On an Apparent Contradiction in the Doctrine of Curved Lines*

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I. We generally believe that Geometry distinguishes itself from the other sciences because all that one proposes about it is founded on the most rigorous demonstrations and that nothing is found there which can give rise to controversy. In effect, since in Geometry one allows only those propositions which are perfectly demonstrated, one will not be able to understand how one can fall into dispute. It is still less possible, that two solidly proven propositions are in contradiction with one another, since truths, far from being contradictory, are always found in the most perfect harmony with one another. And although it frequently happens in the other sciences, that two truths appear to contradict, we are nevertheless well assured that this is only an apparent contradiction, which for the most part draws its origin from less precise ideas, or from a lack of sufficient understanding of things, which we should be taking into consideration. By this reasoning one will be all the more led to believe that in Geometry, such apparent contradictions can not occur, since we are far from contenting ourselves with ill-determined ideas.

II. Nevertheless, I am going to present two propositions from Geometry both of which are rigorously demonstrated, and which appear to lead to an overt contradiction. This difficulty is encountered in the doctrine of curved lines, where for curves of a certain order one knows how many points are needed in order to determine it. Thus, a line of the third order can be described by 9 given points, or 9 points determine such a line of the third order, and that there is but one that can be drawn from these 9 points. But it

*Leonhard Euler, *Sur une contradiction apparente dans la doctrine des lignes courbes*, in *Opera Omnia*, vol. I.26, p. 33 R45 originally in *Mémoires de l'académie des sciences de Berlin* **4** (1748), 1750, p. 219-233. This article is numbered E147 in Eneström's index of Euler's work.

is also proven that two lines of the third order can intersect at 9 points; thus it may occur that two lines of the third order pass through 9 given points, from which it follows that 9 points are not sufficient in order to create a line of the third order, this being contrary to that which was already established. Before explaining and developing this apparent contradiction, it will be convenient to put everything in order, to better understand its importance. To this end I will start with an analysis of these two propositions, which seem to include this contradiction.

III. As a line of the first order, or a straight line, can be drawn through any two given points, so a line of the second order, or a conic section, can be drawn with 5 points, a line of the third order by 9 points, a line of the fourth order with 14 points and in general, a line whose order is indicated by n, can be drawn by $\frac{nn+3n}{2}$ points. Because the general equation for lines of this order:

$$Ay^n+(B+Cx)y^{n-1}+(D+Ex+Fx^2)y^{n-2}+(G+Hx+Ix^2+K^3)y^{n-3}+\text{ \&c. } = 0$$

contains $\frac{nn+3n}{2}+1$ arbitrary coefficients A, B, C, D, etc. Now each given point, through which the curved line must pass, provides given values for the coordinates x and y which, being substituted, will give an equation. Thus $\frac{nn+3n}{2}$ given points lead to such equations, from which all the coefficients A, B, C, etc., are determined, and by consequence the curve itself. For although the number of coefficients is greater by one than the number of equations drawn from $\frac{nn+3n}{2}$ points, since here we are only concerned with mutual agreement amongst the coefficients, we need only $\frac{nn+3n}{2}$ equations in order to determine this agreement.

IV. Two straight lines, or lines of the first order, may intersect in only one point; two conic sections, or two lines of the second order, may intersect in only 4 points. Two lines of the third order can intersect in 9 points; two lines of the fourth order in 16 points, and so forth. And as a line of the order m can be cut, by a straight line in m points, by a line of the second order in $2m$ points, by a line of the third order in $3m$ points; thus one maintains in general that a line of the order m can be cut by a line of order n in mn points. This proposition must be understood as stating that the number of intersections of two curved lines, of which one is of order m and the other of order n can not be larger than mn, even though it is very often smaller: some points of intersection being either at infinity, or at imaginary values.

The demonstration of this proposition is not so simple, and I will speak of it in greater detail in the rest of this discourse.

V. The truth of these 2 propositions being recognized, I will first refer to the consequences which we draw from them which appear to be contradictory. Then I will expose the flaw which is found in these consequences, which consists of a very subtle error in reasoning, which being not so easily discovered, should render us extremely cautious, principally in the other sciences, lest we allow ourselves to deduce similar apparent contradictions. For, if in Geometry we are subjected to such remarkable difficulties, where it is certainly permissible to reduce all ideas to almost the greatest degree of precision, how much more could we be burdened with in the other sciences, where it is not possible to achieve sufficiently precise ideas and where it is infinitely more difficult to protect oneself against similar errors in reasoning? Finally I will bring to light the manner in which it is necessary to understand these two propositions, in applying a certain necessary restriction to them, which being noted, all the contradictions, however solid they might have appeared, will vanish all at once and we will perceive the most beautiful harmony between these two propositions.

