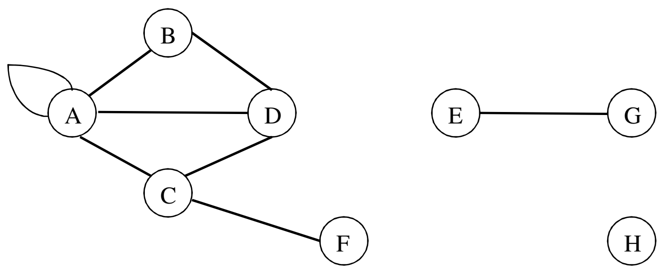
CSC 344 – Algorithms and Complexity

Lecture #12 – Graphs (Extended)

What is a Graph?

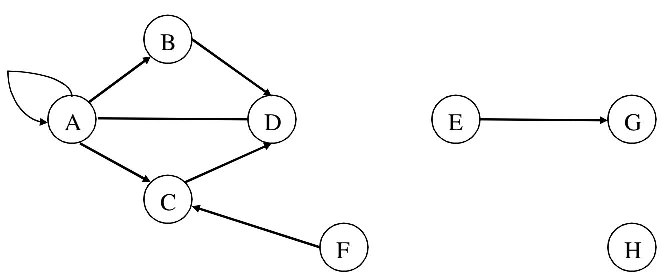
- A graph consists of a set of ***nodes*** (or ***vertices***) and a set of ***arcs*** (or ***edges***).
- Each arc in a graphs is specified by a pair of nodes.
- If the pair of nodes that make up the arcs are ***ordered pairs*** then the graph is a ***directed graph*** or ***digraph***.

Undirected Graph – An Example



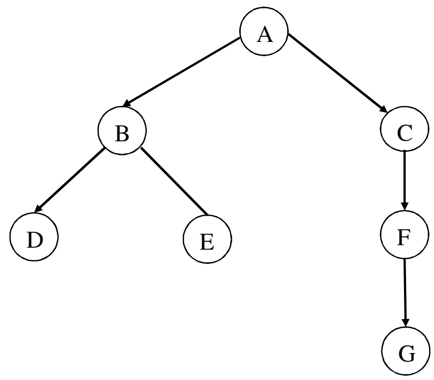
Set of nodes = {A, B, C, D, E, F, G, H}
Set of arcs = {(A, B), (A, D), (A, C), (C, D), (C, F), (E, G), (A, A)}

Directed Graph – An Example

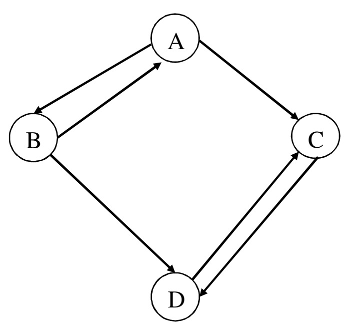


Set of nodes = {A, B, C, D, E, F, G, H}
Set of arcs = {<A, B>, <A, D>, <A, C>, <C, D>, <F, C>, <E, G>, <A, A>}

Digraph – An Example



Digraph – An Example



A graph need not be a tree but a tree must be a graph.

Other Definitions

- A node *n* is incident to an arc *x* if *n* is one of the two nodes in the ordered pair of nodes constituting *x*. We also say that *x* is incident to *n*.
- The ***degree of a node*** is the number of arcs incident to it.
- ***indegree of* n** – the number of arcs with n as the head.
- ***outdegree of* n** – the number of arcs with n as the tail.

Weighted Graphs

- A number may be associated with each arc of a graph. Such a graph is called a ***weighted graph*** or ***network***. The number associated with an arc is called the ***weight***.

Operations Used With Graphs

- *join (a, b)* – adds an arc from node *a* to *b*.
- *joinwt(a, b, x)* – adds an arc from *a* to *b* with weight *x*.
- *remove(a, b)* – removes an arc from *a* to *b* if it exists.
- *removewt(a, b, x)* – removes an arc from *a* to *b* and sets *x* to the weight of the now-defunct arc.

