Trigonometry

Don Crossfield
Roseburg High School
Roseburg, Oregon

Charlyn Shepherd
The Paideia School
Atlanta, Georgia

Robert Stein
California State University
San Bernardino, California

Grace Williams
Maryknoll High School
Honolulu, Hawaii

Historical Modules Project
Sponsored by the Mathematical Association of America
Funded by the National Science Foundation

Table of Contents

Introduction

This module was written with those teachers and students in mind who are engaged in trigonometric ideas in courses ranging from geometry and second-year algebra to trigonometry and pre-calculus. The lessons contain historical and cultural context, as well as developing traditional concepts and skills. They are to be used alongside the class' regular text.

For the convenience of busy teachers, individual lessons are in reproducible form for the classroom and are accompanied by a teacher guide with a description of the unit, prerequisites, materials, and teacher notes. Each lesson is generally independent of the others while respecting the standard sequencing of trigonometry topics. The teacher guide suggests how these historical lessons could introduce, replace, complement, or extend the text lessons.

An introduction to angle ties Euclid's linear concept to circular arc measure, explains the transition from degree to radian measure, and raises the dynamic definition of angle as rotation. To calculate positions of heavenly bodies, the astronomer Ptolemy (c. 85-145 AD) created tables of circular chord lengths, which the students will compare to sine values. Indirect measurement of height and solar angles by shadow math, with hands-on activities, develops the tangent ratio. Students will formulate a basic tangent table.

After defining sine, cosine, and tangent, students can proceed from circle geometry to trace through Ptolemy's calculation of his chord table (with some beautiful pentagon properties along the way). In the process, they will derive the identities for sine and cosine of the sum and difference of angles, and the sine of a half angle. Then the students can develop the Law of Sines from an inscribed triangle according to Brahmagupta (628), the Law of Cosines from Euclid's *Elements*, and Heron's Formula from a theorem of Brahmagupta.

Historical determinations of astronomical distances and sizes, as well as solving right and general triangles, introduce some authentic purposes why past cultures studied trigonometric ratios and the Laws of Sines and Cosines. Likewise, the unit on trigonometric identities establishes the historical importance of elementary identities, sum-to-product, product-to-sum, double angle, half angle, and other identities, including a glimpse at how the Hindu astronomers made a sine table.

The final lesson, on spherical trigonometry, is extensive because high school texts usually omit it. Its historical development, the connections between plane and spherical systems, essential formulas, proofs, and exercises are rounded out with the intriguing problem of *qibla* (in what direction is Mecca).

The very last section, for both reference and reading enjoyment, contains brief biographies and quotations of key mathematicians mentioned in various places throughout the book; derivation of terminology (such as function names); a timeline of important developments in the history of trigonometry; and a bibliography of print and on-line materials.

A guiding principle in creating these historical trigonometry lessons was to forego merely repeating what could be found in the usual classroom text. Also, treatment of trigonometry was not intended to be comprehensive. Navigation and surveying for instance - areas where trigonometry has vital application - are only touched upon. They are complete sciences in themselves whose literature is readily accessible to both teacher and student. . Lastly, in order to maintain the trigonometry focus, many threads of intriguing context, such as the events, politics, art, and personal lives of the famous and everyman of the times, could not be developed more fully.

Rationale

This set of lessons grew from a desire to enhance the teaching and learning of trigonometry by connecting it to people who developed it, and when, where, and why they did so. Further motivations for the historical approach were to clarify the beginnings of trigonometry in the mathematics needed for astronomy, to credit people in other cultures and in the past for their solutions of important mathematical problems, and to trace the development of the trigonometric functions and related terminology. Astronomy, studying the heavens, earthly measuring, surveying, navigation, and formal abstraction (developing generalizations) will benefit students looking toward space age careers as they improve their understanding of the earth and its place in the universe.

All students need to see how mathematics is a human activity not only in its application but in its making, to trace through how it is used as well as how people reason in order to arrive at its validity. In general, a historical approach to trigonometry can excite inquiry into mathematics and history as part of the process of discovery and integrated understanding.

Principles and Standards 2000 published by the National Council of Teachers of Mathematics underscores the wide range of career opportunities provided by a solid foundation in trigonometry. In particular, "Carpenters apply the principles of trigonometry in their work, as do surveyors, navigators, and architects." (p. 288) Furthermore, in its vision for school mathematics, NCTM states that our rapidly changing world needs more than ever "*Mathematics as a part of cultural heritage*. Mathematics is one of the greatest cultural and intellectual achievements of humankind, and citizens should develop an appreciation and understanding of that achievement."

The Standards for Grades 9-12 specifically addressed by this module are:

Algebra Standard: (p. 296)

- "Understand patterns, relations, and functions", including the class of periodic functions."

Geometry Standard: (p. 308)

- "Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships." Specifically "use trigonometric relationships to determine lengths and angle measures."
- "Use ... other coordinate systems, such as ... spherical."

The description of the Geometry Standard includes:

- "Properties of ... trigonometric relationships ... give students additional resources to solve mathematical problems." (p. 309)
- "Right-triangle trigonometry is useful in solving a range of practical problems." (p. 313)
- An example of the problem positioning lights for maximum illumination "draws on students' knowledge of geometric and trigonometric relationships." (p. 317)

Measurement Standard: (p. 320)

- "Understand ... units, systems, and processes of measurement."
- "Apply appropriate techniques, tools, and formulas to determine measurements", including "understand and use formulas for the area."

The description of the Measurement Standard includes:

- Students should extend their "facility with derived measures and indirect measurement."

Problem Solving Standard: (p. 334)

- "Build new mathematical knowledge through problem solving."
- "Solve problems that arise in mathematics and in other contexts."
- "Apply and adapt a variety of appropriate strategies to solve problems."
- "Monitor and reflect on the process of mathematical problem solving."

