Problems for Solution Sets for Systems of Linear Equations

1. Let $f(X) = MX$ where

$$M = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

   Suppose that α is any number. Compute the following four quantities:

$$\alpha X\,,\ f(X)\,,\ \alpha f(X) \text{ and } f(\alpha X)\,.$$

   Check your work by verifying that

$$\alpha f(X) = f(\alpha X)\,.$$

   Now explain why the result checked in the Lecture, namely

$$f(X+Y) = f(X) + f(Y)\,,$$

   and your result $f(\alpha X) = \alpha f(X)$ together imply

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)\,.$$

2. Write down examples of augmented matrices corresponding to each of the five types of solution sets for systems of equations with three unknowns.

3. Let

$$M = \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & & \vdots \\ a_1^r & a_2^r & \cdots & a_k^r \end{pmatrix}, \qquad X = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix}$$

Propose a rule for MX so that $MX = 0$ is equivalent to the linear system:

$$\begin{array}{cccc} a_1^1 x^1 & +a_2^1 x^2 & \cdots +a_k^1 x^k & = 0 \\ a_1^2 x^1 & +a_2^2 x^2 & \cdots +a_k^2 x^k & = 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^r x^1 & +a_2^r x^2 & \cdots +a_k^r x^k & = 0 \end{array}$$

Show that your rule for multiplying a matrix by a vector obeys the linearity property.

Note that in this problem, x^2 does not denote the square of x. Instead x^1, x^2, x^3, etc... denote different variables. Although confusing at first, this notation was invented by Albert Einstein who noticed that quantities like $a_1^2 x^1 + a_2^2 x^2 \cdots + a_k^2 x^k$ could be written in summation notation as $\sum_{j=1}^{k} a_j^2 x^j$. Here j is called a summation index. Einstein observed that you could even drop the summation sign $\sum$ and simply write $a_j^2 x^j$.

4. Use the rule you developed in the problem 3 to compute the following products

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 14 \\ 21 \\ 35 \\ 62 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 42 & 97 & 2 & -23 & 46 \\ 0 & 1 & 3 & 1 & 0 & 33 \\ 11 & \pi & 1 & 0 & 46 & 29 \\ -98 & 12 & 0 & 33 & 99 & 98 \\ \log 2 & 0 & \sqrt{2} & 0 & e & 23 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now that you are good at multiplying a matrix with a column vector, try your hand at a product of two matrices

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hint, to do this problem view the matrix on the right as three column vectors next to one another.

5. The *standard basis vector* e_i is a column vector with a one in the ith row, and zeroes everywhere else. Using the rule for multiplying a matrix times a vector in problem 3, find a simple rule for multiplying Me_i, where M is the general matrix defined there.