On the Cuspidal Point of the Second Kind of Monsieur le Marquis de l'Hôpital*

Leonhard Euler†

1. I have already pointed out in several pieces, which I have had the honor of presenting to the Royal Academy, that even Geometry is not exempt from controversies and apparent contradictions, although we quite often maintain the contrary. But I have also remarked on this great advantage of Geometry: that these difficulties may be smoothed out, so that not the slightest doubt remains, as long as we thoroughly examine the circumstances of the controversial subject. In this treatise, I will once again address the Assembly on the matter of a controversy in pure Geometry, which concerns a certain kind of cuspidal point, resembling the beak of a bird, and formed from two branches of a curve, whose concavities turn in the same direction, as opposed to the ordinary cuspidal point, where the branches curve in opposite senses.

2. Mr. le Marquis de l'Hôpital, in his *Analyse des infiniments petits*[1] calls such cuspidal points of the bird's *of the second kind*, and he holds that there are infinitely many curves, both algebraic and transcendental, which are endowed with such a cuspidal point. He proves this in the following way. Let (Fig. 1) ABC be an arbitrary curve which has a point of inflection at B. Suppose also that we wrap a thread around this curve, which we then unwrap by pulling it successively from the point A, until it becomes detached at the inflection point B, and the extremity A will describe, by this evolution, the arc AD, to which the detached thread BD will be perpendicular to its radius of curvature, which we know by the theory of evolutes. But if we continue this evolution beyond the point B, the thread BD will turn back and, coming to the position MN, its extremity will describe the arc DN, which is, as a

*E169 - *Opera Omnia*, Ser. I, vol. 27, p. 236-252, originally in *Mémoires de l'académie des sciences de Berlin* **5** (1749), 1751, p. 203-221.

[1]See article 109, on p. 102.

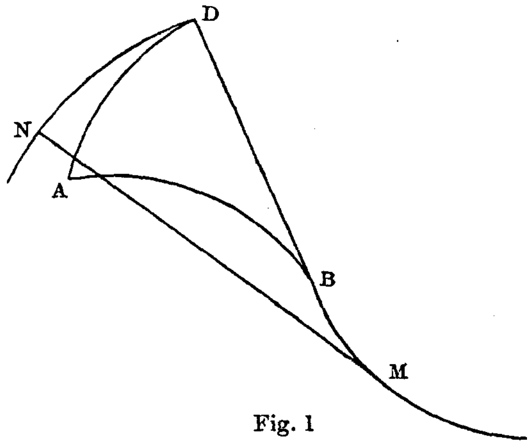

Fig. 1

consequence, the continuation of the arc AD. Now these two arcs AD, DN, which at the point D form an infinitely small angle, are concave in the same direction. Thus, the curve ADN described by the evolution of the line ABC will have, at the point D, a cuspidal point of the second kind, as the Marquis de l'Hôpital calls it.

3. However sound this demonstration may appear, Mr. Guà de Malves[2], in his treatise *Usage de l'Analyse de DesCartes pour découvrir les propriétés des lignes géometriques de tous les Ordres*, is of an entirely different opinion, and holds that there are no curves which, having a branch extending from A to D, may suddenly reverse direction and come to the point N, without also changing curvature; that is, without becoming convex, having previously been concave in the same direction. As for the proof that Mr. le Marquis de l'Hôpital gives, he does not deny that the curve formed in this way has the represented figure, but he holds that the branch DN is not the continuation of the branch AD, even though it is described by the same motion of evolution. He is thus obliged to say, that one may not judge the continuity of a curve by the continuity of its description. Such an exception will no doubt appear rather strange, and I confess that, were it well founded, it would overturn most of the proofs, by which we believe ourselves able to determine with certainty the shape of curved lines.

4. Mr. Guà recognized only analytic equations, from which one may draw a precise understanding of the shape of the curved line, of the number

[2]Jean Paul de Guà de Malves (1712-1785). The book cited here was published in 1740.

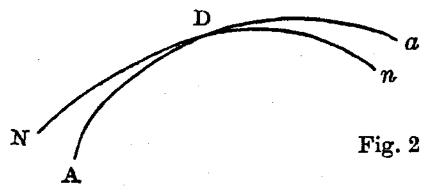

Fig. 2

of branches, and of their continuity. He believes to have demonstrated in his treatise that every time one finds oneself with such a cuspidal point in a curved line, one is nonetheless mistaken, and if one completes the description following the equation that expresses its nature, the figure is always such as is represented in Figure 2. That is, that the bird's beak will be a pair ADN and aDn, and that it is not the arc DN that is the coninuation of the arc AD, but the arc Da, so that the point D is but the intersection of two branches, ADa and NDn, which meet in D in an infinitely small angle, and which are concave in the same direction. One could also say that these are two branches of the same curved line, ADn and NDa, which meet at D, and so it will be uncertain whether the arc Dn or the arc Da will be the continuation of the arc AD.

