Elementary Row Operations

1. (Row Equivalence)

   (a) Solve the following linear system using Gauss-Jordan elimination:

$$\begin{aligned} 2x_1 + 5x_2 - 8x_3 + 2x_4 + 2x_5 &= 0 \\ 6x_1 + 2x_2 - 10x_3 + 6x_4 + 8x_5 &= 6 \\ 3x_1 + 6x_2 + 2x_3 + 3x_4 + 5x_5 &= 6 \\ 3x_1 + 1x_2 - 5x_3 + 3x_4 + 4x_5 &= 3 \\ 6x_1 + 7x_2 - 3x_3 + 6x_4 + 9x_5 &= 9 \end{aligned}$$

   Be sure to set your work out carefully with equivalence signs $\sim$ *between each step, labeled by the row operations you performed.*

   (b) Check that the following two matrices are row-equivalent:

$$\left(\begin{array}{ccc|c} 1 & 4 & 7 & 10 \\ 2 & 9 & 6 & 0 \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 0 & -1 & 8 & 20 \\ 4 & 18 & 12 & 0 \end{array}\right)$$

   Now remove the third column from each matrix, and show that the resulting two matrices (shown below) are row-equivalent:

$$\left(\begin{array}{cc|c} 1 & 4 & 10 \\ 2 & 9 & 0 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 0 & -1 & 20 \\ 4 & 18 & 0 \end{array}\right)$$

   Now remove the fourth column from each of the original two matrices, and show that the resulting two matrices, viewed as augmented matrices (shown below) are row-equivalent:

$$\left(\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 9 & 6 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 0 & -1 & 8 \\ 4 & 18 & 12 \end{array}\right)$$

   Explain why row-equivalence is never affected by removing columns.

   (c) Check that the matrix $\left(\begin{array}{cc|c} 1 & 4 & 10 \\ 3 & 13 & 9 \\ 4 & 17 & 20 \end{array}\right)$ has no solutions. If you remove one of the rows of this matrix, does the new matrix have any solutions? In general, can row equivalence be affected by removing rows? Explain why or why not.

2. (Gaussian Elimination) Another method for solving linear systems is to use row operations to bring the augmented matrix to row echelon form. In row echelon form, the pivots are not necessarily set to one, and we only require that all entries left of the pivots are zero, not necessarily entries above a pivot. Provide a counterexample to show that row echelon form is not unique.

   Once a system is in row echelon form, it can be solved by "back substitution." Write the following row echelon matrix as a system of equations, then solve the system using back-substitution.

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

3. Explain why the linear system has no solutions:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

For which values of k does the system below have a solution?

$$\begin{array}{rcrcrcr} x & - & 3y & & & = & 6 \\ x & & & + & 3z & = & -3 \\ 2x & + & ky & + & (3-k)z & = & 1 \end{array}$$