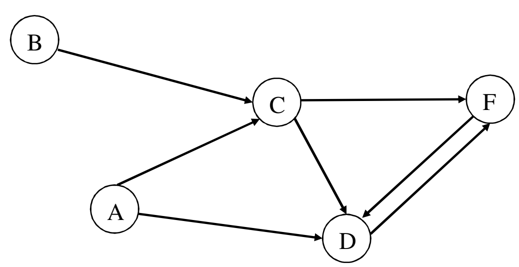
Paths and Cycles

- A path of length *k* from node a to node b is defined as a sequence of $k + 1$ nodes $n_1, n_2, \ldots, n_{k+1}$ such that $n_1 = a$ and $n_{k+1} = b$ and $adjacent(n_i, n_{k+1})$ is true for all i between 1 and k.
- A path from one node to itself is called a <u>***cycle***</u>.
- A graph with a cycle is cyclic; a graph without cycles is acyclic.
- Directed Acyclic Graphs are called ***dags***.

Transitive Closure

- Let's assume that the adjacency matrix (*adj*) completely describes the graph (the nodes contain no data and the graph is unweighted).
  - `if (adj[i][k] && adj[k][j] == true)`
  - `  // we have a 2-arc path from i to j`
- What if the path requires 3 or more arcs?

Sample Graph



adj

	A	B	C	D	E
A	0	0	1	1	0
B	0	0	1	0	0
C	0	0	0	1	1
D	0	0	0	0	1
E	0	0	0	1	0

adj_2

	A	B	C	D	E
A	0	0	0	1	1
B	0	0	0	1	1
C	0	0	0	1	1
D	0	0	0	1	0
E	0	0	0	0	1

adj_3

	A	B	C	D	E
A	0	0	0	1	1
B	0	0	0	1	1
C	0	0	0	1	1
D	0	0	0	0	1
E	0	0	0	1	0

adj_4

	A	B	C	D	E
A	0	0	0	1	1
B	0	0	0	1	1
C	0	0	0	1	1
D	0	0	0	1	0
E	0	0	0	1	1

$$path = adj_1 \mid adj_2 \mid adj_3 \mid adj_4 \mid adj_5$$

	A	B	C	D	E
A	0	0	1	1	1
B	0	0	1	1	1
C	0	0	0	1	1
D	0	0	0	1	1
E	0	0	0	1	1

Graph.h

```
#ifndef __GRAPH__
#define     __GRAPH__
#endif

using namespace std;

const int MaxNodes = 50;
typedef     int NodeStuffType;

struct node {
      NodeStuffType     data;
};

struct arc  {
      bool  adj;
};
```

```
class Graph
{
public:
      Graph(void);
      void  join(int node1, int node2);
      void  remove(int node1, int node2);
      bool  adjacent(int node1, int node2);
      void  transClose(int path[][MaxNodes]);
private:
      void prod(int a[][MaxNodes],
                        int c[][MaxNodes]);
      struct node nodes[MaxNodes];
      struct arc  arcs[MaxNodes][MaxNodes];
};
```

Graph.cpp

```
#include "Graph.h"

Graph::Graph(void)
{
      int i, j;
      for (i = 0; i < MaxNodes;  i++)
            for (j = 0;  j < MaxNodes;  j++)
                  arcs[i][j].adj = false;
}

void  Graph::join(int node1, int node2)   {
      arcs[node1][node2].adj = true;
}
```

```
void  Graph::remove(int node1, int node2) {
      arcs[node1][node2].adj = false;
}
bool  Graph::adjacent(int node1, int node2)     {
      return((arcs[node1][node2].adj == true)?
            true : false);
}
```

```
void  Graph::transClose(int path[][MaxNodes])   {
      int i, j, k;
      int newprod[MaxNodes][MaxNodes],
            adjprod[MaxNodes][MaxNodes];

      for (i = 0;  i < MaxNodes; i++)
            for (j = 0;  j < MaxNodes;  j++)
                  adjprod[i][j] = path[i][j]
                              = arcs[i][j].adj;

      for (i = 1;  i < MaxNodes; i++)     {
        // i represents the number of times adj
        // has been mulitplied by itself to
        // obtain adjprod.  At this point path
        // represents all paths of length i or
        // less
```

```
        prod(adjprod, newprod);
        for (j = 0;  j < MaxNodes;  j++)
            for (k = 0;  k < MaxNodes;  k++)
              path[j][k]
                  = path[j][k] || newprod[j][k];

            for (j = 0;  j < MaxNodes;  j++)
              for (k = 0;  k < MaxNodes;  k++)
                  adjprod[j][k] = newprod[j][k];
      }
}
```

```
void Graph::prod(int a[][MaxNodes],
                   int c[][MaxNodes])     {
   int      i, j, k, val;

   for (i = 0;  i < MaxNodes; i++)
      //pass through rows
      for (j = 0;  j < MaxNodes;  j++)    {
         // pass through columns
         val = false;
         for (k = 0;  k < MaxNodes;  k++)
            val
             = val ||
                 (a[i][k] && arcs[i][j].adj);
            c[i][j] = val;
      }  // for j..
}
```