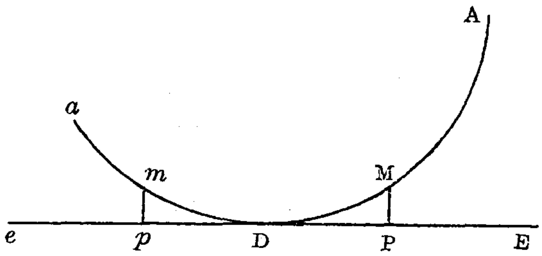

Fig. 3

5. To prove this thesis, he draws from the point D (Fig. 3) the tangent EDe, which he takes as the axis upon which he places the abcissa $DP = x$ and the ordinate $PM = y$. Now, whatever the equation for the curve, if we take the abcissa x to be extremely small, the value of y may always be expressed by a convergent series of this form:

$$y = \frac{xx}{2a} \pm Ax^m \pm Bx^n \pm Cx^k \pm \text{etc.} \ ,$$

where the exponents m, n, k, etc., are increasing, and may be either whole numbers or fractional. And if there are fractions whose denominators are even numbers, the values of these terms will be ambiguous, and this will be the case wherein several branches of the curve meet the axis Ee at the point D. Now the first term, with the lowest power of x, will be $\frac{xx}{2a}$, if the radius of curvature at D is taken to be $= a$.

6. Given all this, he lets the abcissa x be infinitely small, in order to determine the course of the curve at the point D and in this case, he says, the terms Ax^m, Bx^n, Cx^k, etc., all vanish with respect to the first, $\frac{xx}{2a}$, as their exponents are greater than 2. Thus, at the point D, all that remains of the curve is the equation $y = \frac{xx}{2a}$, which expresses the course or path that the curve follows on this side and that of the point D, as long as the abcissa x remains infinitely small. Now in this case, it is clear that the equation $y = \frac{xx}{2a}$ gives the same value, whether x be taken positive or negative, and from this he concludes, that the curve will always have the form MDm near the point D and, as a consequence, the arc AD must take its continuation towards Dm. This should be understood as long as the curvature at D is finite, as it is in our case; for if the curvature should be infinite or infinitely small, the argument loses its validity.

7. These are the arguments with which Mr. Guà de Malves opposes the view of Mr. le Marquis de l'Hôpital on cuspidal points of the second kind, and if one weighs the arguments on each side, one finds them so strong that it seems almost inevitable to find a contradiction between the description made by an evolution, and by the equation of a curve. The one quite clearly tells us that the arc AD (Fig. 2) must turn back by passing along DN, whereas the other persuades us that this continuation must absolutely be along Dn or Da. Nevertheless, I will show by incontestable arguments, that there is not the least dissension between the mechanical description and the calculus, but that there has slipped into the reasoning of Mr. Guà, sound though it might otherwise appear, a small oversight. This being noted, the contradiction will entirely disappear.

8. For although it is true that in an equation such as

$$y = Ax^\alpha + Bx^\beta + Cx^\gamma + Dx^\delta + \text{etc.},$$

where the exponents α, β, γ, δ, etc., are increasing positive numbers, one may ignore the subsequent terms with repsect to the first, when one supposes x to be infinitely small, whether positive or negative, and that in this case, one need only consider the equation $y = Ax^\alpha$, nevertheless one must note with care, that this omission of terms may not take place, except in the case where all of the terms have real values, granted that they be infinitely small. For if a single one of the rejected terms be imaginary, the entire expression, and as a consequence the value of y, will be also, and we would make a great error by replacing it with $y = Ax^\alpha$, which is to say, a real quantity. And thus the customary rule, by which one may ignore subsequent terms with respect to the first one, requires this restriction: that the omitted terms may not be imaginary.

9. Mr. Guà de Malves did not pay attention to this absolutely necessary restriction, and this was the source of his apparent contradiction to the existence of the cuspidal point of the second kind. To make this even more clear, consider the equation

$$y = x + xx\sqrt{-1}.$$

We see, first of all, that the value of y is always imaginary, except in the case $x = 0$, in which case we also have $y = 0$. Therefore, this equation places but a single point, situated on the axis, at the origin of the abcissas x. But had we wished to determine the curve expressed by this equation, following the method of Mr. Guà, by supposing the abcissa x to be infinitely small, we would need only consider the equation $y = x$, and we would conclude that this curve is given near the origin by the straight line expressed by $y = x$, which would however be contrary to the truth. Yet this conclusion would be quite correct if the coefficient of xx, instead of being imaginary, were any real quantity.