Reasoning and Proof Standard: (p. 342)

- "Recognize reasoning and proof as fundamental aspects of mathematics."
- "Develop and evaluate mathematical arguments and proofs."

Connections Standard: (p. 354)

- "Recognize and use connections among mathematical ideas."
- "Understand how mathematical ideas interconnect and build on one another to produce a coherent whole."
- "Recognize and apply mathematics in contexts outside of mathematics."

Representation Standard: (p. 360)

The description of the Representation Standard includes:

- Students "should recognize, for example, that phenomena with periodic features often are best modeled by trigonometric functions."

Historical and Cultural Overview

Trigonometry began outdoors in the open air, over two thousand years ago, when scholars studied the stars and planets moving across the night sky, and the shadows' changing lengths on sunny days. Contrary to the flavor of modern textbook exercises, trigonometry did not start with the right triangle or angles of circular functions. Sine and cosine emerged from astronomers' search for pattern in movement and location of heavenly bodies. Astronomy motivated trigonometry until well into the fifteenth century. On the scale of an 8 a.m. to 3 p.m. school day, this was at 2 p.m. In our metaphor, the first published work-naming trigonometry appeared at 2:30. Tangent and cotangent meanwhile developed in calculating heights of objects from shadow lengths. Secant and cosecant evolved in navigation; astrolabe readings depended on shadows falling across scales. The six trigonometric functions in some form occurred together throughout the history of mathematics before modern terminology, decimal notation, and association of functions with angles.

What cultural forces overall drove trigonometry? Shadow lengths confirmed solstices and agricultural seasons. People surveyed the land, made maps, and navigated for travel and trade. Various cultures scheduled religious rites by eclipses and the sun's position. Heads of state desired heavenly auspices before ventures. In Islam, the moon fixes the calendar; the sun tells the fasting period and the five times and direction to face for daily prayer. Those who lived before us also responded to the large questions about the nature of the cosmos and our place in it.

The Greeks. The early Greeks produced theorems equivalent to modern trigonometric formulas. Euclid's proof of a form of the law of cosines (c. 300 BCE) resembled that of the Pythagorean Theorem. Theorems on chord lengths relate to the modern law of sines. Archimedes' (287-212 BCE) theorem on the broken chord relates to the sine of a sum or difference. In astronomy the Greeks, including Eudoxus (c. 408-335 BCE), had used Babylonian data. To them, the five planets, moon, sun, and stars moved in spheres centered on the fixed Earth. Hipparchus (c. 180-125 BCE) founded trigonometry, publishing 12 books with tables; given a circular arc, they find a subtended chord (that is, a sine) length. The European idea of sine as a ratio in a right triangle came in the sixteenth century. Hipparchus calculated the Earth's radius, the distance to heavenly bodies, and the sun's and moon's diameters. Menelaus of Alexandria (c. 100 CE) wrote the earliest treatises on spherical trigonometry, which developed along side of plane trigonometry.

Around 150 DE, Ptolemy produced his *Mathematical Synthesis*, which the Arab scholars centuries later would call *Almagiste* or "the greatest." From Ptolemy's time, it would serve as the definitive trigonometry for the next millennium. Ptolemy computed chord tables for the sine of (in modern terms) angles from ¼ ^{o} to 90^o in ¼ ^{o} steps on a circle of radius 60 and, in terms of chords, knew $\sin^2 A + \cos^2 A = 1$, the sine of a sum or difference, the half-angle formula for sine, and the law of sines. In the seventeenth century the Ptolemaic model of the universe would be burdened with 77 circles on circles of cycles to match observations. The quest for a better model would provide much driving force for trigonometry; it would be taken up by Copernicus and fulfilled by Kepler in 1609. The Greek intellectual experience dimmed with the Roman conquest and the burning of the Alexandria library.

India. Trigonometry was one of the major contributions of Hindu mathematics. Mathematicians were "expert on the stars." Astronomical treatises had detailed tables for finding angular distances between stars and incorporated poetic verses and aphorisms, all in Sanskrit. Hindu trigonometry was influenced by Greek methods of calculation, but not by the Greeks' devotion to proof. It contained sine, cosine, versine (1 – cosine), the sine of a half angle, methods of solving spherical and plane triangles, and division of angle degrees into minutes and seconds.

About 500 CE Aryabhata gave today's sine concept, in tables of half chords, which were a short cut over Ptolemy's chords. Still, the sine was not a ratio but rather a segment length. For greater decimal precision, Aryabhata chose a circle radius of 3438 units. By the end of the sixth century, work in shadow reckoning displayed today's six trigonometric ratios. In 1150 Bhaskara gave a system for finding the sine of any angle. Pushing for accurate detailed tables, the Hindus produced power series expansions for sine, cosine, and inverse tangent, which predated the European series by two to three centuries. To predict eclipses, they "froze" trigonometric functions in a manner suggesting calculus.

The Hindu trigonometry impacted Chinese mathematics, which sought numerical patterns in eclipses, occultation's, conjunctions, and recurring events. Hindu astronomers working for the Emperor brought reference works such as chord tables and the *Surya Siddhanta* (c. 300-400). These influenced the monk I-Hsing (c. 724), the greatest Chinese astronomer of the time, to make tangent tables, giving formal status to shadow reckoning.

The Islamic World. In the eighth century, the Arabs discovered Hindu trigonometry. They preserved, translated, and added to it topics in arithmetic, algebra, and geometry. The Arabs were the primary transmitters of mathematics to Europe. Trigonometry was second only to algebra as their favorite mathematics, and it remained the servant of astronomy. About 860, the Arabs produced a table of shadows and created tangent and cotangent ratios. Adding new formulas and functions, they synthesized Greek, Hindu, and their own discoveries into a true trigonometry. Al-Battani (c. 850-929) made sine and tangent tables and a formula for the sun's height. The greatest Arab mathematician of the tenth century, Abu-l-Wafa (940-998) placed trigonometry on a unit circle in terms of arcs. The eleventh century astronomer Al-Biruni wrote a treatise on Hindu shadow reckoning and defined sin, cos, tan, cot, sec, and csc by shadows. Islamic scholars used all six quantities from the thirteenth century on. Nasir al-Din al-Tusi (1201-1274) first treated trigonometry independently from astronomy. The Islamic legacy in trigonometry consisted of the six functions, the law of sines and other identities, the formation of trigonometric tables by interpolation, and applications to optics and surveying outside of astronomy.

