\mathbb{H}}$, and let $Q_{B_{\mathbb{H}}}$ be the Q operator on $B_{\mathbb{H}}$ as defined at the beginning of Section* 2.4*. Let I be the identity operator on $C(V(\mathbb{H})) \oplus C(E(\mathbb{H}))$—$C(X)$ is the space of function that map from a set X to the real numbers. Then*

$$\zeta_{\mathbb{H}}(u) = Z_{B_{\mathbb{H}}}(\sqrt{u}) = (1-u)^{\chi(B_{\mathbb{H}})} \det(I - \sqrt{u}A_{B_{\mathbb{H}}} + uQ_{B_{\mathbb{H}}})^{-1},$$

where $\chi(B_{\mathbb{H}}) = |V(B_{\mathbb{H}})| - |E(B_{\mathbb{H}})|$.

Remark 5.3.4. We make a few notes:

1. Despite the square root that appears in this expression, $\zeta_{\mathbb{H}}(u)$ is a rational function. We see this clearly in the previous section, but we can also recover it quickly by recalling that a bipartite graph only has prime cycles of even length.

2. The adjacency operator of $B_{\mathbb{H}}$ can be quickly constructed from the incidence matrix of $\mathbb{H}$ as in (3.3).

3. Similarly, we can construct the operator $Q_{B_{\mathbb{H}}}$ quickly by considering the degrees of vertices in the associated bipartite graph. If x is a vertex which comes from $V(\mathbb{H})$, we have $d(x)$ is the number of hyperedges of which x is a member, counting possible multiplicity. If x comes from $E(\mathbb{H})$, then $d(x)$ is the order of the associated hyperedge. From these two facts, we can easily reconstruct $Q_{B_{\mathbb{H}}}$.

4. We also see that $|V(B_{\mathbb{H}})| = |V(\mathbb{H})| + |E(\mathbb{H})|$. In addition, $|E(B_{\mathbb{H}})|$ can be directly computed in two different ways via

$$|E(B_{\mathbb{H}})| = \sum_{e \in E(\mathbb{H})} |e| = \sum_{v \in V(\mathbb{H})} \sharp\{e \in E(\mathbb{H}); v \in e\}.$$

Example 5.3.5. We compute the generalized Ihara-Selberg zeta function of the hypergraph in Figure 5.6 in two ways. By going through the oriented line graph, we have

$$\zeta_{\mathbb{H}}(u)^{-1} = \det(I - uT) = (1-u)(1 + u + u^2 - 5u^3 - 5u^4 - 5u^5 + 4u^6 + 4u^7 + 4u^8).$$

We can also compute the zeta function of the associated bipartite graph by using Bass's Theorem to realize

$$Z_{B_{\mathbb{H}}}(u)^{-1} = (1-u^2)(1 + u^2 + u^4 - 5u^6 - 5u^8 - 5u^{10} + 4u^{12} + 4u^{14} + 4u^{16}).$$

Thus, we can directly see the relation $\zeta_{\mathbb{H}}(u) = Z_{B_{\mathbb{H}}}(\sqrt{u})$.

We emphasize that Corollary 5.3.3 is mainly useful for computation. In general, the diagonal entries of the Q matrix will not all be the same, making it quite difficult to manipulate the expression for theoretical results. Theorem 5.3.2 makes it clear that the problem of factoring the generalized Ihara-Selberg zeta function is really a problem of factoring the Ihara-Selberg zeta function of a bipartite graph. Fortunately, in [16], Hashimoto deals with this question in great detail. We will define the notation that Hashimoto uses then state his main result in full. Once we have his main result, we will show how it fits within our framework and look at some consequences.

Notation 3. *We let X be a bipartite graph with bipartition $V(X) = V_1 \cup V_2$. Then*

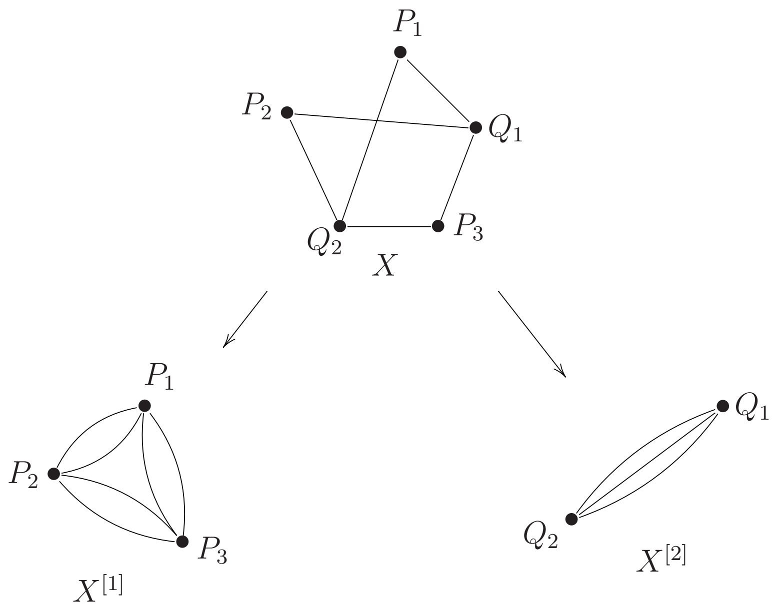

Figure 5.7:

we construct two graphs $X^{[i]}$ $(i = 1, 2)$ *as follows:*

$$V(X^{[i]}) := V_i,$$

$$E(X^{[i]}) := \{c \text{ primitive geodesic};\ |c| = 2,\ o(c), t(c) \in V_i\}. \tag{5.2}$$

