Introduction to Algorithms and Data Structures

Lecture 9 - Deja Vu All Over Again:
An Introduction to Recursion

What is Recursion?

- Recursion involves defining a solution to a problem in terms of another case of the problem except for one simple (or more) simpler cases in which the solution is defined explicitly.
- Example - Factorial

$$n! = n \bullet (n-1)! \qquad \textit{for } n > 0$$
$$\quad = 1 \qquad \textit{for } n = 0$$

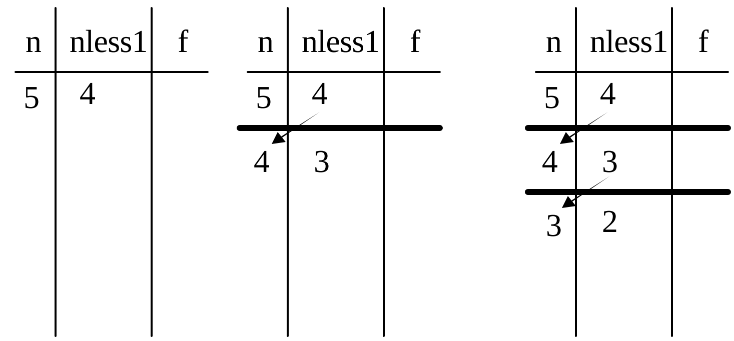
- We can write functions that call themselves recursively as long as they lead to a simple case that we can use as the basis for a solution.

The Original **factorial** Program

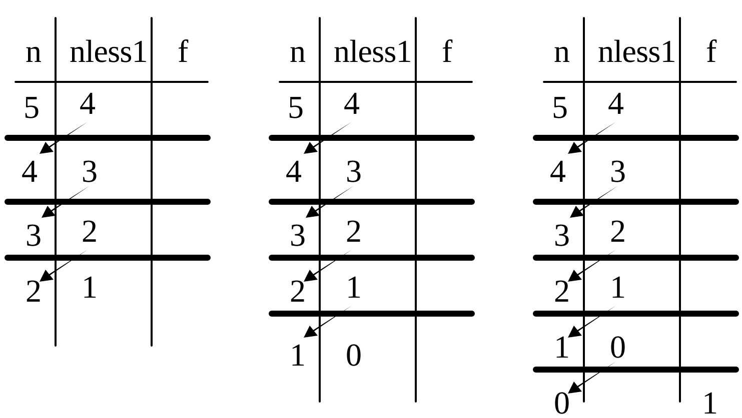
```
public class TestFactorial  {
  // main() - A Driver for our factorial
  //           function
  public static void main(String[] args)
  {
    Factorial f = new Factorial();
    System.out.println(f.factorial(5));
  }
}
```

```
public class Factorial  {
  // factorial() -  A Recursive solution for n!
  public double factorial(int n)  {
    int    nless1;
    double f;
    if (n == 0)
      // The simple case
      return(1.0);
    else {
      // The recursive solution
      nless1 = n - 1;
      f = factorial(n-1);
      return(n*f);
    }
  }
}
```

Tracing **`factorial`**



Tracing **`factorial`** (continued)



Tracing **`factorial`** (continued)

n	nless1	f
5	4	
4	3	
3	2	
2	1	
1	0	1

n	nless1	f
5	4	
4	3	
3	2	
2	1	1

n	nless1	f
5	4	
4	3	
3	2	2

Tracing **`factorial`** (continued)

n	nless1	f
5	4	
4	3	6

n	nless1	f
5	4	24

The Traceable Factorial Program

```
  // factorial() -  A Recursive solution for n!
  public double factorial(int n)  {
    int    nless1;
    double f;
    if (n == 0)
      // The simple case
      return(1.0);
    else {
      // The recursive case
      nless1 = n - 1;
      f = factorial(n-1);
      System.out.println(nless1 + "! = " + f);
      return(n*f);
    }
}
```

Tracing **fact2**

<u>Output</u>

```
0! = 1
1! = 1
2! = 2
3! = 6
4! = 24
120
```

A simpler **`factorial`**

```
// factorial() -  A Recursive solution for n!
 public double factorial(int n)  {
   if (n == 0)
     // The simple case
     return(1.0);
   else
     // The recursive case
     return(n*factorial(n-1));
 }
```

Example - Greatest Common Divisor

- The solution is defined as:

gcd(x, y) =	gcd(y, x)	if x < y
	y	if x MOD y = 0
	gcd(y, x MOD y)	if x >= y & x MOD y ≠ 0