VI. The first contradiction seems at first to occur in the properties of lines of the third order, depending on which of the two general propositions we are considering. Here are the two consequences that we draw:

I. *Since we need, according to the first proposition, nine points in order to determine one line of the third order; with nine given points one can draw only one line of the third order.*

II. *Now, according to the second proposition, two lines of the third order can intersect at nine points. Thus, one will be able to designate nine points, through which two lines of the third order may pass.*

These two consequences overtly contradict each other; since in the first consequence one maintains, that with 9 points being given, one will be able to draw only one line of the third order which passes through each of these 9 points. Nevertheless the second consequence shows us that there are an infinite number of cases, where two curves may be made to pass through 9 given points.

VII. The contradiction becomes even more obvious in lines of the fourth and higher orders. For the fourth order the contradictory consequences are:

I. *According to the first proposition, 14 points are required in order to determine a line of the fourth order. Hence 14 points being given, one will be able to draw only one line of this order, which passes through all of these points.*

II. *However, the second proposition tells us, that two lines of the fourth order intersect each other in 16 points. As a consequence, in this case it will be possible to make two lines of the fourth order pass through 16 given points.*

The contradiction between these two consequences is evident, since if it is possible to describe only one line of the fourth order with 14 given points; it is all the more incomprehensible how one could draw 2 lines of this order, which intersect each other at 16 points.

VIII. For the lines of the fifth order, our two general propositions provide us with two consequences even more self-contradictory:

I. *The first proposition tells us that 20 points suffices in order to determine a line of the fifth order, it follows that with 20 given points we only know how to draw one single line of the fifth order.*

II. *Now the second proposition assures us, that two lines of the fifth order may intersect each other at 25 points. Therefore it will be possible to give 25 points, through which we may draw two lines of the fifth order.*

As a consequence of these cases, where 25 given points are not sufficient in order to determine a single line of the fifth order, and yet the first consequence seems to persuade us that only 20 points are required in order to determine a line of the fifth order. And it is clear that in the curved lines of higher order, the difference between the number of points, which ought to suffice for their determination, becomes even greater.

IX. These contradictions being completely evident, it is absolutely necessary, either that one of the two general propositions be false, or that the consequences, which we have drawn from them, are not justified. Although I know of no sufficiently rigorous proof of the second proposition, we can be quite certain of its truth, as I will illustrate here; and the consequences, which have been drawn here, are so clear that there remains not the smallest doubt on this side of the question. Since if for example, two lines of the fourth order intersect at 16 points, one must absolutely agree, that it is possible to give 16 points, through which not only one, but two lines of the fourth order may pass. Thus, since the second proposition, as well as the consequences which we have drawn from it, are entirely confirmed, we are obliged to conclude that we must seek the fallacy in the first proposition, or in the manner which one draws the consequences from it marked number (I), where we must search for some fallacy.

X. The consequences that we have drawn from the first proposition, are equally well justified: since if 9 points determine a line of the third order, it follows that given 9 points one may draw only one line of this order: just as with two given points one can draw only one straight line, and with five given points only one conic section. Therefore, if the general equation for lines of order n is determined by $\frac{nn+3n}{2}$ points, through which the line must pass, it will not be possible that more than one line of this order passes through as many points, as given by the formula $\frac{nn+3n}{2}$. It is therefore not in the consequences that we have just deduced from the first proposition, that we must seek the source of these contradictions. Therefore, all that remains is to suspect the first proposition itself as the source of the fallacy, however well-founded it previously seemed.

XI. In fact, after carefully considering the first proposition, we will observe that there may be some cases where $\frac{nn+3n}{2}$ given points are not sufficient to determine a curve of order n, which can be drawn through these points. Or, what amounts to the same thing, as $\frac{nn+3n}{2}$ equations do not suffice for the determination of as many coefficients, or for determining the relationship among $\frac{nn+3n}{2}+1$ coefficients even though these coefficients appear in each of the equations and occupy but one single dimension. Moreover, this is the simplest case, where several unknowns must be determined by as many equations, since by the successive elimination of these unknown quantities, one still remains at the first degree, so that we never find more than one value for each unknown, which consequently will always be real. These circumstances appear to confirm the truth of the 1st proposition, and to free it of all exceptions.