We see an example of this construction in Figure 5.7. We should point out that if we start with a hypergraph $\mathbb{H}$, then the two graphs we construct from $B_{\mathbb{H}}$ are exactly $G_{\mathbb{H}}$ and $G_{\mathbb{H}^*}$. This follows from the definition we gave for the adjacency operator on a hypergraph.

We now give Hashimoto's result in full. The interested reader can find the original reference as Main Theorem(III) in [16]:

Theorem 5.3.6 (Hashimoto)**.** *Suppose that X is a finite, connected* $(q_1 + 1, q_2 + 1)$-*biregular bipartite graph with* $q_1 \geq q_2$. *Let* $A^{[i]}$ *be the adjacency matrix of the associated* *graph* $X^{[i]}$ $(i = 1, 2)$, *and let* $n_i = |V(X^{[i]})|$. *Then, one has*

$$\begin{aligned}
&Z_X(\sqrt{u})^{-1} \\
&\quad = (1-u)^{(r-1)}(1+q_2u)^{(n_2-n_1)} \times \det[I_{n_1} - (A^{[1]} - q_2 + 1)u + q_1q_2u^2] \\
&\quad = (1-u)^{(r-1)}(1+q_1u)^{(n_1-n_2)} \times \det[I_{n_2} - (A^{[2]} - q_1 + 1)u + q_1q_2u^2],
\end{aligned}$$

where $r = n_1q_1 - n_2 + 1 = n_2q_2 - n_1 + 1$ *is* $-\chi(X) + 1$.

This theorem gives us a very good expression for biregular bipartite graphs. These bipartite graphs exactly correspond to bipartite graphs associated with (d, r)-regular hypergraphs. We rewrite Theorem 5.3.6 to put it into our notation:

Corollary 5.3.7. *Suppose that* $\mathbb{H}$ *is a finite, connected* (d, r)*-regular hypergraph with* $d \geq r$. *Let* $n_1 = |V(\mathbb{H})|$, $n_2 = |E(\mathbb{H})|$, *and* $q = (d-1)(r-1)$. *Let* A *be the adjacency operator of* $\mathbb{H}$, *and let* A^* *be the adjacency operator of* $\mathbb{H}^*$. *Then one has*

$$\begin{aligned}
&\zeta_{\mathbb{H}}(u)^{-1} \\
&\quad = (1-u)^{-\chi(B_{\mathbb{H}})}(1+(r-1)u)^{(n_2-n_1)} \times \det[I_{n_1} - (A - r + 2)u + qu^2] \\
&\quad = (1-u)^{-\chi(B_{\mathbb{H}})}(1+(d-1)u)^{(n_1-n_2)} \times \det[I_{n_2} - (A^* - d + 2)u + qu^2],
\end{aligned}$$

where $-\chi(B_{\mathbb{H}}) = n_1(d-1) - n_2 = n_2(r-1) - n_1$.

Corollary 5.3.7 will provide the tool we need to produce theoretical results about the generalized Ihara-Selberg zeta function on (d, r)-regular hypergraphs. The condition that $d \geq r$ is actually not a problem. If $\mathbb{H}$ is a (d, r)-regular hypergraph; then, $\mathbb{H}^*$ is (r, d)-regular. Thus, if $d < r$, we simply consider $\mathbb{H}^*$. In the next section, we will explore some of the consequences of this determinant expression, recovering functional equations and a reasonable Riemann hypothesis as we had for regular graphs.

5.4 Consequences of the determinant expression

Our first observation is that the generalized Ihara-Selberg zeta function of a hypergraph is a non-trivial generalization of the Ihara-Selberg zeta function. By this, we mean that we can produce an infinite number of zeta functions which are not the Ihara-Selberg zeta function of any graph. A simple way to produce zeta functions which did not come from a graph is encoded in the next proposition.

Proposition 5.4.1. *Suppose X is a finite graph, and $\mathbb{H}$ is a finite hypergraph. Then,*

1. *The degree of the polynomial $Z_X(u)^{-1}$ is $2|E(X)|$.*

2. *The degree of the polynomial $\zeta_{\mathbb{H}}(u)^{-1}$ is $\sum_{e \in E(\mathbb{H})} |e|$.*

Proof. 1. Let X be a finite graph. Then by Theorem 2.4.1,

$$Z_X(u)^{-1} = (1-u^2)^{|E|-|V|} \times \det(I - uA + u^2 Q).$$

The degree of the determinant term is $2|V|$, and the degree of the explicit polynomial is $2|E(X)| - 2|V(X)|$. Hence, the degree of $Z_X(u)^{-1}$ is $2|E(X)| - 2|V(X)| + 2|V(X)| = 2|E(X)|$.