The GCD `main` program

```
public class TestGcd  {

  // main() -  A Driver for the GCD function
  public static void main(String[] args)  {
    Gcd g = new Gcd();
    final  int x= 81, y = 180;

    System.out.println("The greatest common"
                       " divisor of "
                         +  x + " and " + y
                         + " is " + g.gcd(x, y));
  }
}
```

```
public class Gcd {
  // gcd() - Finds the Greatest Common Divisor
  public int gcd(int x, int y)  {
    if (x < y)
      // Reverse the parameter
      // y cannot be larger
      return(gcd(y, x));
    else if (x % y == 0)
      // The simple case
      return(y);
    else // The Recursive case
      return(gcd(y, x % y));
  }
}
```

Tracing **gcd**

x	y
81	180

x	y
81	180
180	81

x	y
81	180
180	81
81	18

x	y
81	180
180	81
81	18
18	9

Tracing **gcd** (continued)

x	y
81	180
180	81
81	18
18	9
9	0

A Traceable Form of **gcd**

```
public class Gcd {
  // gcd() - Finds the Greatest Common Divisor
  public int gcd(int x, int y)  {
    if (x < y) {
      // Reverse the parameter
      // y cannot be larger
      System.out.println("x = " + x + "\ty = "
                                 + y);
      return(gcd(y, x));
    }
```

```
    else if (x % y == 0) {
      // The simple case
      System.out.println("x = " + x + "\ty = “
                                 + y);
      return(y);
    }
    else {
      // The Recursive case
      System.out.println("x = " + x + "\ty = “
                                 + y);
      return(gcd(y, x % y));
    }
  }
}
```

Tracing `gcd`

Output

```
x = 81      y = 180
x = 180     y = 81
x = 81      y = 18
x = 18      y = 9
The greatest common divisor of 81 and 180 is 9
```

Example - Fibonacci Numbers

- Fibonacci numbers start with 0 and 1 and in each subsequent case are the sum of the two previous numbers.
- Fibonacci numbers are defined by the relationship:

$$\begin{aligned} Fib_n &= Fib_{n-1} + Fib_{n-2} && \text{for } n>1 \\ &= n && \text{for } n \leq 1 \end{aligned}$$

- Fibonacci numbers is different from our previous example because it is doubly-recursive, i.e., requires two recursive calls

TestFib.java

```
import java.util.Scanner;

public class TestFib {
  // main() - Drives the Fibonacci function
  public static void main(String[] args)
                              throws IOException {
    Scanner keyb = new Scanner(Ssytem.in);
    Fib                f = new Fib();
    int n;
    // Input n
    System.out.print("Which Fibonacci number do"
                     + "you want\t?");
    n = keyb.nextInt();
    System.out.println("The number is "
                       + f.fib(n));
  }
}
```

Fib.java

```
public class Fib  {
  // fib() - A Recursive Fibonacci function
  public int fib(int n)  {
    if (n <= 1)
      // The simple case
      return(n);
    else
      // The Recursive case
      return(fib(n-1) + fib(n-2));
  }
}
```

<u>Program Run</u>

```
Which Fibonacci number do you want  ?6
The 6th Fibonacci number is 8
```

The Traceable Fib - `fib.java`

```
public class Fib  {
  // fib() - A Recursive Fibonacci function
  public int fib(int n)  {
    int fibnless1, fibnless2, fibn;
    if (n <= 1)
      return(n);
    else {
      fibnless1 = fib(n-1);
      fibnless2 = fib(n-2);
      fibn = fibnless1 + fibnless2;
      System.out.println("The " + n
             + "th Fibonacci number is " + fibn);
      return(fibn);
    }
  }
}
```