10. Now it may happen that a branch of the curve vanishes, or becomes imaginary, and yet no term of the equation is overtly affected by an imaginary quantity. For, let us consider the curve whose equation is

$$y = x + x\sqrt{x},$$

and suppose x to be ifinitely small, either negative or positive. In the first case, the term $x\sqrt{x}$ being real and infinitely smaller that x, we may ignore it and have $y = x$, which represents the beginning of the curve in the region of positive abcissas. But as soon as we take x to be negative, even though infinitely small, the term $x\sqrt{x}$ becomes imaginary, making the value of y also imaginary, and beginning from here it is no longer permissible to imagine the equation $y = x$ in place of $y = x + x\sqrt{x}$. This curve does not, therefore, extend into the region of negative abcissas, but has two branches in the positive direction, according to the double value of $\pm x\sqrt{x}$, which form a cuspidal point of the first kind. It is clear that this is the second cubic parabola.

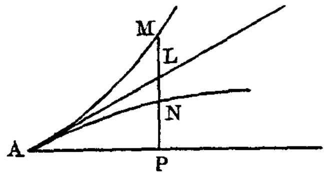

Fig. 4

11. A similar ordinary cuspidal point will also be found on this more general equation:

$$y = \alpha x + \beta x^{1+\frac{m}{n}},$$

whenever m is a postive odd number and n a positive even number. For in this case, the value of the term

$$\beta x^{1+\frac{m}{n}}$$

will be ambigous, and we will have

$$y = \alpha x \pm \beta x^{1+\frac{m}{n}}.$$

To find the shape (Fig. 4) of this curve, one need only draw the straight line AL for which, given the abcissa $AP = x$, the ordinate PL becomes $= \alpha x$. Then, one takes on PL the extendsion

$$LM = LN = \beta x^{1+\frac{m}{n}},$$

which will be infinitely smaller than x when x is infinitely small, and the curve will pass through the points M and N, and join up at the point A. Therefore the curve will have two branches AM and AN, each of which is convex towards AL and which form, as a consequence, a cuspidal point MAN of the first kind, for this curve does not extend into the region of negative abcissas, since taking x to be negative, the ordinate y becomes imaginary.

12. The same occurs with those curves, which appear to Mr. Guà not to admit a cuspidal point of the second kind. One need only consider the equation

$$y = \alpha xx \pm \beta xx\sqrt{x},$$

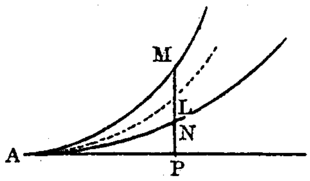

Fig. 5

which becomes imaginary as soon as one makes x negative, and from this it is certain that this curve has no part in the region of negative abcissas. To determine the shape of this curve (Fig. 5), one need only construct on the axis AP a parabola AL, which correpsonds to the equation $y = \alpha xx$ and, letting $AP = x$, we have $PL = \alpha xx$. If we then let $LM = LN = \beta xx\sqrt{x}$, then the points M and N are on the curve we seek. And supposing further that x be infinitely small, the parts LM and LN are infinitely small compared to PL, and both branches AM and AN will meet the origin A with the parabola AL, and will have the same curvature and will, as a consequence, be convex in the same direction with respect to the axis AP. Thus, these two branches MA and NA will form a true cuspidal point of the second kind at A.

13. From this it is clear that, even if we take the abcissa x to be infinitely small, we may not neglect the term $\beta xx\sqrt{x}$ with respect to αxx, except when x is taken to be positive. And in the case where x is negative, the equation $y = \alpha xx$ cannot be used for

$$y = \alpha xx \pm \beta xx\sqrt{x},$$

for this indicates a real value of y, which is instead imaginary, because of the second term. As a consequence, the arguments of Mr. Guà do not apply in this case, for he holds that the equation $y = \alpha xx$ may be used to determine the shape pf the curve near the point A, from which it would follow without a doubt that the curve extends into the region of negative abcissas. However, the neglected term $\beta xx\sqrt{x}$ tells us rather that the ordinates y become imaginary. It is therefore not permissible to neglect this term with respect to αxx, for its value does not become imaginary, whereas it should, even when x is infinitely small.

14. Having thus found a curve that has a cuspidal point of the second kind, it becomes easy to conceive of an infinity of other curves, in which one may fina similar cuspidal point. For one may add as many powers of x as one wishes to the equation $y = \alpha xx \pm \beta xx\sqrt{x}$, as long as these powers be positive

numbers greater than 2, for all such terms will vanish with respect to the first, αxx, as long as one takes x to be infinitely small, yet positive. However, for negative values of x, the single term $\beta xx\sqrt{x}$ renders this hypothesis invalid. Indeed, one may even add the term xxP, to get

$$y = \alpha xx \pm \beta xx\sqrt{x} + xxP,$$

where P represents an arbitrary function of x, which vanishes when $x = 0$.