2. Let $\mathbb{H}$ be a finite hypergraph with associated bipartite graph $B_{\mathbb{H}}$. Then by Theorem 5.3.2,

$$\zeta_{\mathbb{H}}(u)^{-1} = Z_{B_{\mathbb{H}}}(\sqrt{u})^{-1}.$$

From the previous part, we see that the degree of $Z_{B_{\mathbb{H}}}(\sqrt{u})^{-1}$ is $|E(B_{\mathbb{H}})|$. We can compute this explicitly as $|E(B_{\mathbb{H}})| = \sum_{e \in E(\mathbb{H})} |e|$. □

Hence, the reciprocal of the Ihara-Selberg zeta function of a graph will always have even degree. If we wish to exhibit hypergraphs with generalized Ihara-Selberg zeta functions that did not arise from some graph, we need only find a hypergraph for which $\sum_{e \in E(\mathbb{H})} |e|$ is odd. This is quite easy to do; for instance, we might take any finite hypergraph with exactly one 3-edge and as many hyperedges of even orders as we want. However, $\zeta_{\mathbb{H}}(u^2)$ can be realized as the zeta function of a graph, namely of the associated bipartite graph of $\mathbb{H}$.

Example 5.4.2. In example 5.3.5, we computed the generalized Ihara-Selberg zeta function of the hypergraph appearing in Figure 5.6. We see that the reciprocal of the zeta function has odd degree, so this is an example of a hypergraph which produces a zeta function that no graph could produce.

Before we turn to a discussion of the poles of the generalized Ihara-Selberg zeta function of a (d, r)-regular hypergraph, we look at some of the symmetry provided by Hashimoto's expression. We can give several functional equations in the same spirit as Corollary 2.4.8.

Corollary 5.4.3. *Suppose that $\mathbb{H}$ is a finite connected (d, r)-regular hypergraph with $d \geq r$. Let $n_1 = |V(\mathbb{H})|$, $n_2 = |E(\mathbb{H})|$, $q = (d-1)(r-1)$, and $\chi = \chi(B_{\mathbb{H}})$. Let A be the adjacency operator of $\mathbb{H}$, and let A^* be the adjacency operator of $\mathbb{H}^*$. Finally, suppose $p(u)$ is a polynomial in u that satisfies $p(u)^\eta = \pm(qu^2)^\eta p(\frac{1}{qu})^\eta$ for all $\eta \in \mathbb{N}$. Then we have the following functional equations for $\zeta_{\mathbb{H}}(u)$:*

1. $\Lambda_{\mathbb{H}}(u) = p(u)^{n_1}(1-u)^{-\chi}(1+(r-1)u)^{n_2-n_1}\zeta_{\mathbb{H}}(u) = \pm\Lambda_{\mathbb{H}}(\frac{1}{qu})$.
2. $\tilde{\Lambda}_{\mathbb{H}}(u) = p(u)^{n_2}(1-u)^{-\chi}(1+(d-1)u)^{n_1-n_2}\zeta_{\mathbb{H}}(u) = \pm\tilde{\Lambda}_{\mathbb{H}}(\frac{1}{qu})$.

Proof. The strategy is really one of brute force algebra, using Corollary 5.3.7. By Corollary 5.3.7, we can write $\zeta_{\mathbb{H}}(u)$ as

$$\zeta_{\mathbb{H}}(u) = (1-u)^{\chi}(1+(r-1)u)^{(n_1-n_2)} \times \det[I_{n_1} - (A-r+2)u + qu^2]^{-1}.$$

Substituting this expression into $\Lambda_{\mathbb{H}}(u)$, we have

$$\Lambda_{\mathbb{H}}(u) = p(u)^{n_1} \times \det[I_{n_1} - (A-r+2)u + qu^2]^{-1}.$$

We now algebraically manipulate the determinant term:

$$
\begin{aligned}
\det[I_{n_1} - (A-r+2)u + qu^2]^{-1} &= \det[\frac{qu^2}{qu^2} - (A-r+2)\frac{qu^2}{qu} + \frac{qu^2}{1}]^{-1} \\
&= \left(\frac{1}{qu^2}\right)^{n_1} \times \det[\frac{1}{qu^2} - (A-r+2)\frac{1}{qu} + \frac{1}{1}]^{-1} \\
&= \left(\frac{1}{qu^2}\right)^{n_1} \times \det[\frac{1}{1}I_{n_1} - (A-r+2)\frac{1}{qu} + \frac{q}{(qu)^2}]^{-1}.
\end{aligned}
$$