Tracing `fib`

n	fibnless1	fibnless2	fibn
3		1	

n	fibnless1	fibnless2	fibn
3			
2			

Tracing **`fib`** (continued)

n	fibnless1	fibnless2	fibn
3			
2			
1			1

n	fibnless1	fibnless2	fibn
3			
2	1		
1			1

Tracing **`fib`** (continued)

n	fibnless1	fibnless2	fibn
3			
2	1		
0			0

n	fibnless1	fibnless2	fibn
3			
2	1	0	
0			0

Tracing `fib` (continued)

n	fibnless1	fibnless2	fibn
3			
2	1	0	1

n	fibnless1	fibnless2	fibn
3			
2	1	0	1
1			

Tracing `fib` (continued)

n	fibnless1	fibnless2	fibn
3			
2	1	0	1

n	fibnless1	fibnless2	fibn
3	1		
2	1	0	1

Tracing `fib` (continued)

n	fibnless1	fibnless2	fibn
3	1		
1			1

n	fibnless1	fibnless2	fibn
3	1	1	
1			1

Tracing `fib` (continued)

n	fibnless1	fibnless2	fibn
3	1	1	2

Binary Search

- A binary search is a reasonably fast way to search an array of data items or structures to find one with a particular value or a particular value in one field (known as the ***key***).
- Although it can be written without recursion, the recursive version is easier to follow.

BinarySearch.java

```
public class BinarySearch  {

  // The main binarySearch function and the
  // recursive function that it calls, named
  // binSearch

  // binarySearch() -  A Binary Search function
  //                   that calls the recursive
  //                   function binsearch
  public int binarySearch(int key, int[] x,
                                      int n)  {
    int index;
    index = binSearch(key, x, 0, n-1);
    return(index);
  }
```

```
// binsearch() - The recursive binary search
//               function
private int binSearch(int key, int[] x,
                      int low, int high)  {
  int  mid;

  // Find the mid point
  mid = (low + high) / 2;
  if (low > high)
    // We've searched the whole array
    //  - it isn't in it
    return(-1);
  else if (x[mid] == key)
    // We found it
    return(mid);
```

```
  else if (x[mid] < key)
    // Search the upper half of this section
    return(binSearch(key, x, mid+1, high));
  else
    // Search the lower half of this section
    return(binSearch(key, x, low, mid - 1));
 }
}
```

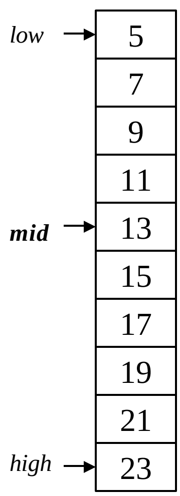
A Driver for **BinarySearch**

```
public class RunBinarySearch  {
  public static void main(String[] args)  {
    TestBinarySearch tbs = new TestBinarySearch();
    tbs.run(13);
  }
}
```

```
import java.util.Scanner;
// TestBinarySearch - A Driver for the binary
//                     search function
public class TestBinarySearch  {
  public static void run(int key)  {
    int i = 0;
    int [] x = new int[10];
    BinarySearch  bs = new BinarySearch();

    for  (i = 0;  i < 10;  i++) {
      x[i] = 2*i + 5;
      System.out.println("x[" + i + "] = "
                                  + x[i]);
    }
    System.out.println("\n\n\n");
    i = bs.binarySearch(key, x, 10);
    System.out.println("x[" + i + "] = 13");
  }
}
```

Tracing **`binSearch`**

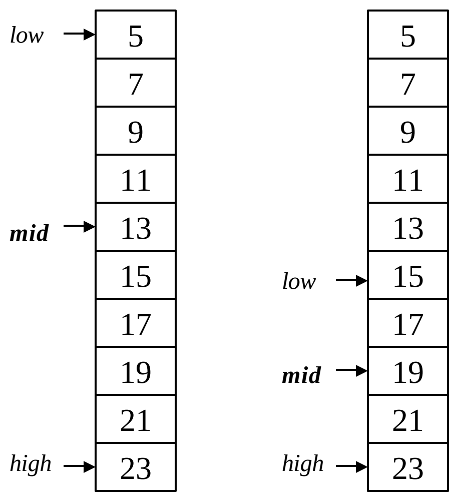


Returns ***4***

`RunBinarySearch` with 19

```
public class RunBinarySearch  {
  public static void main(String[] args)  {
    TestBinarySearch tbs = new TestBinarySearch();
    tbs.run(19);
  }
}
```