Shortcoming in `transClosure()`

- The matrix multiplication that is performed is $O(n^3)$. It is performed n-1 times. That makes the efficiency of the algorithm $O(n^4)$, which is generally unacceptable.

Warshall's Algorithm

- We need a more efficient algorithm.
- Matrix $path_k$ is defined such that $path_k[i][j]$ is true if and only if there is path from node i to j that does not pass through any node numbered higher than k.
- Can we determine $path_{k+1}$ from $path_k$?

Warshall's Algorithm

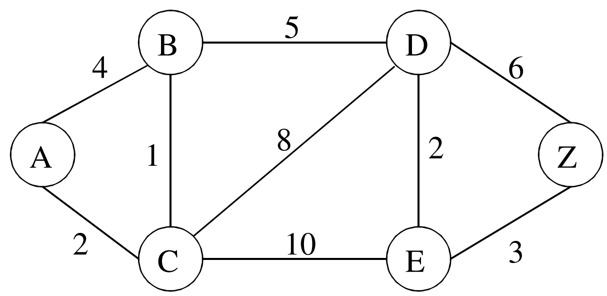
- $path_{k+1}$ will be true if and only if:
  1. $path_k[i][j] == true$
  2. $path_k[i][k+1] == true$

     $\&\& \; path_k[k+1][j] == true$

quickTransClose()

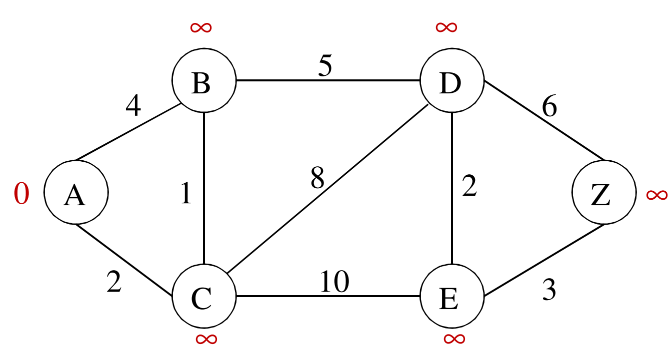
```
void  Graph::quickTransClose
                  (int path[][MaxNodes])  {
  int i, j, k;
  for (i = 0;  i < MaxNodes; i++)
    for (j = 0;  j < MaxNodes;  j++)
      // Path starts off as adj
      path[i][j] = arcs[i][j].adj;

  for (k = 0;  k < MaxNodes; i++)
    for (i = 0;  i  < MaxNodes; i++)
      if (path[i][k] == 1)
        for (j = 0;  j < MaxNodes; j++)
          path[i][j] = path[i][j] || path[k][j];
}
```

Dijkstra's Algorithm



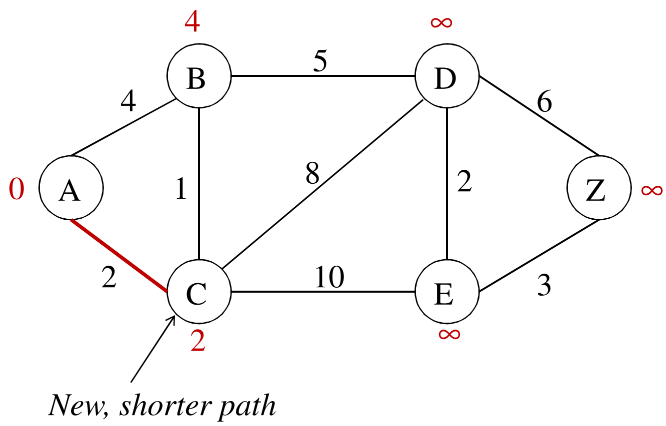
We want to find the shortest path from A to Z