We substitute this back into the expression for $\Lambda_{\mathbb{H}}(u)$ and then use the given condition for $p(u)^{n_1}$:

$$
\begin{aligned}
\Lambda_{\mathbb{H}}(u) &= p(u)^{n_1} \times \left(\frac{1}{qu^2}\right)^{n_1} \times \det[\frac{1}{1}I_{n_1} - (A-r+2)\frac{1}{qu} + \frac{q}{(qu)^2}]^{-1} \\
&= \pm(qu^2)^{n_1} p(\frac{1}{qu})^{n_1} \times \left(\frac{1}{qu^2}\right)^{n_1} \times \det[\frac{1}{1}I_{n_1} - (A-r+2)\frac{1}{qu} + \frac{q}{(qu)^2}]^{-1} \\
&= \pm p(\frac{1}{qu})^{n_1} \times \det[\frac{1}{1}I_{n_1} - (A-r+2)\frac{1}{qu} + \frac{q}{(qu)^2}]^{-1} \\
&= \pm\Lambda_{\mathbb{H}}(\frac{1}{qu}).
\end{aligned}
$$

This completes the first functional equation. The second one is identical, using Hashimoto's second expression. We leave it as an exercise to the reader. □

Remark 5.4.4. Using Corollary 5.4.3, we can write down several explicit functional equations for (d,r)-hypergraphs with $d \geq r$.

1. $\Lambda_{\mathbb{H}}(u) = (1-u)^{n_1-\chi(B_{\mathbb{H}})}(1+(r-1)u)^{(n_2-n_1)}(1-qu)^{n_1}\zeta_{\mathbb{H}}(u) = \Lambda_{\mathbb{H}}(\frac{1}{qu})$.

2. $\tilde{\Lambda}_{\mathbb{H}}(u) = (1-u)^{n_2-\chi(B_{\mathbb{H}})}(1+(d-1)u)^{(n_1-n_2)}(1-qu)^{n_2}\zeta_{\mathbb{H}}(u) = \tilde{\Lambda}_{\mathbb{H}}(\frac{1}{qu})$.

3. $\Xi_{\mathbb{H}}(u) = (1-u)^{-\chi(B_{\mathbb{H}})}(1+(r-1)u)^{(n_2-n_1)}(1+qu^2)^{n_1}\zeta_{\mathbb{H}}(u) = \Xi_{\mathbb{H}}(\frac{1}{qu})$.

4. $\tilde{\Xi}_{\mathbb{H}}(u) = (1-u)^{-\chi(B_{\mathbb{H}})}(1+(d-1)u)^{(n_1-n_2)}(1+qu^2)^{n_2}\zeta_{\mathbb{H}}(u) = \tilde{\Xi}_{\mathbb{H}}(\frac{1}{qu})$.

5.4.1 The Riemann hypothesis and Ramanujan hypergraphs

Now that we have several established functional equations, we turn to the next important question for a zeta function. We will look at the location of the poles and show that they very explicitly detect the Ramanujan condition on a (d, r)-regular hypergraph.

We assume throughout this section that $\mathbb{H}$ is a (d, r)-regular hypergraph with $d \geq r$. We let $n_2 = |E(\mathbb{H})|$, $n_1 = |V(\mathbb{H})|$, and A be the adjacency operator on $\mathbb{H}$. Then, we have that $n_2 \geq n_1$ since $d \geq r$. By Equation (3.5), $-d$ is not an obvious eigenvalue of $\mathbb{H}$. This will simplify our consideration of the Ramanujan condition on $\mathbb{H}$.

We now want to focus on the determinant term in Hashimoto's expression. Since A is symmetric, it is diagonalizable, so suppose Q diagonalizes A. Then,

$$\begin{aligned}
\det[I_{n_1} - (A - r + 2)u + qu^2] &= \det\left(Q[I_{n_1} - (A - (r-2)I_{n_1})u + qu^2 I_{n_1}]Q^{-1}\right) \\
&= \det[QI_{n_1}Q^{-1} - (QAQ^{-1} - (r-2)QI_{n_1}Q^{-1})u + qu^2 QI_{n_1}Q^{-1}] \\
&= \det[I_{n_1} - (QAQ^{-1} - r + 2)u + qu^2] \\
&= \prod_{\lambda \in \mathrm{Spec}(\mathbb{H})} [1 - (\lambda - r + 2)u + qu^2].
\end{aligned}$$

This is the expression we need to fully examine the relation between poles of $\zeta_{\mathbb{H}}(u)$ and eigenvalues of $\mathbb{H}$. The next two propositions detail the connection fully.

Proposition 5.4.5. *Suppose $\mathbb{H}$ is a (d, r)-regular hypergraph with $d \geq r$. Then,*

1. *$\zeta_{\mathbb{H}}(u)$ has a pole at $u = 1$ with multiplicity $n_1(d-1) - n_2 = n_2(r-2) - n_1 = -\chi(B_{\mathbb{H}})$,*

2. *$\zeta_{\mathbb{H}}(u)$ has a pole at $u = -\frac{1}{r-1}$ with multiplicity $n_2 - n_1$.*

Proof. The first set of poles is contributed by the factor $(1-u)^{\chi(B_{\mathbb{H}})}$ given in Corollary 5.3.7. The second set is from the factor $(1 + (r-1)u)^{(n_1 - n_2)}$. □