Europe. By the twelfth century the translation of Greek, Hindu, and Islamic mathematics by Latin scholars gave Europe an appetite for the trigonometry of Islamic astronomy. Fibonacci's *Practica Geometriae* (1220) gathered trigonometry from Islamic works. European trigonometry truly emerged in 1533 with the publication of *De Triangulis* by Regiomontanus (1436-1476), Europe's first systematic treatment of spherical and plane trigonometry. This book heralded a new age of mathematicians and astronomers, focusing in sixteenth century Poland and Germany. Constructing tables for sine and versine was laborious but demanded by astronomers patching up the Ptolemaic model of planetary motion. Copernicus (1473-1543) assembled all the trigonometry he needed for astronomy in *De revolutionibus* (*On the Revolutions of the Heavenly Spheres*) (1545), which asserted that the Earth and planets circled the sun. His student Rheticus (1514-1576) was the first to define trigonometric quantities as ratios of sides of right triangles instead of circular arcs. Rheticus' own publication caused trigonometry to come of age in Europe. His sine and cosine tables took the circle radius to be 10,000,000, so that he could achieve seven-place accuracy. He broke each angle into 10-second steps, so that his table was 90 times as dense as Ptolemy's. Why would anyone want to do this? The scientific community was assisting or resisting the cultural paradigm shift caused by the Copernican theory. The practical demands of surveying, navigation, and calendar making required ever more accurate calculations. Tables to 15 decimal places were achieved by 1700 by hand and without modern decimal notation.

The Marriage of Trigonometry and Algebra. Three years after Rheticus died, the first book was published in western Europe having methods of solving triangles with all six trigonometric functions, by Viète (1540-1603), the founder of today's algebra. Viète approached trigonometry from algebra and functions. Based on work by Regiomontanus (with whom he agreed that trigonometry should be its own study) and Rheticus, Viète raised trigonometry to greater abstraction and scope.

Blending trigonometry with algebra was the final step in developing the analysis through the study of functions. Recurrence in events from heartbeats to seasons created the notion of periodicity and trigonometry took up circular or periodic functions. Planetary orbits, pendulums, sound, light, and the vibration of a violin string were all explored in the sixteenth and seventeenth centuries. As Whitehead said, "Thus trigonometry became completely abstract; and in thus becoming abstract, it became useful. It illuminated the underlying analogy between sets of utterly diverse physical phenomena."

Vast collections of trigonometric identities were coming into use all over Europe. Trigonometric functions became more important than numerical computation. Mathematicians reorganized what was known and made texts available. An identity such as $2 \cos x \cos y = \cos(x+y) + \cos(x-y)$ inspired the technique of "prostaphaeresis," and multiplying trigonometric values could be replaced with easier addition. Islamic scholars knew the formula about 1000 CE, and the Europeans developed additional formulas. Napier (1550-1617) knew trigonometric methods, which may have influenced him to create logarithm tables for changing general multiplication into addition.

Analysis of Trigonometric Functions. The first sine graph, a half arch by Roberval (1635), signified trigonometry's trend away from computation to a function approach. The brothers Jakob (1654-1705) and Johann (1667-1748) Bernoulli extended trigonometry into functions of complex numbers. The true founder of analytic trigonometry was Euler (1707-1783). In his *Introductio* (1748), sine was no longer a chord but a numeric value expressed as a ratio, the ordinate on a unit circle, or a sum of an infinite series. This marked the formal birth of the circular function concept in which sine and cosine are periodic, although Euler had used sine and cosine functions somewhat earlier in papers giving solutions to differential equations. Euler developed the identity $e^{ix} = \cos x + i \sin x$ (where e is the base for natural logarithms,) which mathematics historian Ronald Calinger called "the cardinal formula of analytical trigonometry." A result is $e^{\pi i} + 1 = 0$, one of the most beautiful theorems in all of mathematics. The *Introductio* gave us most of our modern notations. Euler's abbreviations sin, cos, tang, cot, sec, and cosec became standard. Euler brought together zero, unity, π, trigonometry, complex numbers, and infinite series.

Trigonometry explained why the compass and straightedge of the ancient Greeks could not trisect a general angle. Another challenge was whether those tools could construct a regular polygon with 17 sides. In 1796, eighteen-year-old Gauss proved it could be done. His proof involved complex numbers and trigonometry.

The Grand Design, or Back to the Universe. Studies in the physics of heat by Fourier (1768-1830) triggered soul-searching in the mathematics of the late nineteenth and early twentieth centuries. Fourier saw heat as the fundamental property of the universe and sought a model for heat diffusion in objects having specific shape. In *The Analytic Theory of Heat* (1822), his great theorem showed that any function whatsoever could be written as an infinite series adding sine and cosine of multiple angles. In challenging themselves to bring logical rigor and abstract generality to Fourier's work, mathematicians created new areas in function theory and the foundations of calculus. Among these mathematicians were Dirichlet (1805-1859), one of the originator's of today's function definition; Riemann (1816-1866) and Lebesgue (1875-1941), who clarified the concept of integration; and Cantor (1845-1918), who gave meaning to the notion of infinite sets. In the process of giving precise proofs to results about Fourier series, the deep roots of mathematics were examined and a profound synthesis brought forth. The physics of heat, expressed in trigonometric terms, opened mathematics to generalized functions, sets, and the infinite. These in turn yielded applications the ancients could not envision such as quantum mechanics and engineering electrical circuits.

