CSC 344 – Algorithms and Complexity

Lecture #1 – Review of Mathematical Induction

Proof by Mathematical Induction

- Many results in mathematics are claimed true for every positive integer.
- Any of these results could be checked for a specific value of n (e.g., 1, 2, 3, ..) but it would be impossible to check every possible case. For example, let Sn represent the statement that the sum of the first n positive integers is

Proof by Mathematical Induction, (continued)

- Let S_n represent the statement that the sum of the first *n* positive integers is n(n+1)/2

$$S_n : \quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

If $n = 1$, then S_1 is $1 = \frac{1(1+1)}{2}$, which is true.

If $n = 2$, then S_2 is $1 + 2 = \frac{2(2+1)}{2}$, which is true.

If $n = 3$, then S_3 is $1 + 2 + 3 = \frac{3(3+1)}{2}$, which is true.

If $n = 4$, then S_4 is $1 + 2 + 3 + 4 = \frac{4(4+1)}{2}$, which is true.

Proof by Mathematical Induction, (continued)

- Continuing in this way for any amount of time would still not prove that Sn is true for every positive integer value of n.
- To prove that such statements are true for every positive integer value of n, the principle shown on the following slide is often used.

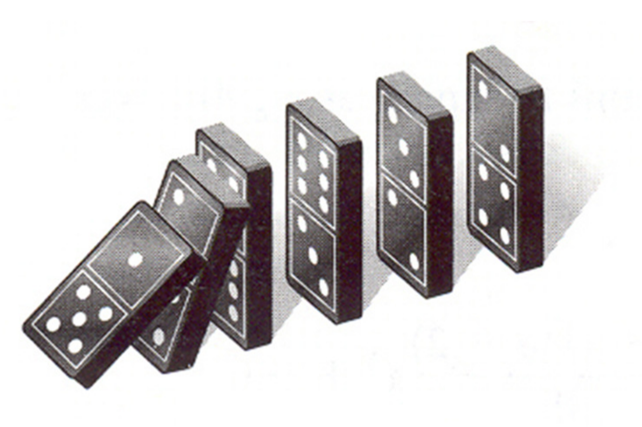
Principle of Mathematical Induction

- Let S_n be a statement concerning the positive integer n. Suppose that
  - 1. S_1 is true;
  - 2. for any positive integer k, $k \leq n$, if S_k is true, then S_{k+1} is also true.
- Then S_n is true for every positive integer value of n.

Principle of Mathematical Induction (continued)

- By assumption (1), the statement is true when n = 1.
- By assumption (2), the fact that the statement is true for n = 1 implies that it is true for n = 1 + 1 = 2.
- Using (2) again, the statement is thus true for 2 + 1 = 3, for 3 + 1 = 4, for 4 + 1 = 5, etc.
- Continuing in this way shows that the statement must be true for every positive integer.

How Does Mathematical Induction Work?



How To Prove by Mathematical Induction

- Step 1
  - Prove that the statement is true for n = 1.
- Step 2
  - Show that, for any positive integer k, $k \le n$, if S_k is true, then S_{k+1} is also true.

Example 1 - Proving An Equality Statement

Let S_n represent the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Prove that S_n is true for every positive integer *n.*

Solution

Step 1 Show that the statement is true when $n = 1$. If $n = 1$, S_1 becomes

$$1 = \frac{1(1+1)}{2},$$ which is true.

Example 1 - Proving An Equality Statement

Step 2 Show that S_k implies S_{k+1}, where S_k is the statement

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2},$$

and S_{k+1} is the statement

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2}.$$

Example 1 - Proving An Equality Statement

Step 2 Start with S_k and assume it is a true statement.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2},$$

Add $k + 1$ to both sides of this equation to obtain S_{k+1}.

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

Example 1 - Proving An Equality Statement

Step 2

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=(k+1)\left(\frac{k}{2}+1\right) \quad \text{Factor out } k+1.$$

$$=(k+1)\left(\frac{k+2}{2}\right) \quad \text{Add inside the parentheses.}$$

$$=\frac{(k+1)[(k+1)+1]}{2} \quad \text{Multiply; } k+2=(k+1)+1.$$

Example 1 - Proving An Equality Statement

- This final result is the statement for n = k + 1; it has been shown that if S_k is true, then S_{k+1} is also true.
- The two steps required for a proof by mathematical induction have been completed, so the statement S_n is true for every positive integer value of n.