Proposition 5.4.6. *Suppose* $\mathbb{H}$ *is a* (d, r)*-regular hypergraph with* $d \geq r$*. Let* $q = (d-1)(r-1)$*, then* $\mathbb{H}$ *is a Ramanujan hypergraph if and only if the poles of* $\det[I_{n_1} - (A - r + 2)u + qu^2]^{-1}$ *are distributed as below:*

1. *There is a simple pole at* $u = 1$ *and at* $u = \frac{1}{q}$*.*

2. *All other poles lie on the circle in the complex plane given by* $|r| = \frac{1}{\sqrt{q}}$*.*

Proof. Since $\mathbb{H}$ is a (d, r)-regular hypergraph, there is an eigenvalue $\lambda = d(r-1)$. We first rewrite the factor for this eigenvalue as

$$\begin{aligned} f(u) &= qu^2 - (\lambda - r + 2)u + 1 \\ &= qu^2 - (q+1)u + 1 \\ &= (1-u)(1-qu). \end{aligned}$$

We can then see the roots at $u = 1$ and at $u = \frac{1}{q}$ as claimed in part 1. We note that if $\mathbb{H}$ is Ramanujan, no other eigenvalue is $d(r-1)$, so these poles are simple as claimed.

We now look at the eigenvalues which satisfy $\lambda \neq d(r-1)$. Then the polynomial $f(u) = qu^2 - (\lambda - r + 2)u + 1$ has roots at

$$u = \frac{(r - 2 - \lambda) \pm \sqrt{(\lambda - r + 2)^2 - 4q}}{2q}.$$

Then u has $\text{Im}(u) \neq 0$ if and only if $(\lambda - r + 2)^2 \leq 4q$. This is true if and only if $|\lambda - r + 2| \leq 2\sqrt{q}$, which is true if and only if $\mathbb{H}$ is Ramanujan, by Definition 3.3.2 (there are no obvious eigenvalues to consider by our assumption on d and r). In this case, we can calculate the modulus of the roots by

$$\begin{aligned} |u|^2 &= \frac{(\lambda - r + 2)^2}{4q^2} + \frac{4q - (\lambda - r + 2)^2}{4q^2} \\ &= \frac{4q}{4q^2} = \frac{1}{q}. \end{aligned}$$

□

This gives us a complete characterization of the relation between the poles of the generalized Ihara-Selberg zeta function and the Ramanujan condition on a hypergraph. Where the naive zeta function had far more considerations and was more delicate, this zeta function provides a very clean solution that accurately mirrors the case for graphs. We propose a Riemann hypothesis as follows:

Definition 5.4.7. *Let $\mathbb{H}$ be a (d, r)-regular hypergraph with $d \geq r$ and $q = (d-1)(r-1)$. We then consider $\zeta_{\mathbb{H}}(q^{-s})$. We say that $\zeta_{\mathbb{H}}(q^{-s})$ satisfies the* modified hypergraph Riemann hypothesis *if and only if for*

$$\mathrm{Re(s)} \in (0,1), \quad \frac{(1 + (\mathrm{r}-1)\mathrm{q}^{-\mathrm{s}})^{(\mathrm{n}_2-\mathrm{n}_1)}}{\zeta_{\mathbb{H}}(\mathrm{q}^{-\mathrm{s}})} = 0 \Longrightarrow \mathrm{Re(s)} = \frac{1}{2}.$$

We state the connection between this "Riemann hypothesis" and the generalized zeta function as a theorem:

Theorem 5.4.8. *For a connected (d, r)-regular hypergraph $\mathbb{H}$, $\zeta_{\mathbb{H}}(q^{-s})$ satisfies the modified hypergraph Riemann hypothesis if and only if $\mathbb{H}$ is a Ramanujan hypergraph.*

Proof. The proof follows from the definition of the modified hypergraph Riemann hypothesis and the previous two propositions detailing where the poles of the zeta function lie. Every connected (d, r)-regular hypergraph has $d(r-1)$ as a simple eigenvalue, so there is always a simple pole at $u = 1$ and at $u = \frac{1}{q}$. Hence, there is no possibility that ζ has the correct complex poles but not the simple real poles required in Proposition 5.4.6. □

This result and the ability to produce hypergraphs with new zeta functions are what motivate us to prefer this zeta function to the naive zeta function. We show one more interesting theoretical property of the generalized Ihara-Selberg zeta function that partially generalizes a nice property of the Ihara-Selberg zeta function in the next section.

5.5 Unimodular hypergraphs and the generalized Ihara-Selberg zeta function

Before we move on and show how the generalized Ihara-Selberg zeta function can be interpreted as a graph zeta function with a restricted cycle set, we show how some well-known hypergraph properties fit into this framework. In particular, we will be interested in the case when the generalized Ihara-Selberg zeta function is an even function. A graph is bipartite if and only if its Ihara-Selberg zeta function is even, and we will see that the generalized zeta function indicates some of the generalizations of "bipartite" to hypergraphs. The hypergraph theorems we refer to are all from Chapter 20, Section 3 of Berge [3].