Mathematicians now living are still writing chapters in the history of trigonometry. Perhaps a young person whom you know, perhaps even yourself, will add to it. Girls in the time of Sofia Kovalevskaya (1850-1891) were discouraged from studying mathematics. To understand a physics book, she built the needed trigonometry on her own. She eventually became one of the greatest female mathematicians.

The *UNESCO Courier* devoted its November 1989 magazine to a tour of mathematics in different times and cultures: "Mathematics forms part of our cultural heritage and history." And an article in the *American Scientist* magazine of July-August 1996 showed that the history of discovery teaches us unity, "in the ceaseless borrowing connecting diverse traditions and disciplines. ... However they may differ, the multitudinous projects of science share in and emerge from a common history."

To learn and teach trigonometry from a historical approach is to enter an exploration of how we as mathematics inquirers - and humankind - have arrived at where and as who we are today.

The Concept of Angle
Degree and Radian Measures
Teacher Notes

Description of Unit: The material in this unit is designed mainly for teachers. The commentary examines the historical development of angular measurement and how it is related to the chords of Ptolemy and explains how Cotes came up with radian measure. Most of the information in this unit is not found in standard high school texts. Since many introductions to trigonometry start with the radian measure of angles, a teacher could integrate material from this unit at that time. The topic of why there are 360 degrees in a circle is presented as a possible teacher script and is written in italics.

Prerequisites: Much of the unit requires only the knowledge of basic geometric terms. Although references are made to infinite series, derivatives of sine and cosine, and $e^{i\pi}$, material beyond the background of the students may be easily omitted.

Materials: Problems converting radians to degrees and vice versa as well as the relationship of angles measured in radians to arc lengths and circle radii are found in standard textbooks.

The Concept of Angle
Degree and Radian Measures
Student Pages

Hunters, stargazers, sailors, builders, farmers, and many others must surely have used angles informally long before any careful attempts were made to define the concept of angle. Over the centuries, as people began to treat angles more systematically, units of angular measure were introduced, instruments for measuring angles were made, and eventually the study of angles became part of geometry.

Euclid's *Elements,* which contain a systematic treatment of much of the plane geometry of his time, attempts to define the concept of angle and related notions in Book 1 as follows:

> Definition 8. A **plane angle** is the inclination to one another of two lines in a plane, which meet one another and do not lie in a straight line.
>
> Definition 9. And when the lines containing the angles are straight, the angle is called **rectilinear.**
>
> Definition 10. When a straight line meeting another straight line makes the adjacent angles equal to one another, each of the angles is **right** and the first straight line is called a **perpendicular** to the second line.
>
> Definition 11. An **obtuse** angle is greater than a right angle.
>
> Definition 12. An **acute** angle is less than a right angle.

Euclid's statements may be effective at some intuitive level, but they certainly raise questions. Clearly Euclid associates angles with intersecting lines (figure 1), but what is meant by "the inclination?"

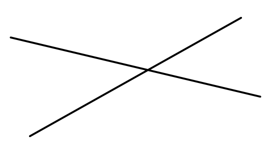


figure 1

Are the angles represented by figure 2 the same as the angles in figure 1, or are they in some sense smaller, since they are only a part of figure 1?

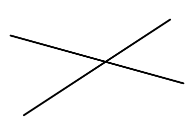


figure 2

Proclus, who taught in Athens some seven centuries after Euclid, commented extensively on Euclid's definitions, but he did not fundamentally clarify them. Attempts to give clear definitions of mathematical concepts and terms were an important part of the formalization process. Indeed, Euclid's *Elements* begins with definitions of basic concepts such as point and line. It is now realized that any attempt to define all terms, which may once have seemed an appealing goal, is doomed to failure. Despite the inherent difficulties, we often still try to define basic terms and concepts in mathematics, and Euclid's attempts in this area do seem to have some intuitive value in describing what he had in mind.

A second ancient view of angles is associated with the lengths of arcs on circles. Specifically, an angle, regarded as a central angle in a circle, is determined by, and can be defined by, the arc length that it subtends on the circle. Of course, this arc length depends on the size of the circle as well as on the central angle, a fact that caused complications for those who calculated trigonometric tables since they always had to specify the size of the circle being used. Ptolemy is a most prominent example.

Using lengths of circular arcs to define angles seems to fit naturally with the system of degrees, minutes, and seconds, still widely used today to measure angles. This system of units, whose origin is unknown, was used in ancient Babylonia approximately 4000 years ago. Speculation about the origins of this system of angular measure often centers on the fact that 360 is a very convenient number from a practical view.

Among the reasons given for this division of a circle into 360 parts are that 360 has many divisors, that it is the closest "round" number to the number of days in the year, and that the Babylonians used a base-60 place value system. Another reason is given by Otto Neugebauer: "In early Sumerian times there existed a large distance unit, a sort of Babylonian mile, equal to about seven of our miles. Since the Babylonian mile was used for measuring longer distances, it was natural that it should also become a time unit, namely the time required to travel a Babylonian mile. Later, some time in the first millennium B.C., when Babylonian astronomy reached the stage in which systematic records of celestial phenomena where kept, the Babylonian time-mile was adopted for measuring spans of time. Since a complete day was found to be equal to 12 time-miles, and one complete day is equivalent to one revolution of the sky, a complete circuit was divided into 12 equal parts. But, for convenience, the Babylonian mile had been subdivided into 30 equal parts. We thus arrive at (12)(30) = 360 equal parts in a complete circuit." (Newsom & Eves, *Introduction to College Mathematics,* 2nd edition)

Eli Maor in *Trigonometric Delights* states that "the word *degree* originated with the Greeks. According to the historian of mathematics David Eugene Smith, they used the word μοιρᾳ (moira), which the Arabs translated into *daaraja* (akin to the Hebrew *dar'ggah*, a step on a ladder or scale); this in turn became the Latin *de gradus*, from which came the word degree." The Greeks divided the degree into sixty "first parts" and each of these into sixty "second parts". In Latin this translates to "partes minutae primae" (first small parts) and "partes minutae secunda" (second small parts). Thus occurs our English division of degrees into minutes and seconds. By the fourteenth century the word "degree" was prevalent enough for Chaucer to write in *The Canterbury Tales* "the yonge sonne that in the Ram is foure degrees vp ronne."