Tracing **`RunBinarySearch`** with 19

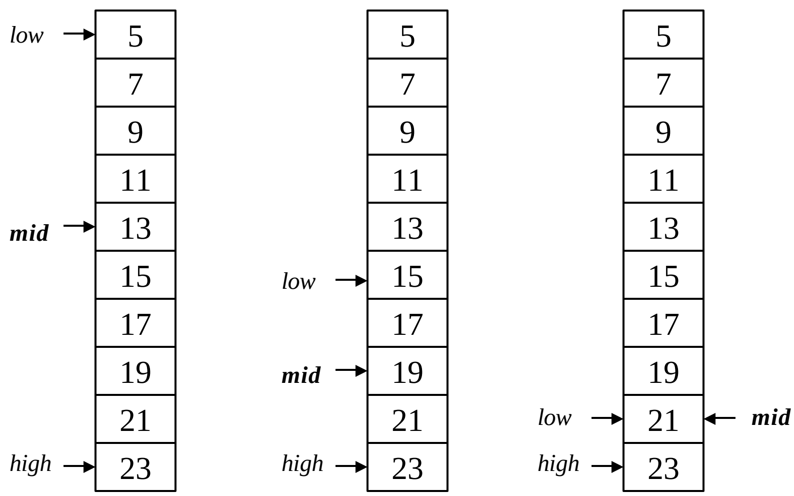


Returns ***7***

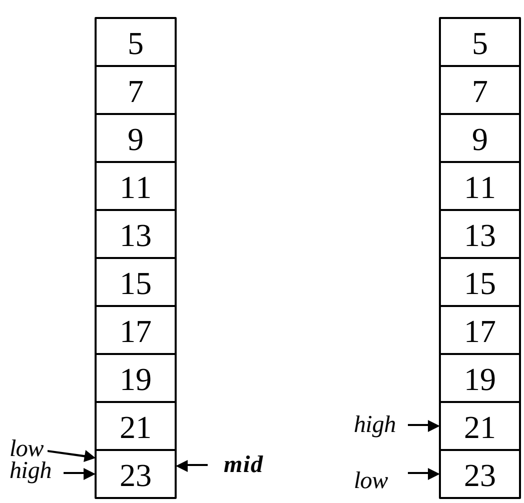
`RunBinarySearch` with 22

```
public class RunBinarySearch  {
  public static void main(String[] args)  {
    TestBinarySearch tbs = new TestBinarySearch();
    tbs.run(22);
  }
}
```

Tracing **`RunBinarySearch`** with 22



Tracing **`RunBinarySearch`** with 22 (continued)



Returns ***-1***

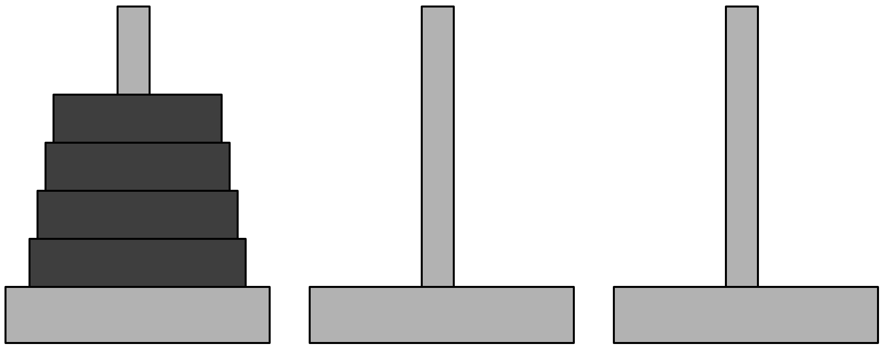
Towers of Hanoi

- The Towers of Hanoi is a classical example of a problem that is difficult to solve without recursion.
- The Towers consists of three pegs, one of which has a series of disks placed on it, with the disks stacked with smaller disks on top of larger disks.
- The challenge is to move them from one peg to another using an extra peg:
  – moving them one at a time
  – no disk can be placed on top of a smaller disk.

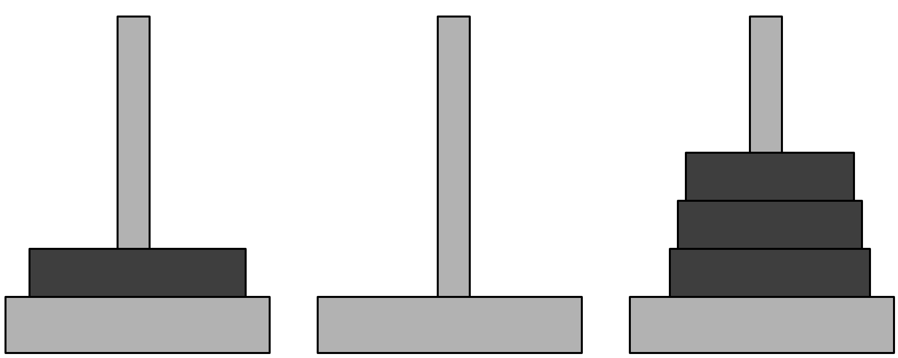
Solving the Towers of Hanoi

- Our solution involves three steps:
  1. Move all but the bottom disk to the extra peg
  2. Move the bottom disk to its final location.
  3. Move the other disks on top of the bottom disk.
- This is the basis of our recursive solution.