15. One should note that all such curves, which are described by this general equation, will have finite curvature at the cuspidal point A, that is the radius of the evolute y will be finite; just as occurs with the curves that Mr. le Marquis de l'Hôpital constructed to demonstrate the existence of the cuspidal point of the second kind. It is appropriate to make note here of several very simple curves of this type

$$\begin{array}{rcll} y &=& \alpha xx + \beta xx\sqrt{x} & \text{of the 5th degree} \\ y &=& \alpha xx + \beta x^3\sqrt{x} & \text{of the 7th degree} \\ y &=& \alpha xx + \beta x^4\sqrt{x} & \text{of the 9th degree} \\ y &=& \alpha xx + \beta x^5\sqrt{x} & \text{of the 11th degree} \\ && \text{etc.,} & \end{array}$$

which all have a cuspidal point of the second kind at the origin, where $x = 0$, and the radius of the evolute at this point is a finite quantity.

16. One may render these formulas even more general without increasing the number of terms, by using a different radical sign, whose exponent is an even number, so that its value is ambiguous. Thus, we have these equations:

	or	
$y = \alpha xx + \beta xx\sqrt[4]{x}$		$y = \alpha xx + \beta xx\sqrt[6]{x}$
$y = \alpha xx + \beta xx\sqrt[4]{x^3}$		$y = \alpha xx + \beta xx\sqrt[6]{x^5}$
$y = \alpha xx + \beta x^3\sqrt[4]{x}$		$y = \alpha xx + \beta x^3\sqrt[6]{x}$
$y = \alpha xx + \beta x^3\sqrt[4]{x^3}$		$y = \alpha xx + \beta x^3\sqrt[6]{x^5}$
etc.		etc.

All of these formulas are included in the general formula

$$y = \alpha xx + \beta x^{2+\frac{m}{n}},$$

where m is any positive odd number and n is any positive even number. The radius of the evolute will be finite at the cuspidal point in all of these curves.

17. Instead of the first term αxx, one may also take a higher power of x, and then the radius of the evolute at the cuspidal point will be infinitely large, as in the formulas

$$\begin{array}{rcl} y &=& \alpha x^3 + \beta x^{3+\frac{m}{n}} \\ y &=& \alpha x^4 + \beta x^{4+\frac{m}{n}} \\ y &=& \alpha x^5 + \beta x^{5+\frac{m}{n}} \\ && \text{etc.} \end{array} \tag{1}$$

or, more generally,

$$y = \alpha x^k + \beta x^{k+\frac{m}{n}},$$

where k is an integer greater than 1 and the values of m and n are as in the previous paragraph. For if $k = 1$, the cuspidal point will be of the first kind. It is not in fact necessary that k be a whole number; it may be a fractional number, as long as the power x^k does not take on an ambiguous value, which may be either postive or negative.

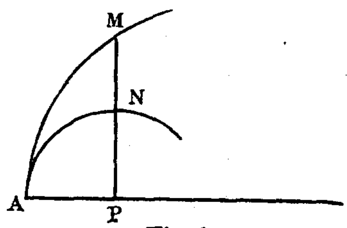

Fig. 6

18. The exponent k may also be a fractional number less than 2, as long as it be greater than 1. In this case, the radius of the evolute at the point A, where the cuspidal point of the second kind is located, will be infinitely small. One may even let k be smaller than unity, as long as the power x^k does not become ambiguous; this will be the case if k is a fraction whose denominator is odd. But here the tangent to the curve (Fig. 6) at the cuspidal point A will be perpendicular to the axis AP, in contrast to the preceding cases, where it lies along the axis AP itself. In this case, one must note that the radius of the evolute at the point A will be finite only if $k = \frac{1}{2}$; it will be infinitely small if $k > \frac{1}{2}$ and infinitely large is $k < \frac{1}{2}$. Thus, the equation

$$y = \alpha\sqrt[3]{x} + \beta\sqrt[2]{x}$$

represents a curve MAN, which has a cuspidal point of the second kind at A, such that the radius of the evolute is infinitely large.

19. It is also possible that the exponent k be $= \frac{1}{2}$ without the power $\alpha x^{\frac{1}{2}}$ becoming ambiguous, the ambiguity being destroyed by the following term, as in the equation,

$$y = \alpha\sqrt{x} + \beta\sqrt{x\sqrt{x}},$$

where the first term $\sqrt{x}$ is found in second, after the radical sign, so that if we take the first term $\sqrt{x}$ to be negative, the second becomes imaginary. And in effect, since x in the second term has a higher power than in the first, this curve also has a cuspidal point of the second kind at the origin. Now this curve is of the fourth order, as the equation, with the radicals eliminated, becomes

$$y^4 - 2\alpha\alpha xy^2 - 4\alpha\beta\beta xxy + \alpha^4 xx - \beta^4 x^3 = 0,$$

which seems to be the simplest curve endowed with such a cuspidal point.