Dijkstra's Algorithm



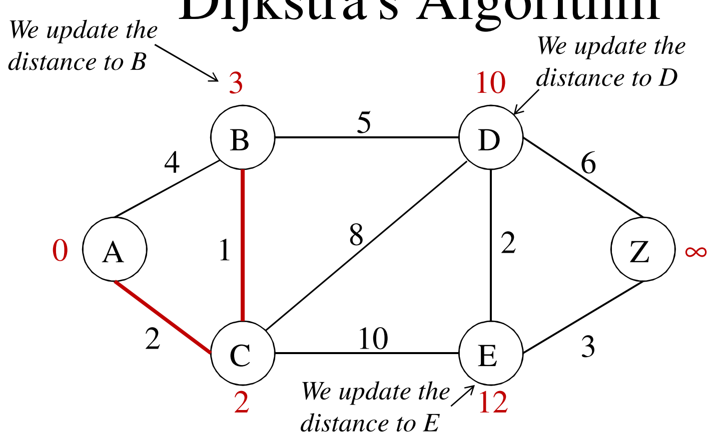
A's distance is 0; everyone else is initialize to ∞

Dijkstra's Algorithm



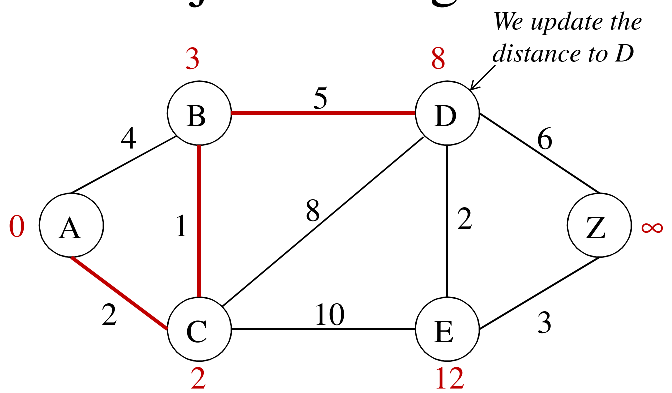
We update A's neighbor's

Dijkstra's Algorithm



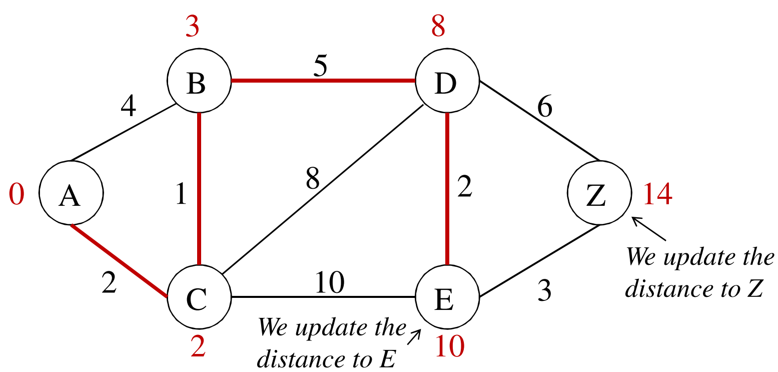
We update C's neighbor's
since it is closer to A

Dijkstra's Algorithm



We update C's neighbor's
since it is closer to A

Dijkstra's Algorithm



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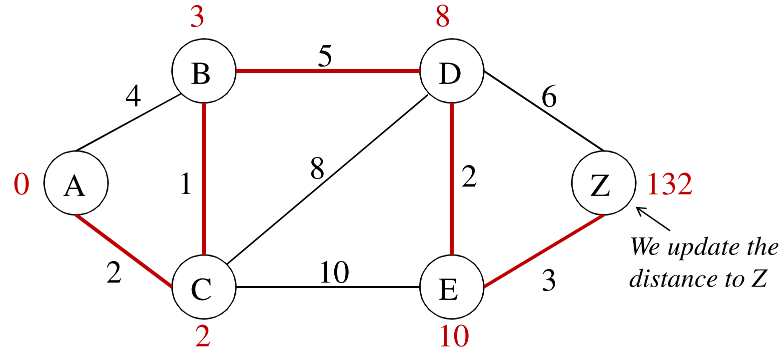
Dijkstra's Algorithm



We update C's neighbor's
since it is closer to A