Math 253 **Exam 1** **10/21/2015**

Name: ____________________

Answer the questions on the exam and not on a separate sheet of paper. No work is necessary for the Sometimes/Always/Never questions. For all other questions, please circle your answers and justify your work for full credit. There are 15 questions for a total of 100 points.

Sometimes/Always/Never: Read the statement and decide whether the statement is sometimes true, always true, or never true. No work is necessary.

_______ 1. (5 points) The set $\{(a, b) \in \mathbb{R}^2 \mid a, b \geqslant 0\}$ with the usual addition and scalar multiplication is a vector space.

A. Sometimes B. Always C. Never

_______ 2. (5 points) The matrix $\begin{pmatrix} 1 & -\pi \\ 0 & t \end{pmatrix}$ is invertible.

A. Sometimes B. Always C. Never

_______ 3. (5 points) Let L be a linear transformation and P a particular solution to $Lx = b$. If H is a homogeneous solution, then $P + 4H$ is also a particular solution.

A. Sometimes B. Always C. Never

_______ 4. (5 points) The following is a two dimensional hyperplane in $\mathbb{R}^4$:

$$\left\{ \begin{pmatrix} 1 \\ \frac{3}{5} \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ \frac{1}{5} \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{3}{2} \\ -1 \\ \frac{6}{5} \end{pmatrix} s \;\middle|\; t, s \in \mathbb{R} \right\}.$$

A. Sometimes B. Always C. Never

_______ 5. (5 points) If A and B be $n \times n$ matrices with real entries, then $AB = BA$.

A. Sometimes B. Always C. Never

Multiple Choice. Circle your answers, not work is necessary.

6. (5 points) Circle all of the following matrices that are in reduced row echelon form.

$$\begin{pmatrix}1 & 0 & 3 & 1\\ 0 & 1 & 2 & 4\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad \begin{pmatrix}1 & 4 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \begin{pmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Short Answer. Make sure and justify your answer for full credit.

7. (5 points) Assume that M is a square matrix. Give 3 equivalent statements to the following:

   M is an invertible matrix.

   (a)

   (b)

   (c)

8. (5 points) Let $A = \begin{pmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{pmatrix}$, $B = \begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$, and $C = \begin{pmatrix}3 & 5\\ 2 & -6\end{pmatrix}$. Compute, if possible, $B + C$, AB, BA.

9. Let $M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

(a) (5 points) Determine whether or not M is invertible. If so, find the inverse.

(b) (5 points) If possible, use the inverse to find the solution to $Mx = b$. Write the solution set for the system in the form $S = \{X_0 + \sum_i \mu_i Y_i \mid \mu_i \in \mathbb{R}\}$.

10. Let $M = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) (2 points) Find the dot product between MX and X.

(b) (3 points) Compute the lengths of MX and X.

(c) (5 points) Find the angle between MX and X.

11. (5 points) If $p(3x) = x^2$ and $p(2x) = x$, is it possible that p is a linear function from the polynomials to the polynomials? Explain your answer. (Hint: Fractions are scalors too!!)

12. (5 points) For any $a, x, y \in \mathbb{R}$ define $x \oplus y = (x^3 + y^3)^{1/3}$ and $a \otimes x = \sqrt[3]{a}x$. Show that $a \otimes (x \oplus y) = (a \otimes x) \oplus (a \otimes y)$.

13. (10 points) For what values k and h does the linear system

$$\begin{aligned} x + 2y &= k \\ 4x + hy &= 5 \end{aligned}$$

have

(a) no solution;

(b) unique solution;

(c) infinite solution.

14. (10 points) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator such that

$$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} \text{ and } L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix}.$$

(a) Show that any vector $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$ is a sum of multiples of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$.

(b) Use part (a) to show how L acts on any vector $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$.

(c) How many points do you need to define a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?

15. (10 points) Let $B = (x^2, x, 1)$ be an ordered basis for a vector space V of polynomials of degree 2 or less.

    (a) Write $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B$ as a polynomial in V.

    (b) Write $x^2 - x \in V$ as a vector with ordered basis B.

    (c) Find the matrix $\dfrac{d}{dx}$ acting on the vector space V with ordered basis B.

    (d) Rewrite the differential equation $\dfrac{d}{dx}p(x) = x$ as a matrix equation. **DO NOT SOLVE!!**