Problems for What is Linear Algebra?

1. Let M be a matrix and u and v vectors:
   $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, v = \begin{pmatrix} x \\ y \end{pmatrix}, u = \begin{pmatrix} w \\ z \end{pmatrix}.$

   (a) *Propose* a definition for $u + v$.

   (b) *Check* that your definition obeys $Mv + Mu = M(u + v)$.

2. *Matrix Multiplication:* Let M and N be matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } N = \begin{pmatrix} e & f \\ g & h \end{pmatrix} ,$$

and v a vector

$$v = \begin{pmatrix} x \\ y \end{pmatrix} .$$

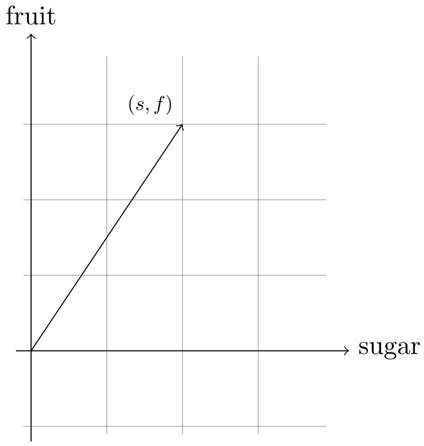
Compute the vector Nv using the rule given above. Now multiply this vector by the matrix M, *i.e.*, compute the vector $M(Nv)$.

Next recall that multiplication of ordinary numbers is associative, namely the order of brackets does not matter: $(xy)z = x(yz)$. Let us try to demand the same property for matrices and vectors, that is

$$M(Nv) = (MN)v \, .$$

We need to be careful reading this equation because Nv is a vector and so is $M(Nv)$. Therefore the right hand side, $(MN)v$ should also be a vector. This means that MN must be a matrix; in fact it is the matrix obtained by multiplying the matrices M and N. Use your result for $M(Nv)$ to find the matrix MN.

3. Pablo is a nutritionist who knows that oranges always have twice as much sugar as apples. When considering the sugar intake of schoolchildren eating a barrel of fruit, he represents the barrel like so:



Find a linear transformation relating Pablo's representation to the one in the lecture. Write your answer as a matrix.

Hint: Let λ represent the amount of sugar in each apple.

4. There are methods for solving linear systems other than Gauss' method. One often taught in high school is to solve one of the equations for a variable, then substitute the resulting expression into other equations. That step is repeated until there is an equation with only one variable. From that, the first number in the solution is derived, and then back-substitution can be done. This method takes longer than Gauss' method, since it involves more arithmetic operations, and is also more likely to lead to errors. To illustrate how it can lead to wrong conclusions, we will use the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned}$$

   (a) Solve the first equation for x and substitute that expression into the second equation. Find the resulting y.

   (b) Again solve the first equation for x, but this time substitute that expression into the third equation. Find this y.

   What extra step must a user of this method take to avoid erroneously concluding a system has a solution?