Definition 5.5.1 (Berge)**.** *We let $\mathbb{H} = (V, E)$ be a finite hypergraph where the hyperedge set $E = \{E_i : i \in I\}$. An* equitable q-coloring *of $\mathbb{H}$ is a partition $(S_1, \cdots, S_q)$ of the hypervertices into q classes such that for each $i \in I$ and for $j, j^{'} \leq q$,*

$$-1 \leq |E_i \cap S_j| - |E_i \cap S_{j'}| \leq 1.$$

The smallest number $q \geq 2$ for which there exists an equitable q-coloring is the equitable chromatic number $\kappa(\mathbb{H})$ *of $\mathbb{H}$. $\mathbb{H}$ is* unimodular *if for each $S \subset V$, the induced subhypergraph $\mathbb{H}_S$ admits an equitable bicoloring.*

A graph is unimodular if and only if it is bipartite, so this definition is a generalization of bipartite for hypergraphs. We now look at what it means for the generalized Ihara-Selberg zeta function to be an even function:

Proposition 5.5.2. *Let $\mathbb{H}$ be a hypergraph. Then, $\zeta_{\mathbb{H}}(u) = \zeta_{\mathbb{H}}(-u)$ for all $u \in \mathbb{C}$ if and only if every primitive cycle in $\mathbb{H}$ has even length.*

Proof. We consider the power series expansion of the zeta function given as an Euler product in Definition 5.1.2. Then u appears to an odd power if and only if there is a prime cycle of odd length. Hence, the zeta function must be even on a disk about the origin. Since it continues to the reciprocal of a polynomial, it must be even throughout the complex plane. □

This is all we need to reframe several of the results cited in [3]:

Theorem 5.5.3. *Suppose $\mathbb{H}$ is a hypergraph with $\zeta_{\mathbb{H}}(u)$ an even function. Then, $\mathbb{H}$ is unimodular.*

Proof. This follows immediately from Theorem 10 in Chapter 20 of [3]. The main idea is that if ever primitive cycle has even length, $\mathbb{H}$ is unimodular. □

Corollary 5.5.4. *Suppose $\mathbb{H}$ is a hypergraph. Then $\zeta_{\mathbb{H}}(u)$ is even if and only if each hypergraph $\mathbb{H}'$, defined by taking hyperedges to be subsets of hyperedges of $\mathbb{H}$ and the hypervertex set to be the union of all the new hyperedges, satisfies $\kappa(\mathbb{H}') \leq 2$.*

Proof. The result follows from the definition of unimodular since subhypergraphs must admit equitable bicolorings. □

Finally, we will look at a different sort of application for the generalized Ihara-Selberg zeta function. In the next section, we will look at two 3-regular graphs which are cospectral. The Ihara-Selberg zeta function of two cospectral regular graphs is always the same, making it a very poor graph invariant when it comes to determining if two regular graphs are isomorphic or not. We will show a way to distinguish the graphs with zeta function considerations, improving the use of zeta functions as a tool for graph theorists.

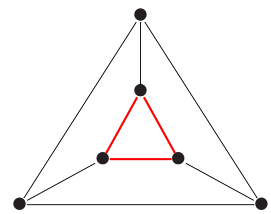

Figure 5.8: A graph with a triangle singled out.

5.6 Distinguishing cospectral graphs

The ideas in this section are motivated, in part, by the question of determining if two given graphs are isomorphic. We saw earlier that using the Ihara-Selberg zeta function as a graph invariant yields mixed results when one wants to distinguish the graphs. For k-regular graphs, being cospectral and having the same Ihara-Selberg zeta function are equivalent. This suggests that, in this case, the cycle structure is very heavily influenced by the spectral properties. Our idea is to try to restrict the cycles we consider so that we are forced to rely more heavily on the actual graph structure instead of the spectral properties. We refer to Figure 5.8 to illustrate how the generalized Ihara-Selberg zeta function might be used to do this. We will start with the set of all prime cycles in this graph. Then we can throw out any prime cycle that uses two red edges in a row. We actually will be throwing out infinitely many prime cycles when we do this. We could now define a new zeta function using this smaller set of prime cycles in the same way as before. It turns out that this is exactly the generalized Ihara-Selberg zeta function for the hypergraph formed by replacing the red triangle with a 3-edge on the same vertices.

We could perform the same sort of construction for other graphs by replacing cliques of any size with a hyperedge on the respective vertices. In this way, we would hope that the path structure would more accurately mirror the structure of the graph and not be as influenced by its spectrum.

Throughout the rest of this section, we will focus our attention on Figure 5.9.

