The Ihara-Selberg Zeta Function

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Outline

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- Now: a lot of people going in a lot of different directions.

Graphs

Definition

A graph $X = (V, E)$ is

- a set V of vertices
- and a set E of unordered pairs of vertices, called edges.

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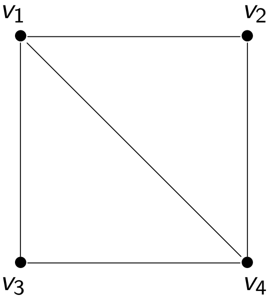

The Adjacency Matrix of a Graph

Definition

Two vertices u and v are adjacent, written $u \sim v$, if $\{u, v\}$ is an edge.

We can use the adjacency relation to associate a matrix A with a graph as follows: the rows and columns of A are parametrized by the vertices. The (v_i, v_j) entry of A is 1 if v_i is adjacent to v_j and 0 otherwise.

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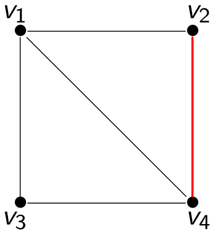

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

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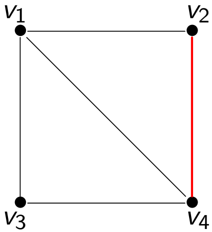

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Question: What does the (v_i, v_j) entry of A^k represent when k is a positive integer?

The Spectrum

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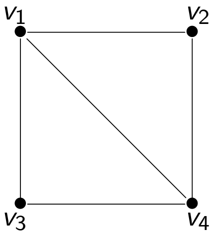

The spectrum of this graph is $\{\frac{1}{2}\left(1+\sqrt{17}\right), 0, -1, \frac{1}{2}\left(1-\sqrt{17}\right)\}$.

Some Properties associated with the eigenvalues

Remarkably, the spectrum of a graph contains quite a bit of useful information regarding the graph. In general, it's associated with more intangible properties like "expansion", but there are some concrete properties encoded in them as well. We list a few very basic properties:

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- The degree of a vertex u, denoted $d(x)$, is the number of vertices to which it is adjacent. If Δ is the maximum degree of all of the vertices in X, then

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$$\Delta \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|} \geq -\Delta.$$
- In particular, if X is k-regular, $\lambda_1 = k$.
- X is bipartite if and only if $\lambda_1 = -\lambda_{|V|}$.

The Second Eigenvalue

Assume X is a k-regular graph; then, $\lambda_1 = k$. The key question is: how large, in absolute value, are the other eigenvalues?

Let's first see why this question is important before we give an answer. To do this, we need to know what the (v_i, v_j)-entry of powers of the adjacency matrix represents. Let's look at an example:

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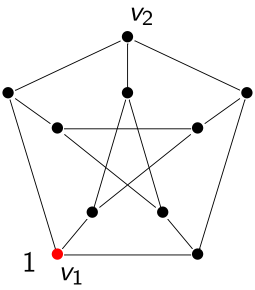

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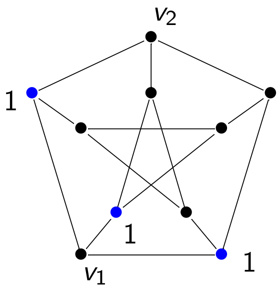
$k = 2$



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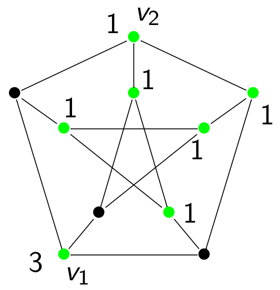

The $(1, 2)$-entry of A^3 is: 1

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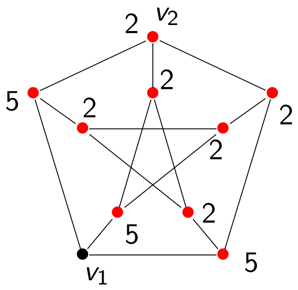

The $(1, 2)$-entry of A^4 is: 2

The Second Eigenvalue

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As you raise A to higher and higher powers, you will be forced into the eigenspace corresponding to the largest eigenvalue in absolute value. The speed at which you reach that eigenspace is determined by how large the next eigenvalue is. If you want good mixing, expansion, etc., you are interested in having all of the other eigenvalues as small as possible.

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Definition

A k-regular graph is Ramanujan if

$$|\lambda| \leq 2\sqrt{k-1}$$

with the exception of $\lambda_1 = k$.

The Riemann Zeta Function

Let $s = \sigma + it$. The Riemann zeta function is defined by

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Remark

There is a different way to express the zeta function which illustrates how it connects to the prime numbers. We can write $\zeta(s)$ as an Euler Product Expansion by

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The Riemann hypothesis

After a lot of complex analysis, it's possible to analytically continue $\zeta(s)$ to the entire complex plane, excepting a simple pole at $s = 1$. This means there is some other function $\hat{\zeta}(s)$ such that $\hat{\zeta}(s)$ agrees exactly with $\zeta(s)$ on $\sigma > 1$ and is analytic throughout the complex plane (minus the pole).