For many centuries, the notion of angle as an arc of a circle to be measured in degrees was universally accepted and seems to have coexisted with Euclid's definition. For astronomy and navigation, Ptolemy's *Almagest*, which uses degrees, was the paradigm, a role-played in theoretical geometry by Euclid's *Elements*.

The question of what an angle really is, or, more precisely, what unit of measure should be used for angle measure was raised again by Roger Cotes (1682-1716) in a 1714 paper entitled *Logometria.* This paper, the only one Cotes published in his short lifetime, was nominally about logarithms but ranged broadly over many related topics. In the preface to Part II of *Logometri*a, Cotes remarked on "that Harmony of Measures, which is so strong that I propose a single notation to serve to designate measures, whether of ratios [logarithms] or of angles." Then he considered measures of angles, thus revisiting a topic, which seems not to have been addressed explicitly for some two millennia. Cotes reasoned that the arc of a circle contained between the sides of an angle would be an obvious candidate for the measure of an angle, were it not dependent on the size of the circle. For example, looking at figure 3, arc *AD* is larger than arc *BC*, even though both could be used to measure the same angle.

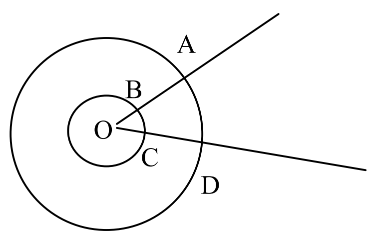


figure 3

Cotes also realized that the difference in the size of the circles could be neutralized if he measured angles not just by the size of their circular arcs but also by the size of the arc divided by the radius of the circle. For example, in figure 3, if *OD* is three times as large as *OC*, then arc *AD* is three times as long as arc *BC*, so that

$$\frac{\text{length of arc } AD}{\text{radius } OD} = \frac{\text{length of arc } BC}{\text{radius } OC}$$

Under this system an angle of measure 1 is an angle for which the circular arc equals the radius of the circle. The number of degrees in this angle is 180/π, which Cotes approximated as 57.295. Cotes' system of angle measure is now called radian measure. However, It was not until 1871 that James Thomson, brother of William Thomson (Lord Kelvin), formally named and defined the radian in his private papers. He first used the unit publicly in a final exam he gave at Queen's College in Belfast, Ireland, in 1873. Thomson and others had earlier considered the terms "rad" and "radial."

When an arc length equals the radius, its central angle is defined as having measure 1 radian. It's that simple. The word radian is an abbreviation for <u>radi</u>us-<u>an</u>gle. Just as the symbol for degree is a raised symbol °, so is the symbol for radian a raised $^{\mathrm{R}}$. Some people also refer to a radian as the length of the arc on the unit circle subtended by an angle of 1^{R}.

This turned out to be a very useful unit for theoretical work and today it is universally used for such purposes in mathematics and physics even though for more mundane applications degree measure remains the favorite. When no angle measure is specified, radians are understood to be the unit desired.

It is only when radians are used as the unit of angular measure that the derivative of the sine is the cosine, and that the derivative of the cosine is the negative sine. When graphing an equation such as $y = x + \sin x$, it is universally assumed that x is given in radian measure. Only when the angle x is measured in radians is it true that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ or that $\cos x + i\sin x = e^{ix}$. These results were discovered and stated, in a somewhat different form, long before the radian was invented. The sine series was known in India by the early fifteenth century, and many other remarkable relationships involving trigonometric and exponential functions and their inverses were discovered in the following three centuries. Euler, in his *Introductio in analysin infinitorum* presented these results, including many important new ones, in a systematic treatment in 1748.

Angles have also been measured in other units. The United States Army used the mil, originally introduced by the Swiss in 1864, in 1900 as an angle that subtends one yard at a thousand yards. A "true mil" is the angle subtended at the center of a circle by an arc equal in length to 0.001 of the radius. The gradian is a metric measure of angles. There are 100 gradians in a right angle. Fractional parts are computed decimally. This measurement was also used by military artillery and by civil engineers building railroads (banking of curves). On many nongraphing scientific calculators there is still a DGR button for degree, gradian, radian.

The question of how best to define an angle is still alive today. In the 1960s, the National Council of Teachers of Mathematics endorsed defining an angle as "the union of two rays with a common endpoint." This is essentially a reworked version of Euclid's definition. It was widely adopted in U.S. geometry textbooks, but it left blurred the same points as Euclid's old definition. For example, according to this definition the geometric shape in figure 4 qualifies as an angle, but when we ask how large the angle is (what its "measure" is) the confusion emerges. Is it 90°, 270°, or something else, such as 450°?