XII. However, we will no longer doubt, that we must apply a certain restriction to the first proposition, without which it would not be true in general, as soon as one considers the following remarks. First of all, to begin with the simplest of cases, I say that it may occur that two equations are insufficient to determine the values of two unknowns, even though both appear in each of these two equations, and yet determine only a single dimension. For one need only to consider the two equations $3x - 2y = 5$ and $4y = 6x - 10$ and one will see at first, that it is not possible to determine here the two unknowns x and y: for in eliminating the variable x, the other variable disappears, and one obtains the equation of an identity, from which one can determine nothing. The reason for this occurrence now becomes clear, since the second equation can be written as $6x - 4y = 10$, which being nothing other than the first, $3x - 2y = 5$ doubled, does not differ from it. This is why, when one says that in order to determine two unknown quantities, it is enough to have two equations, it is necessary to add this restriction to this proposition: that these two equations are different from one another, or that the one is not already present in the other and it is only with this restriction, that the aforementioned proposition is admissible.

XIII. Secondly, one will easily realize, that three equations may not suffice to determine three unknown quantities. For if it should happen as in the previous example, one of the three equations is present in one of the other two, in which case the three equations amount only to two; it will then be impossible to determine the three unknowns. For instance, if one had these three equations:

$$\begin{aligned} 4x - 6y + 10z &= 16 \\ 3x - 5y + 7z &= 9 \\ 2x - 3y + 5z &= 8 \end{aligned}$$

it is clear that the first, not differing from the third, contributes nothing to the determination of the three unknowns x, y and z. But there are also some cases, where one of the three equations is conjointly contained in the other two, for example if one had these three equations:

$$\begin{aligned} 2x - 3y + 5z &= 8 \\ 3x - 5y + 7z &= 9 \\ x - y + 3z &= 7 \end{aligned}$$

where the sum of the second and the third is the double of the first. In this case one can omit any of these three equations that one wishes, and it is the same as though there were only two equations. Thus when one says that in order to determine three unknowns, it is enough to have three equations, it is necessary to add the restriction, that these three equations differ such that none are already present in the others.

XIV. It is the same with four equations, which do not suffice to determine four unknown quantities, except in the case that they are all different from each other, or that none is already contained in the others. For if one is already contained in the other three, these four equations must be regarded as if there were only three. It may even occur that two equations are already contained in the two others, and thus there will be only two equations, which remain in the calculation, and by consequence two unknowns will remain undetermined. For instance if one were to come across these four equations:

$$\begin{aligned} 5x + 7y - 4z + 3v - 24 &= 0 \\ 2x - 3y + 5z - 6v - 20 &= 0 \\ x + 13y - 14z + 15v + 16 &= 0 \\ 3x + 10y - 9z + 9v - 4 &= 0 \end{aligned}$$

they amount to only two. For having drawn from the third the value of $x = -13y + 14z - 15v - 16$, and having substituted it into the second to have:

$$y = \frac{33z - 36v - 52}{29} \quad \& \quad x = -\frac{23z + 33v + 212}{29}$$

these two values of x and y being substituted into the first and fourth equations will lead to identities, so that the quantities z and v will remain undetermined.

XV. The same circumstance can take place in any number of equations that one wishes, the reason being that although one could have as many equations as there are unknowns, they may not be sufficient to determine all of them. For one of the unknown quantities will remain undetermined, if one of the given equations is contained in the others. In addition, two or more unknown quantities will remain undetermined if there are two or three amongst the equations, which are already contained in the others, and which consequently contribute nothing to the determination of the unknowns. This is why when one maintains, that in order to determine n unknown quantities, it is enough to have n equations, which explain their mutual relationship, it is necessary to make the following restriction: that all the equations are different from each other, or that there are none which are already contained in the others.

