CSC 344 – Algorithms and Complexity

Lecture #9 – Recursive Algorithms

What is Recursion?

- **Recursion** - when a method calls itself
- Classic example - the factorial function:
  n! = 1· 2· 3· ··· · (n-1)· n
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

Recursion – An Example

- As a Java method:

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
    if  (n  ==  0)
         // basis case
         return  1;
    else
         // recursive case
         return  n  *  recursiveFactorial(n- 1);
   }
```

Linear Recursion

- ***Test for base cases.***
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls ***must*** eventually reach a base case, and the handling of each base case should not use recursion.

Linear Recursion (continued)

- ***Recur once.***
  - Perform a single recursive call. (This recursive step may involve a test that decides which of several possible recursive calls to make, but it should ultimately choose to make just one of these calls each time we perform this step.)
  - Define each possible recursive call so that it makes progress towards a base case.

A Simple Example of Linear Recursion

Algorithm LinearSum(A, n):

Input:

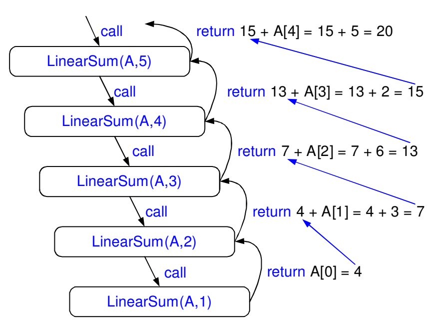
A integer array A and an integer $n = 1$, such that A has at least n elements

Output:

The sum of the first n integers in A

if $n = 1$ **then**
  return $A[0]$
else
  return LinearSum(A, n - 1) + $A[n - 1]$

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

Swap $A[i]$ and $A[j]$

ReverseArray(A, $i + 1$, $j - 1$)

return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional parameters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(A, i, j), not ReverseArray(A).

Computing Powers

- The power function, p(x,n)=x^n, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x, n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$

$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$

$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$

$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$

A Recursive Squaring Method

Algorithm Power(x, n):
  Input: A number x and integer n = 0
  Output: The value x^n
  if $n = 0$ **then**
    return 1
  if n is odd **then**
    y = Power(x, (n - 1)/ 2)
    return $x \cdot y \cdot y$
  else
    y = Power(x, n/ 2)
    return $y \cdot y$

Analyzing the Recursive Squaring Method

Algorithm Power(x, n):
  Input: A number x and integer n = 0
  Output: The value x^n
  if $n = 0$ **then**
    return 1
  if n is odd **then**
    y = Power(x, (n - 1)/ 2)
    return $x \cdot y \cdot y$
  else
    y = Power(x, n/ 2)
    return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we used a variable twice here rather than calling the method twice.

Tail Recursion

- ***Tail recursion*** occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).

Tail Recursion

- Example:

Algorithm IterativeReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

Swap $A[i]$ and $A[j]$

$i = i + 1$

$j = j - 1$

return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: the DrawTicks method for drawing ticks on an English ruler.

```
---- 0          ----- 0         --- 0
-               -               -
--              --              --
-               -               -
---             ---             --- 1
-               -               -
--              --              --
-               -               -
---- 1          ----            --- 2
-               -               -
--              --              --
-               -               -
---             ---             --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

A Binary Recursive Method for Drawing Ticks

```
// drawOneTick() - draw a tick with no label
public static void  drawOneTick(int  tickLength)  {
  drawOneTick(tickLength,  - 1);
}

// drawOneTick() - draw one tick with a label
public static void  drawOneTick
      (int  tickLength,  int  tickLabel)  {
    for  (int i =  0; i <  tickLength;  i++)
         System.out.print("-");

    if  (tickLabel  >=  0)
         System.out.print(" "  +  tickLabel);
    System.out.print("\n");
}
```

```
public static void  drawTicks(int  tickLength)  {
  // draw ticks of given length
    if  (tickLength  >  0)  {
      // stop when length drops to 0
      // recursively draw left ticks
      drawTicks(tickLength- 1);

      // draw center tick
      drawOneTick(tickLength);

      // recursively draw right ticks
      drawTicks(tickLength- 1);
   }
}
```

```
//drawRuler() - Draw a ruler
public static void  drawRuler
       (int  nInches,  int  majorLength)  {

    // draw tick 0 and its label
    drawOneTick(majorLength,  0);

    for  (int i =  1; i <=  nInches;  i++)      {
        // draw ticks for this inch
        drawTicks(majorLength- 1);

        // draw tick i and its label
        drawOneTick(majorLength,  i);
    }
}
```

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(A, i, n):

Input: An array A and integers i and n

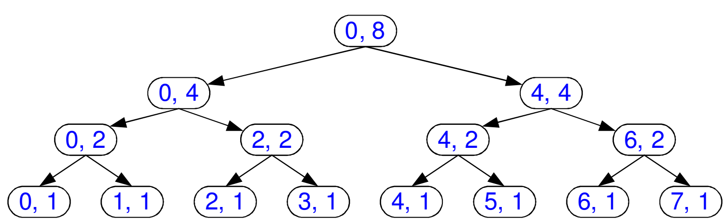
Output: The sum of the n integers in A starting at index i

if $n = 1$ **then**

return $A[i]$

return BinarySum(A, i, $n/2$) + BinarySum(A, $i + n/2$, $n/2$)

Example Trace:



Computing Fibanacci Numbers

- Fibonacci numbers are defined recursively:

$F_0 = 0$

$F_1 = 1$

$F_i = F_{i\text{-}1} + F_{i\text{-}2}$ for $i > 1$.

Computing Fibanacci Numbers

- As a recursive algorithm (first attempt):

Algorithm BinaryFib(k):

Input: Nonnegative integer k

Output: The kth Fibonacci number F_k

if $k = 1$ **then**

return k

else

return BinaryFib(k - 1) + BinaryFib(k - 2)

Analyzing the Binary Recursion Fibonacci Algorithm

- Let n_k denote number of recursive calls made by BinaryFib(k). Then
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that the value at least doubles for every other value of n_k. That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead:

Algorithm LinearFibonacci(*k*):
  Input: A nonnegative integer *k*
  Output: Pair of Fibonacci numbers (F_k, F_{k-1})
 if $k = 1$ **then**
  return (k, 0)
 else
  (i, j) = LinearFibonacci(k - 1)
  return ($i + j$, i)

- Runs in $O(k)$ time.