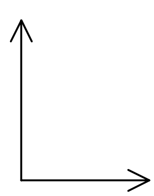
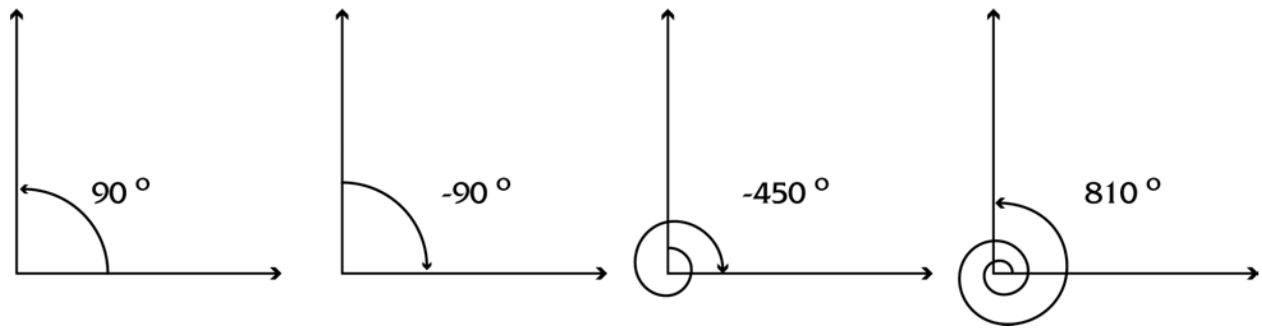


figure 4

In the meantime, the view of angles as associated with circular arcs has evolved to identify angles with rotations. Angles (read: rotations) are considered positive if they are counterclockwise and negative if clockwise. This point of view was used in trigonometry classes even as the NCTM definition was used in geometry classes in the same schools. From this point of view, figure 4 does not define an angle at all, since it does not describe a rotation. To describe an angle with a picture, it is necessary to supply curved arrows or other visual cues to describe the rotation completely.


90 o
-90 o
-450 o
810 o

The Beginnings of Trigonometry: Sine Values Found in Ptolemy's Table Teacher Notes

Description of Unit: In the Greek model of the universe, the heavenly bodies made circular orbits around the fixed Earth. After Hipparchus, the astronomer Ptolemy compiled numeric tables relating central angles to their chords. In this unit, a student handout provides exercises so that students will apply the sine ratio to a central angle and its chord. They will examine how accurate Ptolemy's chord values are. They will derive a formula expressing a chord in terms of sine and understand why Ptolemy's table is said to involve sine. Before using sine, however, students will look up a central angle's chord in Ptolemy's table and change the chord from base 60 to base 10. Students will then see How Ptolemy's Table Helped the Greeks Solve Triangles, that is, why the central angle-chord method helps evaluate unknown parts of any triangle whatsoever. At the end of the unit are two Extension Exercises, problems for inquiry, challenge, and enjoyment, i.e. extra credit or projects; their solutions are not provided here.

Teacher Notes at the end of the unit more detailed background for the teacher, tidbits, hints, and solutions and answers to the exercises.

Throughout this unit, the primary purpose is to practice applying sine and to develop mathematical concepts and skills. A later section in this module, on the Development of Ptolemy's Table, is suitable as a project for advanced students who would be interested in recreating Ptolemy's geometric steps deriving sin $\frac{1}{2}^{o}$.

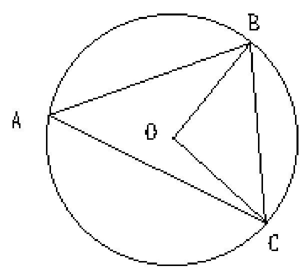
Prerequisites: Before starting this unit, students should have some practice with sin, cos, tan, sec, csc, and cot for right triangles from the textbook. They may need reminding on circle geometry: chord, central angle, inscribed angle, inscribed (right) triangle. A teacher note, rather scripted, reviews the base-10 system and then explains the base-60 system.

Materials: Calculator.

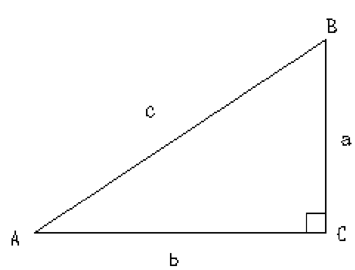
The Beginnings of Trigonometry: Sine Values Found in Ptolemy's Table Student Pages

The early Greeks thought that Earth was the unmoving center of the universe, as recorded by Eudoxus (408-335 BCE). The stars were fastened to an immense crystal sphere, which the Greeks considered to be the perfect shape. The Sun, the Moon, and the five visible planets (Mercury, Venus, Mars, Jupiter, and Saturn) also were attached to spheres. All the heavenly bodies moved in great circles around the Earth.

Trying to understand this, Hipparchus (180-125 BCE), one of the greatest astronomers of antiquity, created the mathematics, which eventually became trigonometry. In his work he dealt with triangles that were inscribed in circles. Because he was often dealing with triangles on the heavenly sphere, he developed spherical trigonometry at the same time as he was developing plane trigonometry. A basic problem was to evaluate the three angles and three sides of the inscribed triangle. The solution involved this: given a central angle BOC, find the length of the intercepted chord BC.



To do that, Hipparchus made tables of numbers where he could look up the chord for an angle. The tables evolved into what we know today as the sine relationship between an acute angle and two sides of a right triangle. Today's definition of sine is: For an acute angle of a right triangle, the sine is the ratio of the length of the side opposite the angle to the length of the hypotenuse. This was first written by Rheticus (1514-1574).



$$\sin A = \frac{a}{c}$$

Around 150 CE, the astronomer Ptolemy extended the work of Hipparchus in an astronomical work called the *Mathematical Collection*. Arab scholars who studied this work centuries later called it *Almagest*, meaning The Greatest. The first chapter of the *Almagest* discussed Ptolemy's version of trigonometry. Below is part of a copy of Ptolemy's table of chords. It is in the base-60 system, which was the standard at that time. The base-60 system will be explained during exercises assigned below. The source of the table is *Trigonometric Delights* by Eli Maor. The left side of the table is written in the original Greek, with the numbers in base-60, while the right side is the translation into Indo-Arabic numerals, though still in that base. Ptolemy followed the custom of using a circle whose radius was 60 units. He calculated chords for arcs from ½ ° to 180° in steps of ½ °. The sixtieths column was for interpolating, that is finding chords for angles in between the steps of ½ °. Thus, for example, the table tells us that the chord subtended by an arc of 4° in a circle of radius 60 units is 4;11, 16, which means 4 units plus 11/60 of a unit plus $16/60^2 = 16/3600$ of a unit. In the decimal system, that amounts to 4.18778 units.