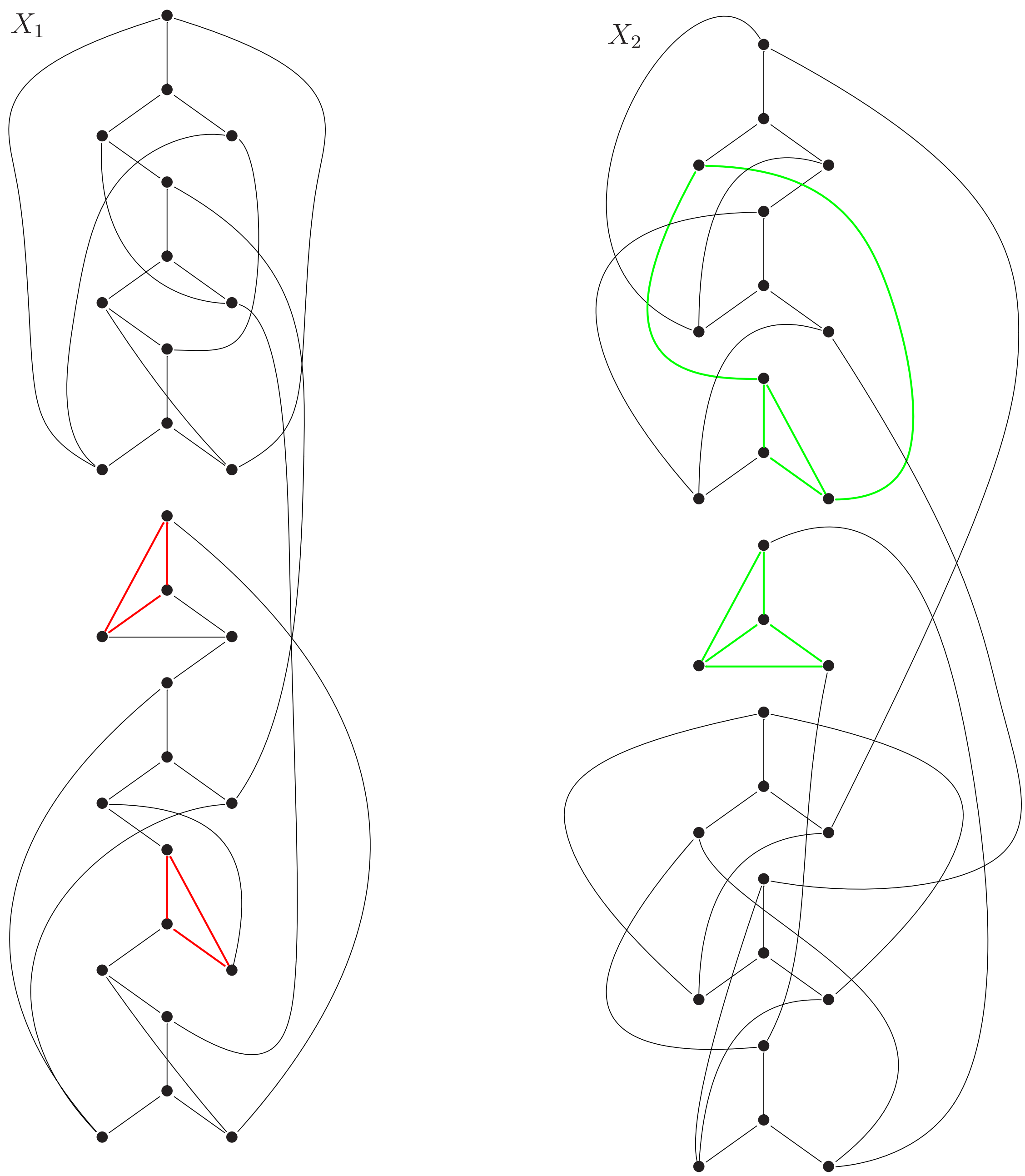

Figure 5.9: Two cospectral 3-regular graphs constructed by Stark and Terras in [35] by zeta function and covering considerations.

Stark and Terras constructed these graphs as 5-fold covers of the complete graph on 4 vertices. They have the same Ihara-Selberg zeta function and are thus cospectral.

It is fairly straightforward to check that, though these graphs are cospectral, they are not isomorphic. We will offer a different proof of this fact as an application of the generalized Ihara-Selberg zeta function.

When we compute the Ihara-Selberg zeta function of X_1 (and thus of X_2), we find that the coefficient of u^3 in the polynomial is -8. Hence, by Theorem 2.5.9, each graph has exactly 4 triangles. We can find them quickly by inspection. In X_1, we've highlighted two of the triangles in red. Each of the red triangles intersects another triangle. In X_2, we've highlighted all four triangles in green.

We now suppose that X_1 and X_2 are isomorphic. We change X_1 into a hypergraph in the following way. We delete each red triangle. We then replace the triangle with a hyperedge on the three vertices. The advantage to doing this is that we can now look at the generalized Ihara-Selberg zeta function on this hypergraph. This will correspond to looking at a product over all primitive cycles in X_1 that do not use two red edges in a row. In essence, we've restricted slightly our set of prime cycles.

If X_1 and X_2 are isomorphic, we should be able to repeat the transformation from graph to hypergraph in X_2 and have isomorphic hypergraphs. There are four possible ways to create a hypergraph from X_2 in the same manner as we did for X_1. For each green subgraph, we have a choice of two triangles to focus on, and there are two such green subgraphs.

Now a simple comparison of generalized Ihara-Selberg zeta functions distinguishes the graphs. All four of the hypergraphs constructed from X_2 actually have the same generalized Ihara-Selberg zeta function. However, the hypergraph we constructed from X_1 has a different zeta function. Hence, these two graphs are not isomorphic.