20. For it is evident enough that such a cuspidal point cannot be located in lines of the third degree, for it is always possible to draw a straight line near such a cuspidal point which intersects the curve in 4 places, which requires a fourth degree line, at very least. Now I soon will show that there are an infinity of lines of the fourth degree, which have such a cuspidal point, and it will still be uncertain which among these can be considered the simplest. If we wish to consider only the equation between the coordinates in making this judgement, the decision will be no more difficult, once we consider the general equation of the fourth degree, which includes all curves of this type. From this, it will be easy to see that in the fifth and higher degrees, the number of such curves infinitely greater still, so that we will no longer have the slightest doubt about the exisitence of this cuspidal point of the second kind of Mr. le Marquis de l'Hôpital.

21. This last formula,

$$y = \alpha\sqrt{x} + \beta\sqrt{x\sqrt{x}},$$

differs from the others only insofar as the axis AP is perpendicular to the curve, as opposed to the previous equations, where it is, in fact, tangent. For one need only exchange y and x to have

$$x = \sqrt{y} + \sqrt{y\sqrt{y}},$$

and, taking squares of both sides, one has

$$xx = y + 2y\sqrt[4]{y} + y\sqrt{y}.$$

So if x and y are infinitely small, we may ignore the last term, and letting $y = xx$, the term $y\sqrt[4]{y}$ with this substitution becomes $xx\sqrt{x}$, so that

$$xx = y + xx\sqrt{x} \quad \text{or} \quad y = xx \pm xx\sqrt{x},$$

which is one of the preceding equations. Similarly, in the general equation (§17)

$$y = \alpha x^k + \beta x^{k+\frac{m}{n}},$$

changing the coordinates x and y, we have

$$x = y^k + y^{k+\frac{m}{n}}.$$

Now, as x and y are taken to be infinitely small, one has

$$y^k = x \quad \text{and} \quad y = x^{\frac{1}{k}},$$

and so

$$x = y^k + x^{1+\frac{m}{nk}} \quad \text{and} \quad y = \sqrt[k]{x - x^{1+\frac{m}{nk}}} = x^{\frac{1}{k}} - \frac{1}{k}x^{\frac{m+n}{nk}}$$

or

$$y = \alpha x^{\frac{1}{k}} + \beta x^{\frac{m+n}{nk}},$$

the equation of a curve which has a cuspidal point of the second kind at the origin, perpendicular to the axis.

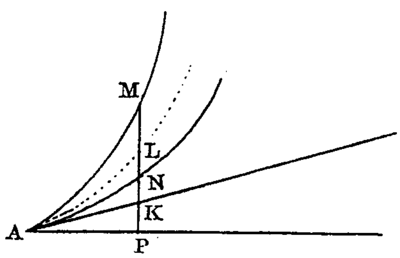

Fig. 7

22. From these formulas, it is easy to draw others, which have cuspidal points of the second kind, whose tangents make an oblique angle with the axis AP. To begin with the simplest cases, consider the equation

$$y = \alpha x + \beta xx + \gamma xx\sqrt{x}.$$

One sees, first of all, the the line AK (Fig. 7) is tangent to this curve at the point A, where $AP = x$ and $PK = \alpha x$. Next, if one constructs the parabola AL such that $PL = \alpha x + \beta xx$, and one takes

$$LM = LN = \gamma xx\sqrt{x},$$

one obtains two branches AM and AN, which meet at A, which have there the line AK as tangent, and whose curvature is the same as the parabola AL, so that the convexity of both branches at A turn towards the axis AP. Now, if the coefficient β is negative, the equation

$$y = \alpha x - \beta xx + \gamma xx\sqrt{x}.$$

will represent a cuspidal point MAN (Fig. 8), whose concavity will turn towards the axis AP.

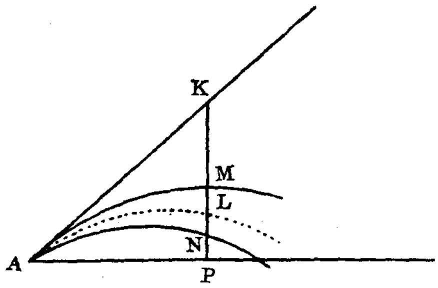

Fig. 8

23. This formula may be rendered more general in the following manner:

$$y = \alpha x + \beta xx + \gamma x^{2+\frac{m}{n}},$$

taking m to be an odd number and n and even number. One may further add as many other powers of x as one wishes, as long as their exponents are greater than 2, and in all these cases the radius of the evolute at the cuspidal point will be finite. Now it will be infinitely large in the equation

$$y = \alpha x + \beta x^k + \gamma x^{k+\frac{m}{n}},$$

if the exponent k is greater than two, and infinitely small if k is less than 2, and yet greater than 1. We remark that k may even be fractional, as long as the power x^k does not become ambiguous.