Κανόνιον τῶν ἐν κύκλῳ εὐθειῶν			Table of Chords		
περιφε-ρειῶν	εὐθειῶν	ἑξηκοστῶν	arcs	chords	sixtieths
∠'	ō λα κε	ō α β ν	½°	0;31,25	0;1,2,50
α	α β ν	ō α β ν	1°	1;2,50	0;1,2,50
α∠'	α λδ ιε	ō α β ν	1½°	1;34,15	0;1,2,50
β	β ε μ	ō α β ν	2°	2;5,40	0;1,2,50
β∠'	β λζ δ	ō α β μη	2½°	2;37,4	0;1,2,48
γ	γ η κη	ō α β μη	3°	3;8,28	0;1,2,48
γ∠'	γ λθ νβ	ō α β μη	3½°	3;39,52	0;1,2,48
δ	δ ια ις	ō α β μζ	4°	4;11,16	0;1,2,47
δ∠'	δ μβ μ	ō α β μζ	4½°	4;42,40	0;1,2,47
ε	ε ιδ δ	ō α β μς	5°	5;14,4	0;1,2,46
ε∠'	ε με κζ	ō α β με	5½°	5;45,27	0;1,2,45
ς	ς ις μθ	ō α β μδ	6°	6;16,49	0;1,2,44
ς∠'	ς μη ια	ō α β μγ	6½°	6;48,11	0;1,2,43
ζ	ζ ιθ λγ	ō α β μβ	7°	7;19,33	0;1,2,42
ζ∠'	ζ ν νδ	ō α β μα	7½°	7;50,54	0;1,2,41
⋮	⋮	⋮	⋮	⋮	⋮
ροδ∠'	ριθ να μγ	ō ō β νγ	174½°	119;51,43	0;0,2,53
ροε	ριθ νγ ι	ō ō β λς	175°	119;53,10	0;0,2,36
ροε∠'	ριθ νδ κζ	ō ō β κ	175½°	119;54,27	0;0,2,20
ρος	ριθ νε λη	ō ō β γ	176°	119;55,38	0;0,2,3
ρος∠'	ριθ νς λθ	ō ō α μζ	176½°	119;56,39	0;0,1,47
ροζ	ριθ νζ λβ	ō ō α λ	177°	119;57,32	0;0,1,30
ροζ∠'	ριθ νη ιη	ō ō α ιδ	177½°	119;58,18	0;0,1,14
ροη	ριθ νη νε	ō ō ō νζ	178°	119;58,55	0;0,0,57
ροη∠'	ριθ νθ κδ	ō ō ō μα	178½°	119;54,24	0;0,0,41
ροθ	ριθ νθ μδ	ō ō ō κε	179°	119;59,44	0;0,0,25
ροθ∠'	ριθ νθ νς	ō ō ō θ	179½°	119;59,56	0;0,0,9
ρπ	ρκ ō ō	ō ō ō ō	180°	120;0,0	0;0,0,0

On the next page, there is a more complete version of Ptolemy's table, written entirely in our own characters, but with values still given in the sexagesimal system. You will use both of these tables in the exercises that follow.

TABLE OF CHORDS

Arcs	Chords	Sixtieths	Arcs	Chords	Sixtieths
½	0 31 25	1 2 50	23	23 55 27	1 1 33
1	1 2 50	1 2 50	23½	24 26 13	1 1 30
1½	1 34 15	1 2 50	24	24 56 68	1 1 26
2	2 5 40	1 2 50	24½	25 27 41	1 1 22
2½	2 37 4	1 2 48	25	25 58 22	1 1 19
3	3 8 28	1 2 48	25½	26 29 1	1 1 15
3½	3 39 52	1 2 48	26	26 59 38	1 1 11
4	4 11 16	1 2 47	26½	27 30 14	1 1 8
4½	4 42 40	1 2 47	27	28 0 48	1 1 4
5	5 14 4	1 2 46	27½	28 31 20	1 1 0
5½	5 45 27	1 2 45	28	29 1 50	1 0 56
6	6 16 49	1 2 44	28½	29 32 18	1 0 52
6½	6 48 11	1 2 43	29	30 2 44	1 0 48
7	7 19 33	1 2 42	29½	30 33 8	1 0 44
7½	7 50 54	1 2 41	30	31 3 30	1 0 40
8	8 22 15	1 2 40	30½	31 33 50	1 0 35
8½	8 53 35	1 2 39	31	32 4 8	1 0 31
9	9 24 54	1 2 38	31½	32 34 22	1 0 27
9½	9 56 13	1 2 37	32	33 4 35	1 0 22
10	10 27 32	1 2 35	32½	33 34 46	1 0 17
10½	10 58 49	1 2 33	33	34 4 55	1 0 12
11	11 30 5	1 2 32	33½	34 35 1	1 0 8
11½	12 1 21	1 2 30	34	35 5 5	1 0 3
12	12 32 36	1 2 28	34½	35 35 6	0 59 57
12½	13 3 50	1 2 27	35	36 5 5	0 59 52
13	13 35 4	1 2 25	35½	36 35 1	0 59 48
13½	14 6 16	1 2 23	36	37 4 55	0 59 43
14	14 37 27	1 2 21	36½	37 34 47	0 59 38
14½	15 8 38	1 2 19	37	38 4 36	0 59 32
15	15 39 47	1 2 17	37½	38 34 22	0 59 27
15½	16 10 56	1 2 15	38	39 4 5	0 59 22
16	16 42 3	1 2 13	38½	39 33 46	0 59 16
16½	17 13 9	1 2 10	39	40 3 25	0 59 11
17	17 44 14	1 2 7	39½	40 33 0	0 59 5
17½	18 15 17	1 2 5	40	41 2 33	0 59 0
18	18 46 19	1 2 2	40½	41 32 3	0 58 54
18½	19 17 21	1 2 0	41	42 1 30	0 58 48
19	19 48 21	1 1 57	41½	42 30 54	0 58 42
19½	20 19 19	1 1 54	42	43 0 15	0 58 36
20	20 50 16	1 1 51	42½	43 29 33	0 58 31
20½	21 21 11	1 1 48	43	43 58 49	0 58 25
21	21 52 6	1 1 45	43½	44 28 1	0 58 18
21½	22 22 58	1 1 42	44	44 57 10	0 58 12
22	22 53 49	1 1 39	44½	45 26 16	0 58 6
22½	23 24 39	1 1 36	45	45 55 19	0 58 0

Source: *Episodes in the Mathematics of Medieval Islam* by J. L. Berggen.