Prove By Mathematical Induction

- Please note that the left side of the statement S_n ***always*** includes all the terms up to the nth term, as well as the nth term.

Example 2 - Proving An Inequality Statement

Prove: If x is a real number between 0 and 1, then for every positive integer n,

$$0 < x^n < 1.$$

Solution

Step 1 Here S_1 is the statement

$$\text{if } 0 < x < 1, \text{ then } 0 < x^1 < 1,$$

which is true.

Example 2 - Proving An Inequality Statement

Step 2 Here S_k is the statement

$$\text{if } 0 < x < 1, \text{ then } 0 < x^k < 1.$$

To show that this implies that S_{k+1} is true, multiply all three parts of $0 < x^k < 1$ by x to get

$$x \times 0 < x \times x^k < x \times 1$$

Example 2 - Proving An Inequality Statement

Step 2 (Here the fact that $0 < x$ is used.) Simplify to obtain

$$0 < x^{k+1} < x.$$

Since $x < 1$,

$$0 < x^{k+1} < x < 1$$

and thus

$$0 < x^{k+1} < 1.$$

This work shows that if S_k is true, then S_{k+1} is true. Since both steps for a proof by mathematical induction have been completed, the given statement is true for every positive integer *n.*

Generalized Principle of Mathematical Induction

- Some statements S_n are not true for the first few values of *n,* but are true for all values of *n* that are greater than or equal to some fixed integer *j*.
- The following slightly generalized form of the principle of mathematical induction takes care of these cases.

Generalized Principle of Mathematical Induction

- Let S_n be a statement concerning the positive integer *n*. Let *j* be a fixed positive integer. Suppose that
  - ***Step 1*** S_j is true;
  - ***Step 2*** for any positive integer *k*, $k \geq j$, S_k implies S_{k+1}.
  - Then S_n is true for all positive integers *n,* where $n \geq j$.

Example 3 - Using The Generalized Principle

Let S_n represent the statement $2^n > 2n + 1$. Show that S_n is true for all values of n such that $n \geq 3$.

Solution

Step 1 Show that S_n is true for $n = 3$. If $n = 3$, then S_n is

$$2^3 > 2 \times 3 + 1$$

or

$$8 > 7.$$

Thus, S_3 is true.

Example 3 - Using The Generalized Principle

Let S_n represent the statement $2^n > 2n + 1$. Show that S_n is true for all values of n such that $n \geq 3$.

Solution

Step 2 Now show that S_k implies S_{k+1}, where $k \geq 3$, and where

$$S_k \text{ is } 2^k > 2k + 1,$$

and

$$S_{k+1} \text{ is } 2^{k+1} > 2(k+1) + 1.$$

Example 3 - Using The Generalized Principle

Step 2

Multiply both sides of $2^k > 2k + 1$ by 2, obtaining

$$2 \times 2^k > 2(2k + 1)$$

$$2^{k+1} > 4k + 2.$$

Rewrite $4k + 2$ as $2k + 2 + 2k = 2(k + 1) + 2k$.

$$2^{k+1} > 2(k+1) + 2k \qquad (1)$$

Since k is a positive integer greater than 3,

$$2k > 1. \qquad (2)$$

Example 3 - Using The Generalized Principle

Step 2

Adding $2(k + 1)$ to both sides of inequality (2) gives

$$2(k+1) + 2k > 2(k+1) + 1. \qquad (3)$$

From inequalities (1) and (3),

$$2^{k+1} > 2(k+1) + 2k > 2(k+1) + 1,$$

or $$2^{k+1} > 2(k+1) + 1,$$ as required.

Example 3 - Using The Generalized Principle

Step 2

Thus, S_k implies S_{k+1}, and this, together with the fact that S_3 is true, shows that S_n is true for every positive integer value of *n* greater than or equal to 3.