24. To render these formulas more general still, let P be an arbitrary funtion of x, which has no ambiguity, and which becomes $= 0$ in taking $x = 0$. Further, let Q be an ambiguous function, which takes both a positive and a negative value, or which is $\pm Q$, but which becomes imaginary when x is negative, or at least when such x is infinitely small, and, further, that Q vanishes when $x = 0$. Given all this, it is clear that the equation

$$y = \alpha x + xP(1 \pm Q)$$

designates a curve which has a cuspidal point of the second kind at the origin. This equation may be made even more general by taking

$$y = \alpha x + xP(1 \pm Q)^n$$

where n is any number, either postive or negative. The function P may even be ambiguous, as long as its ambiguity be destroyed by the function Q, which is the case if it includes the term $\sqrt{P}$, since then P may not take on a negative value.

25. One may render this formula yet more general by adding functions of y to y, which vanish with repect to Px if one takes x to be infinitely small, but I will not go that far, for this formula already suffices for finding an infinity of such curved lines of all orders above the third. I wil remark only that in taking

$$n = -1,\ P = x,\ Q = \sqrt{x},$$

one obtains yet another curve of the fourth degree:

$$y = \frac{\alpha xx}{1 \pm \beta\sqrt{x}},$$

which reduces to the equation

$$(y - \alpha xx)^2 = \beta\beta yyx,$$

which appears to be simpler that the preceeding. Changing the constants for the sake of uniformity, this becomes

$$aayy - 2ayxx + x^4 = bxyy.$$

26. By means of these formulas, it would be easy enough to furnish an infinity of curved lines, which have a cuspidal point of the second kind. But the same cannot be said for the inverse question: that of judging whether or not, given the equation of a curved line, that curve has such a cuspidal point. To find an ordinary cuspidal point, the custom is to seek a point on the curve where the second differential of one of the other of the coordinates either vanishes or becomes infinite, and at such location, the radius of curvature is either infinite, or equal to zero. But with the cuspidal point of the second kind, the radius of curvature is sometimes finite, sometimes infinitely large, and sometimes infinitely small, so that there is no specific property of the second differential attached to such a point. Nor is there anything one can conclude from the third or higher order differentials, for one may always assign whatever values one wishes to the higher differentials at such a cuspidal point.

27. Seeing, therefore, no certain method involving the properties of differentials for discovering cuspidal points of this kind, one is obliged to content oneself with finitary analysis, or to base one's judgement on the equation of the curve in finite terms. First of all, it is evident that such a cuspidal point is a double point of the curve, and so one must begin by seeking all double points on the given curve, and then the determination of whether such a point is also a cuspidal point of the second kind will not be too difficult. Now to locate the double points, one may make use of differentials for, having differentiated the given equation between the coordinates x and y, one would set equal to zero separately, both the terms containing dx, as well as those containing dy, and the cases where both of these equations, as well as the equation of the curve itself, have the same roots, will indicate all of the double points to be found on the given curve.

28. Having discovered, by this method, all double points on the given curve, one may determine whether or not each one is a cuspidal point of the second kind, by the following method. One draws the axis as an arbitrary straight line through the double point, whih is the origin. Starting from here, one takes the abcissas, which are $= x$, and then the ordinates, which are $= y$, are taken either perpendicular, or inclined at some arbitrary angle to, the

axis. This being done, the equation of the curve between x and y will always be such that there will be neither a constant term, nor terms involving the first power of x and y, and so the equation which results will always take the following general form:

$$0 = \alpha x^2 + \beta xy + \gamma y^2 + \delta x^3 + \epsilon x^2 y + \zeta xy^2 + \eta y^3 + \theta x^4 + \iota x^3 y + \text{etc.}$$

For any time an equation is lacking the constant term and the terms in x and y, it is a certain indicator that the curve has a double point at the origin $x = 0$ and $y = 0$; that is, the intersection of two branches.