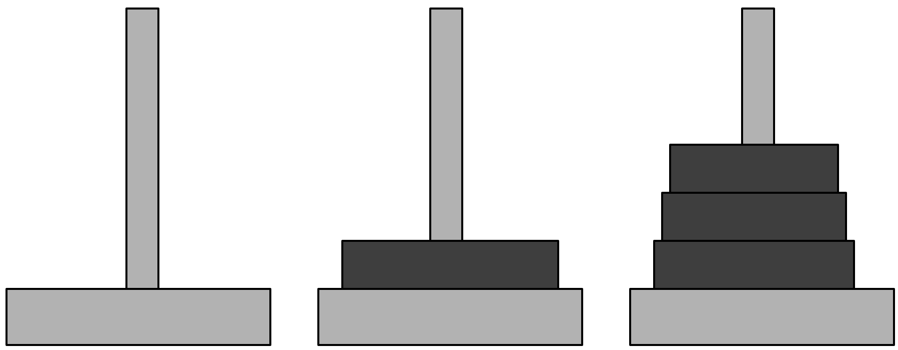
Our Recursive Solution - Initially



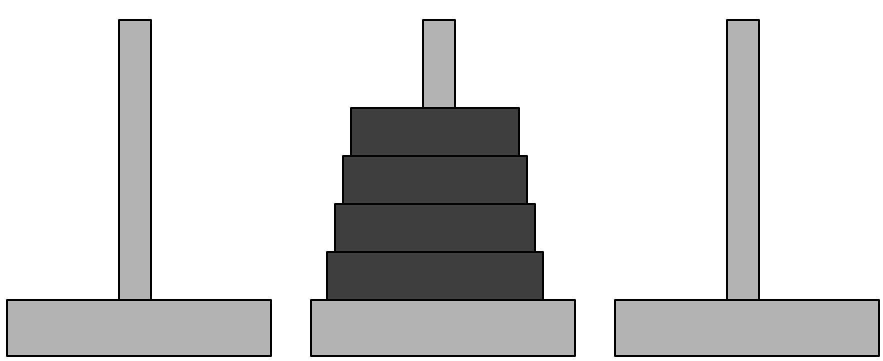
Our Recursive Solution - After Step 1



Our Recursive Solution - After Step 2



Our Recursive Solution - After Step 3



TestTowers.java

```
import java.util.Scanner;
public class TestTowers  {
  // main() -  Runs the recursive solution to
  //           the famous Towers of Hanoi problem

  public static void main(String[] args)  {
    int  numDisks;
    Scanner keyb = new Scanner(System.in);
    Towers  tower = new Towers();
    // Get the number of disks to be used
    System.out.println("How many disks\t(1-9)?");
    numDisks = keyb.nextInt();

    // Solve the problem
    tower.towersOfHanoi('A', 'B', 'C', numDisks);
  }
}
```

Towers.java

```
public class Towers  {

  // TowersOfHanoi() - The recursive solution to
  //                   the famous Towers of
  //                   Hanoi problem
  public void towersOfHanoi(char origin,
        char destination, char via, int disks)  {
    // Use the recursive solution for more than
    // one disk
    if (disks > 1) {
      // Move all but the bottom to the auxiliary
      // peg "via”
      // Move the bottom disk to the
      // destination peg
      // Move the rest on top of the bottom peg
```

```
      towersOfHanoi(origin, via, destination,
                                    disks - 1);
      System.out.println("Move disk #" + disks
                       + " from " + origin + " to “
                       + destination);
      towersOfHanoi(via, destination, origin,
                                     disks - 1);
    }
    else
      // Move the only disk into position
      System.out.println("Move disk #1 from "
                       + origin + " to "
                       + destination);
  }
}
```

Output from `towers`

```
How many disks? 3
Move disk #1 from A to B
Move disk #2 from A to C
Move disk #1 from B to C
Move disk #3 from A to B
Move disk #1 from C to A
Move disk #2 from C to B
Move disk #1 from A to C
```