Example 4 - Sum of Odd Integers

- Proposition: 1 + 3 + … + (2n-1) = n2
  for all integers n≥1.
- Proof (by induction):
  1. Basis step:
     The statement is true for n=1: 1=12 .
  2. Inductive step:
     Assume the statement is true for some k≥1
     (***<u>inductive hypothesis</u>***)
     show that it is true for k+1 .

Example 4 - Sum of Odd Integers (continued)

The statement is true for k:

$$1+3+\ldots+(2k-1) = k^2 \qquad (1)$$

We need to show it for k+1:

$$1+3+\ldots+(2(k+1)-1) = (k+1)^2 \qquad (2)$$

Showing (2):

$$1+3+\ldots+(2(k+1)-1) = 1+3+\ldots+(2k+1)$$
$$= 1+3+\ldots+(2k-1)+(2k+1)$$
$$= k^2+(2k+1)$$
$$= \mathbf{(k+1)^2}$$

We proved the basis and inductive steps,
so we conclude that the given statement true.

Example 5 - The Geometric Series

- Any sum of the form: $1 + r + r^2 + r^3 + ... + r^n$ is called a ***Geometric Series***.
- Thus, $1 + 2 + 4 + 8 + 16 +...+ 2^n$ is a geometric series.
- To find the sum of this series, consider:

$$S = 1 + r + r^2 + r^3 +...+ r^n.$$

So $$-rS = -r - r^2 - r^3 - ... - r^{(n+1)}$$

and $$(1-r)S = 1 - r^{(n+1)}$$

- Therefore, $1 + r + r^2 +...+ r^n = \dfrac{1 - r^{n+1}}{1 - r}$

Proof of the Geometric Series

- Prove: $1 + r + r^2 + ...+ r^n = [r^{(n+1)} - 1] / (r-1)$
- Proof: (by Induction)
- Basis: Show true for n = 0:

  LHS = 1

  $$\text{RHS} = \frac{r^{(0+1)} - 1}{r - 1} = \frac{r - 1}{r - 1} = 1$$
- Therefore LHS = RHS

Proof of the Geometric Series (continued)

- Induction:

  Assume $1 + r + r^2 + ... + r^k = \dfrac{r^{k+1} - 1}{r - 1}$
- Show:

  $$1 + r + r^2 + ... + r^k + r^{k+1} = \frac{r^{k+2} - 1}{r - 1}$$
- Now:

  $$1 + r + r^2 + ... + r^k + r^{k+1}$$
  $$= \frac{r^{k+1} - 1}{r-1} + r^{k+1}$$

Proof of the Geometric Series (continued)

$$1 + r + r^2 + ... + r^k + r^{k+1}$$
$$= \frac{r^{k+1} - 1}{r-1} + r^{k+1}$$
$$= \frac{r^{k+1} - 1 + (r-1)r^{k+1}}{r-1}$$
$$= \frac{r^{k+1} - 1 + r \cdot r^{k+1} - r^{k+1}}{r-1}$$
$$= \frac{r^{k+2} - 1}{r-1}$$

QED

Divisibility Property

- Proposition: For any integer $n \geq 1$,
  $7^n - 2^n$ is divisible by 5. (P(n))
- Proof (by induction):

1. Basis:
   The statement is true for $n = 1$: (P(1))
   $7^1 - 2^1 = 7 - 2 = 5$ is divisible by 5.

Divisibility Property (continued)

- We are given that

$P(k)$:

$7^k - 2^k$ is divisible by 5. (1)

Then

$7^k - 2^k = 5a$ for some $a \in \mathbf{Z}$.

(by definition) (2)

Divisibility Property (continued)

- We need to show:

$P(k+1)$:

$7^{k+1} - 2^{k+1}$ is divisible by 5. (3)

$7^{k+1} - 2^{k+1}$

$= 7\cdot 7^k - 2\cdot 2^k = 5\cdot 7^k + 2\cdot 7^{k-2}\cdot 2^k$

$= 5\cdot 7^k + 2\cdot(7^k - 2^k) = 5\cdot 7^k + 2\cdot 5a$

(by (2))

$= 5\cdot(7^k + 2a)$ which is divisible by 5.

(by def.)

Thus, $P(n)$ is true by induction.

Two Proofs to Try

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=0}^{n} (2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$