XVI. After these remarks one will easily acknowledge that the number of points which according to the first proposition suffice for the determination of the curve of a certain degree, may in certain cases be insufficient, since this determination reduces to the determination of a certain number of coefficients by as many equations, which, as we have just seen, may sometimes be insufficient for this purpose. In order to understand these cases, I will first consider the general equation for the straight line of the first order:

$$\alpha x + \beta y + \gamma = 0,$$

and being given two points, through which a line of this order must pass. Having chosen an arbitrary line for the axis, let a and b be the coordinates of the first point and c and d, for the other. That is, for the first point one will have $x = a$ and $y = b$ and for the other $x = c$ and $y = d$, from which we can draw the two equations:

$$\begin{aligned} \alpha b + \beta a + \gamma &= 0 \quad \text{and,} \\ \alpha d + \beta c + \gamma &= 0. \end{aligned}$$

The difference of these gives us:

$$\frac{\alpha}{\beta} = \frac{c-a}{b-d}.$$

The relationship therefore between α and β will always be determined, as long as it is not the case that $c = a$ and $d = b$, in which case the two points coincide with one another. Therefore two points determine a straight line, provided that they are not coincident. This restriction extends to the description of all the other curved lines, for two points which coincide, are considered as only one single point.

XVII. The general equation for curves of the second degree being:

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \zeta = 0,$$

for which five points are given, through which one must describe a curve of the second order. In order to simplify this calculation, let us take a straight line, which passes through two of these points, as the axis. Another line drawn through one of these two points and a third represents the inclination of the ordinates, since it does not matter whether they are perpendicular to the axis or not. Given this, let the values of x and y for these five given points be:

		I	II	III	IV	V
x	$=$	0	a	0	c	e
y	$=$	0	0	b	d	f

From this we will have the five following equations:

$$\begin{array}{lrcl}
\text{(I)} & \zeta & = & 0 \\
\text{(II)} & \alpha a^2 + \delta a + \zeta & = & 0 \\
\text{(III)} & \gamma b^2 + \varepsilon b + \zeta & = & 0 \\
\text{(IV)} & \alpha c^2 + \beta cd + \gamma d^2 + \delta c + \varepsilon d + \zeta & = & 0 \\
\text{(V)} & \alpha e^2 + \beta ef + \gamma f^2 + \delta e + \varepsilon f + \zeta & = & 0
\end{array}$$

From the first three we have the following:

$$\zeta = 0; \ \delta = -\alpha a; \ \ \& \ \ \varepsilon = -\gamma b,$$

these values being substituted in the last two give these equations

$$\begin{array}{rcl}
\alpha c^2 + \beta cd + \gamma d^2 - \alpha ac - \gamma bd & = & 0 \\
\alpha e^2 + \beta ef + \gamma f^2 - \alpha ae - \gamma bf & = & 0
\end{array}$$

which will determine the desired curve, as long as they are not equivalent. Now this case will occur, when the two values of β which we derive from them are equal

$$\beta = \frac{\alpha c(a-c) + \gamma d(b-d)}{cd} = \frac{\alpha e(a-e) + \gamma f(b-f)}{ef}$$

that is, if

$$\frac{a-c}{d} = \frac{a-e}{f} \ \ \& \ \ \frac{b-d}{c} = \frac{b-f}{e},$$

or if

$$a = \frac{cf-de}{f-d} \ \ \& \ \ b = \frac{de-cf}{e-c}.$$

We see here, that the five given points can be so disposed, that a curve of the second order is not determined from them, and since a coefficient remains undetermined, it follows that one may make an infinity of curves of the second order pass through these five given points.

XVIII. If we consider this case more closely, where the five given points may be insufficient for determining the curve of the second order, we will easily notice that this occurs, when four of the five given points lie in a straight line. This will become clear enough, if from the equations

$$\frac{a-c}{d}=\frac{a-e}{f} \quad \& \quad \frac{b-d}{c}=\frac{b-f}{e}$$

we draw the values

$$f=\frac{d(a-e)}{a-c}=b-\frac{e(b-d)}{c}$$

which give $(c-e)(ad-ab+bc)=0$. When $e=c$, we also have $f=d$ and two points will coincide; thus the case will disappear by itself. Therefore, let $ad-ab+bc=0$ or

$$d=b-\frac{bc}{a}$$

and one will find that

$$f=b-\frac{be}{c}.$$

We need only to consider these formulas geometrically, in order to assure ourselves that four points are situated in a straight line. It would not have been difficult to guess this case, since curves of the second order also include the case of two straight lines situated in any arbitrary manner, we recognize that if three of the five given points are in a straight line, this line will be part of a curve of the second order, and the other line conjoined to it will be determined by the two other points. Now, if four points lie on a straight line, this will be part of the desired curve, but the other part, or the other straight line, will pass through the fifth point and, having no other determination, could be drawn arbitrarily. If all five given points were situated in a straight line, this same straight line with any other arbitrary line will satisfy the question; therefore, this case will be even less determined than the previous one.