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Neat fact: when you analytically continue and get $\hat{\zeta}(s)$, you get functional equations that let you evaluate the function for certain values. One well known value is:

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The Riemann Hypothesis states that if $\hat{\zeta}(s) = 0$; then s is a negative even integer or $\mathrm{Re}\, s = \frac{1}{2}$.

A Graph Zeta Function

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Throughout, we will let X be a finite, connected graph such that the degree of every vertex is at least 2.

What is a prime cycle?

Definition

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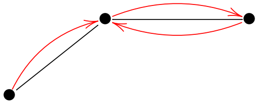

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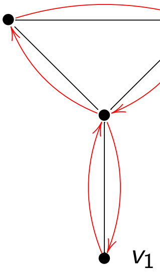

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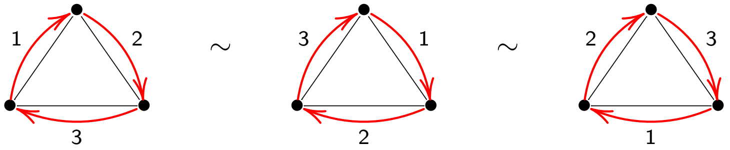

The Ihara-Selberg Zeta Function

We define the Ihara-Selberg Zeta Function for a finite graph X by

$$Z_X(u) = \prod_{[c]} \left(1 - u^{l(c)}\right)^{-1},$$

with $u \in \mathbb{C}$. Here, the product runs over all prime cycles, and $l(c)$ is the length of the cycle c.

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Some Properties:

- Generally the product is infinite.
- This actually turns out to be a rational function.
- There are nice, explicit factorizations.

The Oriented Line Graph

To show that the zeta function is a rational function, we will realize it as a determinant expression. To define the appropriate operators, we must first change our framework. A graph is actually a pretty hard model to really work with here.

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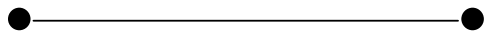

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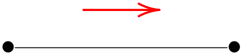

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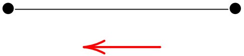
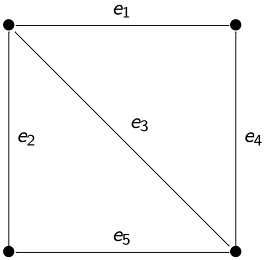


A prime cycle could use this edge by going from left to right, or... by going from right to left.

Oriented Line Graph Construction

To simplify matters, we construct an oriented line graph via



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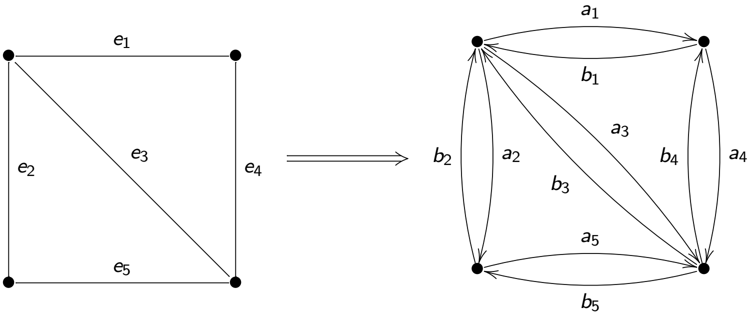

Oriented Line Graph Construction

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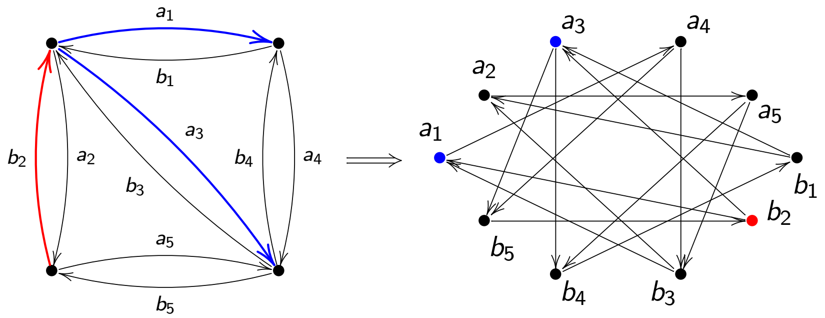
- Now we construct a new graph L via

$$V_L = E(X_o),$$
$$E_L^o = \{(e_i, e_j) \in E(X_o) \times E(X_o);\ \bar{e}_i \neq e_j,\ t(e_i) = o(e_j)\}.$$



Oriented Line Graph Properties

The oriented line graph has several important properties:

- It is strongly connected.
- It exactly mimics the prime cycle structure of the original graph.
- The zeta function of a strongly connected, oriented graph is easy to factor!