29. Now one supposes x and y to be infinitely small, and derives the equation

$$0 = \alpha xx + \beta xy + \gamma yy,$$

whose roots

$$y = -\frac{\beta x}{2\gamma} \pm x\sqrt{\frac{\beta\beta}{4\gamma\gamma} - \frac{\alpha}{\gamma}}$$

denote the tangents of the branches which intersect one another at the origin. Thus, if the double point is a cuspidal point, it is necessary that these two tangents be equal, which occurs if $\beta\beta = 4\alpha\gamma$. As a consequence, if it is not the case that $\beta\beta = 4\alpha\gamma$, or if the equation, when x and y are taken to be infinitely small, does not have two equal roots, this is a certain indicator that the double point is not a cuspidal point, neither of the first kind nor of the second kind. So, this leaves only case of two equal roots for the determination of whether or not there is a cuspidal point of the second kind.

30. But if the two tangents coincide, one may take this common tangent as the axis, and then the equation for the curve will have the form

$$0 = Ayy + Bx^3 + Cx^2 y + Dxy^2 + Ey^3 + Fx^4 + Gxy^3 + \text{etc.}$$

Now it must be determined whether the two branches turn their curvature in the same direction. For this to occur, it must be the case that if x be taken as infinitely small, then neglecting those terms that vanish with respect to the others, the value of y does not become ambiguous. That is to say, it must be the case that $y = \alpha xx$, or $y = \alpha x^3$, or $y = \alpha x^4$, etc. If $y = \alpha xx$, which is the case where the radius of the development is finite, then we have, in negelecting the powers larger than the fourth,

$$0 = \alpha^2 Ax^4 + Bx^3 + \alpha Cx^4 + Fx^4,$$

and thus it must be that $B = 0$ and that α does not take on a double value in the equation

$$\alpha^2 A + \alpha C + F = 0,$$

or that $CC = 4AF$, so that

$$Ayy + Cxxy + Fx^4$$

becomes a perfect square.

31. Therefore, as long as the equation does not have the form

$$(y - \alpha xx)^2 = Bxy^2 + Cy^3 + Dx^3y + Exxyy + Fxy^3 + \text{etc.},$$

the curve will not have a cuspidal point of the second kind. But this does not yet suffice; in addition, it is necessary that the lowest power of x in the expression $Bxy^2 + Cy^3 +$ etc. be odd, when one puts $y = \alpha xx$. From this, one sees easily enough that if the equation be

$$(y - \alpha x^3)^2 = Bxy^2 + \text{etc.}$$

this part must still have the same property, that after having set $y = \alpha x^3$, the lowest power of x be odd and larger than x^6, and this same maxim must be be observed in more complicated equations, so that the determination of the cuspidal point of the second kind will never be difficult when proceeding in this way.

32. From this we may give a general equation for all fourth degree lines which have a cuspidal point of the second kind, which is

$$(y - \alpha xx)^2 = Axy^2 + By^3 + Cx^3y + Dxxyy + Exy^3 + Fy^4.$$

For at the origin, where $y = \alpha xx$, the second member $Axy^2 + By^3 + Cx^3y +$etc. will become equal to

$$\alpha^2 Ax^5 + \alpha^3 Bx^6 + \alpha Cx^5 + \alpha^2 Dx^6 + \alpha^3 Ex^7 + \alpha^4 Fx^8,$$

and from there, by neglecting the higher powers of x, the equation at the origin will be

$$(y - \alpha xx)^2 = \alpha(\alpha A + C)x^5 \quad \text{or} \quad y = \alpha xx \pm xx\sqrt{\alpha(\alpha A + C)x}.$$

Thus, as long as $\alpha A + C$ is not equal to zero, the lines described by the given general equation will have a cuspidal point of the second kind. As a consequence, this property will hold any time it is not the case that $C = -\alpha A$, and so the cuspidal point will be of the type

$$y = \alpha xx + \beta xx\sqrt{x}.$$

33. However, when $C = -\alpha A$, there will also be the cases, where the equation

$$(y - \alpha xx)^2 = Axy(y - \alpha xx) + By^3 + Dxxyy + Exy^3 + Fy^4$$

contains curves endowed with such cuspidal points. To find this case, one need only assume that, x being infinitely small,

$$y = \alpha xx + \beta x^3 + \gamma x^3\sqrt{x}$$

and, after having determined β and γ, one arrives at the equation

$$(y - \alpha xx - \frac{1}{2}Axy)^2 = Byy(y - \alpha xx) + Exy^3 + Fy^4$$

which once again includes an infinity of curves, all of which have such a cuspidal point. Thus, with regards to the simplicity of the equation, there is no doubt that the line

$$(y - \alpha xx)^2 = Axxyy \quad \text{or} \quad y = \frac{\alpha xx}{1 - \sqrt{A}x}$$

is the simplest that has this property.