XIX. The general equation of curves of the third order being:

$$\alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3+\varepsilon x^2+\zeta xy+\eta y^2+\theta x+\iota y+\kappa=0,$$

it is necessary to define nine coefficients in terms of the tenth, so that the equation or the curve of this order be determined. Thus if nine points are given, through which this curve must pass, each supplying an equation, the curve will be determined, as long as these nine equations are all different from each other, and there are none which are already contained in the others. Now one will easily understand, that there is an infinity of cases, where not only one, but also two, or many, of the nine equations may already be contained in the others: and by consequence the curve in this case will not be determined by the nine given points, but one will again add here a tenth, and even an eleventh, or a twelfth, so that the determination becomes complete. It is however rather difficult to define these general cases, as I did for curves of the second order, since the calculation becomes too complicated due to the large number of points and coefficients. Nevertheless it is not difficult to discover many particular cases, where this defect in the determination will take place. From these, one will easily conclude that the number of such cases may be infinite, which is sufficient for my purposes.

XX. One knows that the general equation of the third order also includes, besides the curves of this order, either three straight lines, or one straight line joined to a conic section. Thus, if four of the nine given points are arranged in a straight line, the straight line drawn by these four points will constitute a part of the desired figure, since curves of this order do not contain four points in a straight line, and the other five points will determine the other part, which will be either one conic section, or two straight lines; thus a lack of determination exists, as we already have observed in the preceding case. But let us suppose that 5 points are arranged in a straight line, which will constitute a part of the figure, and it is clear that the other 4 points are not sufficient for determining the other part. In this case therefore we will need ten points in order to determine the figure: now if there are 6 of the given points here arranged in a straight line, 5 more are required, and therefore 11 points in all in order to completely determine the question. Moreover from here it is evident that it may occur, that some great number of points is not sufficient for the determination of one curve of the third order, which must pass through all these points.

XXI. Since these cases can only occur in straight lines which are included in the general equation of the curve of the third order, one may doubt if the same thing may occur in curves of the third order, or if nine points may be insufficient for determining a curve of this order. To prove that this is so, I will consider only one case, where the nine given points a, b, c, d, e, f, g, h, i are arranged in a square:.

a .	b .	c .
d .	e .	f .
g .	h .	i .

Let the axis be drawn through the points d, e,f, and let the point e be the origin. Denoting the distance between two points $= a$, we will have corresponding to the abscissa $x = 0$ three values of the ordinate y, which are 0, $+a$, and $-a$, and these same three values will also correspond to the abscissa $x = a$, and to $x = -a$. The following equation will correspond to these values:

$$my(yy - aa) = nx(xx - aa),$$

where the relationship between the coefficients m and n may be arbitrary, so that an infinite number of curves of the third order may be designated, which all pass through these given points.

XXII. It is certainly true that this equation also includes straight lines, and conic sections: because if $n = 0$, one will have three straight lines ac, df, and gi. If $m = 0$, one will have three straight lines ag, bh, and ci. If $m = n$, one will have a straight line aei, and an ellipse drawn through the points b, c, f, d, g, h. If $m = -n$, the curve of the third order will be composed of a straight line ceg and of an ellipse which passes through the points a, b, d, f, h, i. But in all other relationships that one poses between m and n, one will always have a true curved line, which passes through the nine given points. And from this one will easily understand that every time that two curves of the third order intersect at 9 points, these points will be such that they do not completely determine a curve of the third order, and that in the general equation, after one had applied it to these nine points, a coefficient will remain undetermined. In these cases, therefore, there will be not only two curves of the third order, but an infinity of curves of this order, which can all be described by these nine points.

XXIII. When two lines of the fourth order intersect at 16 points, since 14 points, which lead to equations different from one another, are sufficient for determining a curve of this order, these 16 points will be always such that three or more of the equations, which result from them, are already contained in the others. So that these 16 points do not determine any more than if there had been 13 points, or 12, or even fewer, and hence in order to determine the entire curve, one could sill add one or two points to these 16. The same thing will occur if two curves of the fifth order intersect one another at 25 points which, not being sufficient to determine the curve, are worth no more than 19, or even 18, so that 6 or 7 points are superfluous: and hence these 25 points are always so disposed, that as soon as the curve passes through 19 of these points, it will automatically pass through the others, or it will be impossible that it passes through 19 points, without passing through all 25 points at the same time. These reflections being well formed, one will easily resolve all the other difficulties, which could come from the comparison of the two general propositions, that I reported in the beginning of this discussion.