10 Problems: Inverse Matrix

1. Find formulas for the inverses of the following matrices, when they are not singular:

   (a) $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$

   When are these matrices singular?

2. Write down all 2×2 bit matrices and decide which of them are singular. For those which are not singular, pair them with their inverse.

3. Let M be a square matrix. Explain why the following statements are equivalent:

   (a) $MX = V$ has a *unique* solution for every column vector V.

   (b) M is non-singular.

   (In general for problems like this, think about the key words:

   First, suppose that there is some column vector V such that the equation $MX = V$ has two distinct solutions. Show that M must be singular; that is, show that M can have no inverse.

   Next, suppose that there is some column vector V such that the equation $MX = V$ has no solutions. Show that M must be singular.

   Finally, suppose that M is non-singular. Show that no matter what the column vector V is, there is a unique solution to $MX = V$.)

Hints for Problem 3

4. *Left and Right Inverses:* So far we have only talked about inverses of square matrices. This problem will explore the notion of a left and right inverse for a matrix that is not square. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) Compute:

i. AA^T,

ii. $\left(AA^T\right)^{-1}$,

iii. $B := A^T\left(AA^T\right)^{-1}$

(b) Show that the matrix B above is a *right inverse* for A, *i.e.*, verify that

$$AB = I\,.$$

(c) Does BA make sense? (Why not?)

(d) Let A be an $n \times m$ matrix with $n > m$. Suggest a formula for a left inverse C such that

$$CA = I$$

Hint: you may assume that A^TA has an inverse.

(e) Test your proposal for a left inverse for the simple example

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,,$$

(f) True or false: Left and right inverses are unique. If false give a counterexample.

Left and Right Inverses