This example suggests that the generalized Ihara-Selberg zeta function, by being considered as a zeta function on a graph with a restricted set of prime cycles, can bring more leverage to the problem of distinguishing non-isomorphic graphs. We can restrict our cycle structures by disallowing consecutive edges from a particular j-clique in a graph. This allows us to escape from Quenell's result that k-regular graphs are cospectral if and only if they have the same path structures.

We should mention that there is a drawback to this method as well. We were fortunate that our example had a relatively small number of triangles. As the number of non-disjoint triangles grows, we have to consider more and more potential hypergraphs. Here, we only had to consider 4 potential hypergraphs constructed from X_2; however, this was a graph with quite a small number of triangles.

Other options would be to make every possible triangle into a hyperedge; then, you would only have to compare one generalized Ihara-Selberg zeta function for each initial graph. For this example, changing all four triangles into hyperedges and then computing the generalized Ihara-Selberg zeta function of the resulting hypergraphs also distinguishes the graphs.

All zeta functions referenced in this section are available from the author by request.

Chapter 6

Conclusion

6.1 Future research

We'd like to outline some very broad thoughts for further research in this section. For the most part, these are quite large ideas that could be difficult to approach but would have a very large pay-out in terms of results if they can be resolved. We will look at three main areas: the Ihara-Selberg zeta function as a graph invariant, Ramanujan graph construction, and the Graph Isomorphism problem.

6.1.1 The Ihara-Selberg zeta function as an invariant

We've already seen some results in this direction, but there is a lot of work to be done. We will briefly summarize what is known:

1. Two k-regular graphs are cospectral if and only if they have the same Ihara-Selberg zeta function.

2. Two (d, r)-biregular bipartite graphs are cospectral if and only if they have the same Ihara-Selberg zeta function.

3. Non regular cospectral (or isospectral w.r.t. the laplacian) graphs can have

different zeta functions.

4. The Ihara-Selberg zeta function determines if a graph is k-regular or not and can determine what k if the graph is regular.

5. The coefficients of the reciprocal of the zeta function relate to concrete graph properties.

This list of work leaves open the following interesting questions:

1. Characterize all graphs that have the same Ihara-Selberg zeta function. How many non-isomorphic graphs are there with the same zeta function?

2. Characterize completely the coefficients of the reciprocal of the zeta function.

3. When a graph is not regular, the complex poles of the Ihara-Selberg zeta don't necessarily lie on a circle. What proportion of them do as the size of the graph grows large? Does this relate to other known models?

These questions are all fairly concrete and seem to be approachable with what is currently known. We can hope that the next few years will yield complete solutions.

6.1.2 Ramanujan graph constructions

In this section, we will outline a potential approach to constructing Ramanujan graphs that does not seem to have been explored. There are several places where work would need to be done for this idea to be fruitful, and we will try to point them out as we go along. By Corollary 2.4.3, a finite k-regular graph is Ramanujan if and only if its poles satisfy a Riemann hypothesis. The poles are, in fact, related directly to the spectrum of the Perron–Frobenius operator T that first arises in the factorization of the zeta function.

One strategy to construct Ramanujan graphs would be to try to construct matrices T that yield zeta functions which satisfy the Riemann hypothesis. In particular, this would require constructing square matrices with an even number of rows and columns that satisfy:

1. Every entry is zero or one.
2. Every row and column sums to k.
3. If the i, j-entry is one, the j, i-entry is zero.
4. The eigenvalues would be 1 (with a specific multiplicity), k, and eigenvalues on the circle given by $|r| = \sqrt{k-1}$.

If we could construct such a matrix, the next problem would be to construct a simple graph from which it came. Cooper has a result which can reconstruct the graph from the Perron–Frobenius operator T if it is known that T came from a graph [8]. As part of this, we would need to characterize all possible matrices T which can arise as Perron–Frobenius operators of oriented line graphs of simple graphs. This would certainly add some more conditions to the matrices we are looking for.

The reason we propose this is because we feel there has been a great deal of attention paid to rather direct constructions of Ramanujan graphs. Namely, certain structures are examined and then, primarily through number theoretic methods, their spectrum is shown to satisfy the appropriate bounds. The line of thought proposed here offers a definite shift of perspective. It may be that the matrices T are equally as hard to work with, but often, problems on directed graphs are much simpler than on general graphs, so we have some hopes.

6.1.3 The graph isomorphism problem

We saw in the final chapter that the generalized Ihara-Selberg zeta function has some application to distinguishing non-isomorphic graphs with triangles. It would be quite interesting to develop criteria for when two graphs with triangles have the same generalized Ihara-Selberg zeta function.

The graph isomorphism problem, broadly stated, is, given a specific graph, to determine quickly if another graph is isomorphic to it. For us to have an application to this problem for graphs with triangles, we would have to find a way to avoid needing to construct multiple hypergraphs to check. It may be interesting to replace every triangle by a hyperedge and see what happens. Then there would only be one hypergraph to check on each side, and computing determinants is relatively quick. We hope to continue this line of thought in the future.

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