34. In order to give an example of the method I have just explained, let the following line of the fifth degree be proposed, whose equation is:

$$\left(1 - \frac{x}{a}\right)(aa - zz)^2 = 2aaxx - x^4 - 2xxzz.$$

To determine whether this curve has a cuspidal point of the second kind, I first determine whether it has a double point. To this end, the given equation gives, upon differentiation, in supposing x constant

$$-4\left(1 - \frac{x}{a}\right)(aa - zz)zdz + 4xxzdz = 0.$$

Now, in supposing z constant, we have

$$-\frac{dx}{a}(aa - zz)^2 = 4aaxdx - 4x^3dx - 4xzzdx.$$

These three equations are all satisfied by the values

$$x = 0 \quad \text{and} \quad z = \pm a.$$

35. The curve therefore has two double points, and to determine the nature of each, I let $z = a - y$, for it is clear from the given equation that, as it has a diameter on which the abcissas x are taken, these two double points are similar. Thus, in taking $z = a - y$, we have the equation

$$\left(1 - \frac{x}{a}\right) yy(2a - y)^2 = 2aaxx - x^4 - 2xx(a - y)^2,$$

or rather

$$\begin{gathered} 4aayy - 4ay^3 + y^4 - 4axyy \\ +4xy^3 - \frac{xy^4}{a} - 2aaxx + x^4 - 4axxy + 2xxyy + 2aaxx = 0 \end{gathered}$$

and so

$$(2ay - xx)^2 = 4ay^3 + 4axyy - y^4 - 4xy^3 - 2xxyy + \frac{xy^4}{a}.$$

From this it is clear that the curve has two cuspidal points of the second kind.

36. To understand the shape the this curve, one need only find the value of zz in the given equation, which one finds to be

$$zz = aa - \frac{axx \pm x^2\sqrt{ax}}{a - x}$$

or

$$zz = aa - \frac{xx\sqrt{a}}{\sqrt{a} \mp \sqrt{x}}.$$

From this, one finds the places where the ordinate z vanishes, by means of the equation

$$aa\sqrt{a} \mp aa\sqrt{x} - xx\sqrt{a} = 0 \quad \text{or} \quad (aa - xx)^2 = a^3x,$$

which has two real root, which are:

$$x = 0.5248876a \quad \text{and} \quad x = 1.4902162a.$$

Further, the value of dz will be zero in the case

$$\sqrt{x} = \frac{4}{3}\sqrt{a} \quad \text{or} \quad x = \frac{16}{9}a = 1.7777a,$$

while the value of dx will be $= 0$ when $z = 0$, i.e. where the curve meets the axis. But if we let $x = \frac{16}{9}a$, we will have either

$$zz = \frac{283}{27}aa \quad \text{or} \quad zz = -\frac{67}{189}aa,$$

of which the second value is imaginary. Finally, one observes that the curve also has an asymptote perpendicular to the axis, where $x = a$.

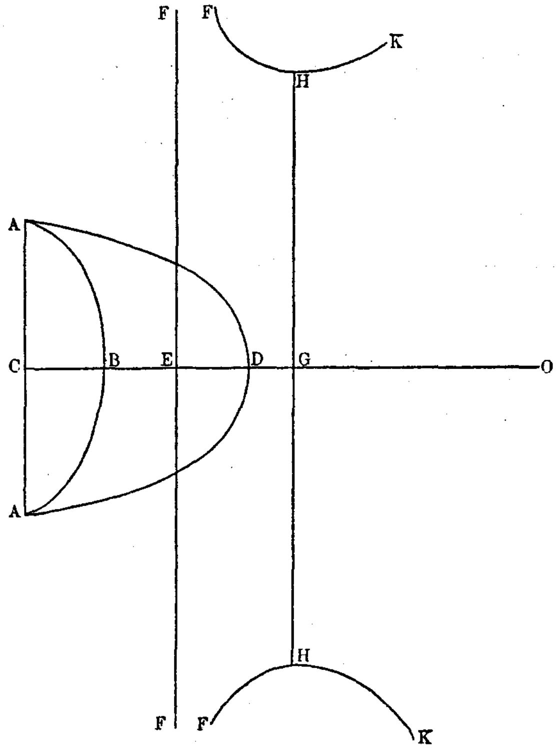

Fig. 9

37. This curve is represented in Figure 9, where CO is the axis, and at the same time the diameter of the curve, and C is the origin of the abcissas x. The points A, A are the two cuspidal points of the second kind, which

we have just found, with $AC = a$. The two branches which give rise to these cuspidal points make the lune $ABADA$. In addition, letting $CE = a$, the straight line FEF, perpendicular to the axis CO, will be the asymptote of the two other branches FHK and FHK which, from K, K diverge to infinity, and having at H, H, their closest approach to the axis CO. One also sees that one may draw a straight line, which cuts the curve at 5 points, as is the nature of lines of the fifth order.