Perron-Frobenius Operator

Definition

For a strongly connected, oriented graph the Perron-Frobenius operator T is a matrix given by setting the i, j-entry to 1 if there is an oriented edge with v_i as the start and v_j as the terminus, and setting it to be zero otherwise. This is an oriented version of the adjacency operator of a graph.

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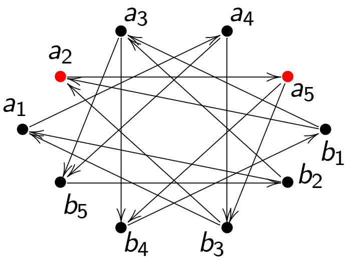
Then,

$$Z_X(u) = \det\left(I - uT\right)^{-1}.$$

Finishing our example



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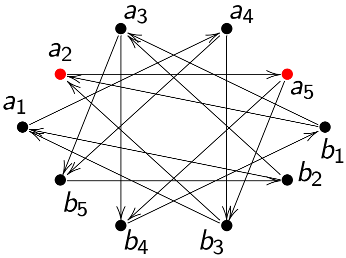
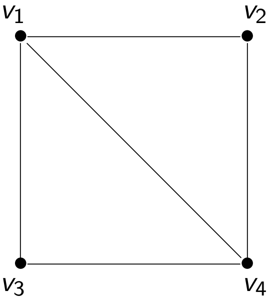


$$T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Still finishing our example...

Now that we have the T operator, we can work out the zeta function explicitly:



Then,

$$Z_X(u) = \frac{1}{1 - 4u^3 - 2u^4 + 4u^6 + 4u^7 + u^8 - 4u^{10}}.$$

Bass' Factorization

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Theorem (Bass, 1992)

Let X be a finite, connected graph as before. Then

$$Z_X(u) = (1-u^2)^{\chi} \det(I - uA + u^2 Q)^{-1},$$

where $\chi = |V| - |E|$ is the Euler Number of X, A is the adjacency matrix, and Q is a diagonal matrix with entries $d(v_i) - 1$.

Remark

As a corollary to Bass' theorem, when X is k-regular, we get functional equations which relate the value at u to the value at $\frac{1}{(k-1)u}$.

The Poles of the zeta function

Let's take a closer look at Bass' factorization, particularly when our graph is $(q+1)$-regular, and look for poles:

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- All other poles are in a specified interval of the real line.

The Complex Poles

Let's take a closer look at the polynomial $f(u) = qu^2 - \lambda u + 1$. The discriminant is

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This relation looks very familiar!

Graph Riemann Hypothesis

We could state a Riemann Hypothesis for this zeta function as follows:

Definition

Let X be a $(q+1)$-regular graph. Then $Z_X(u)$ is said to satisfy the Riemann Hypothesis if the only real poles are a simple pole at $u = \frac{1}{q}$ and poles with absolute value 1.

Equivalently, let $u = q^{-s}$. Then $Z_X(q^{-s})$ satisfies the Riemann Hypothesis if whenever $Z_X(q^{-s}) = 0$ and Re $s \in (0, 1)$, we have Re $s = \frac{1}{2}$.

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Theorem

A $(q+1)$-regular graph satisfies the Riemann Hypothesis if and only if X is a Ramanujan graph.

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There are two directions this question can take:

- Someone hands me the zeta function, what physical properties (number of triangles, colourability, ...) can I attribute to the graph?
- Someone gives me two graphs which have the same (or different) zeta function. What conclusions can I draw about the graphs?

Distinguishing graphs

We will look very briefly at the second question. What can I saw about graphs which have the same zeta function?

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Theorem (Mellein, 2001)

Suppose X and Y are both k-regular graphs. Then $Z_X(u) = Z_Y(u)$ if and only if $Spec(X) = Spec(Y)$.

Remark

If we remove the regularity condition, all bets are off. There are plenty of examples of graphs with the same adjacency matrix spectrum or the same laplacian spectrum that have different zeta functions. There aren't currently any known examples of graphs with the same zeta function but different operator spectrums, though.

Beating the theorem...

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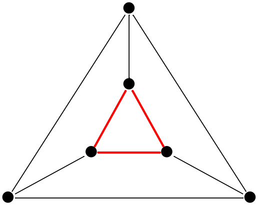

Can we take prime cycles as before but throw away any prime cycle that uses two red edges in a row? When we form the product, what happens? This has the potential to be quite interesting and to broaden the theory.

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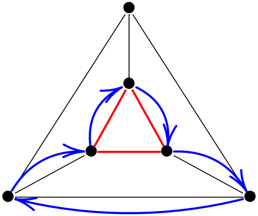

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Distinguishing Cospectral Graphs