Exercises

1. For a 6^o central angle, the length of the intercepted chord is __?__. Since Ptolemy's table uses the base-60 system, express the chord length in our own base-10 system.

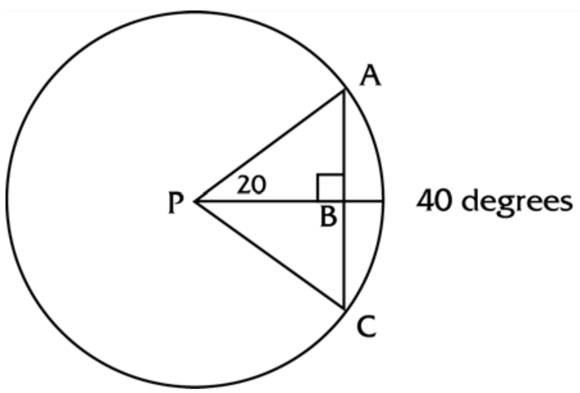
2. If the central angle is 40^o, in a circle with radius 60, find the length of the intercepted chord.

3. Now you can compute the sine of 20° from the chord length of the 40^o central angle. In the diagram, $\angle APC = 40^o$.

   a) Express *AB* in terms of AC and in terms of the sine of 20°.

   b) Determine *AB* numerically from the value for *AC* found in exercise 2.

   c) Calculate sin 20° from your expressions in a and b. Compare this with your calculator's value.



4. With a partner, calculate sin(½ *A*) in two ways. Make up 2 central angles. For each angle, one partner will use Ptolemy's table to find the chord of the angle, then determine the ratio which produces sin(½ *A*) and finally calculate that value. The second partner will find sin(½ *A*) by using the calculator. Then the two of you should compare your answers.

5. Use Ptolemy's value that Crd 36^o = 37;4,55. Calculate sin 18° from this value and check its accuracy against your calculator.

6. Develop the formula relating the chord of a central angle θ in a circle with radius *r* to the sine of ½θ.

Mathematical and Historical Notes

The exercises helped you to see why Ptolemy was a major figure in the beginnings of trigonometry's history.

Nowadays calculators give us sine values easily. How did Ptolemy construct his table? (If you are curious and like challenges, see the unit titled The Development of Ptolemy's Table for a worksheet, which steps you through his process.)

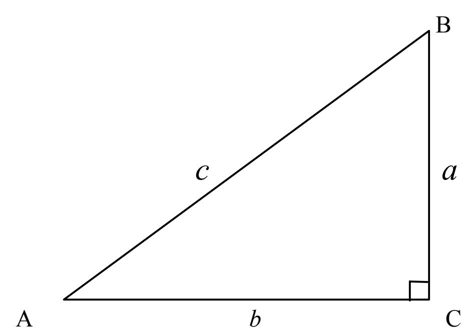
Here is a summary. Ptolemy inscribed the regular polygons with 3, 4, 5, 6, and 10 sides in a circle of radius 60. Then, using only Euclidean geometry, he calculated the side of each polygon. The results were chords for central angles of 120^o, 90^o, 72^o, 60^o, and 36^o respectively. He discovered how to calculate the chord of half the arc of a known chord, added and subtracted known arcs and chords, and thus built his chord table.

How Ptolemy's Table Helped the Greeks Solve Triangles

In the second part of this unit, you will learn how the Greeks solved triangles. Solving a triangle means that, given certain information about some sides and angles, find all 3 sides and 3 angles. Then you will learn how the Greeks' mathematical discoveries, wonderful as they were, raised other questions. As astronomers strived toward answers, the process would lead to the continuation of the story of trigonometry.

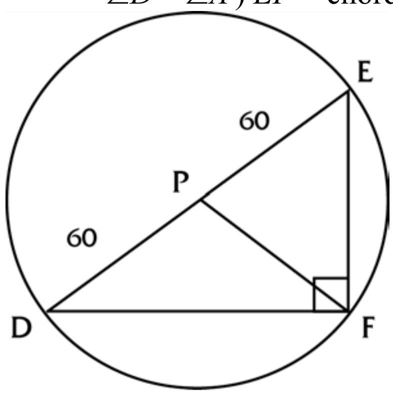
The Greeks focused on solving right triangles. (To solve an acute or obtuse triangle, they would break it down into right triangles.) Here is one procedure the Greeks used.

Problem:
Given the hypotenuse c, and an acute angle A,
Find the legs a, b, and the other acute angle B.



Solution:
First consider a circle with standard radius 60.
Draw right triangle DEF in the circle, so that $\angle A = \angle D$. From geometry, remember that when a right triangle is inscribed in a circle, the hypotenuse is a diameter of the circle.
The diameter here is 120.
Draw radius PF.
Then $\angle EPF = 2\angle D$. Why?

The next step would be to look up $\angle EPF$ in Ptolemy's table and find the chord, which is just side *EF*. The rest will be easier to understand if we keep in mind that EF = chord of $2\angle D$, and therefore (by substituting $\angle D = \angle A$) EF = chord of $2\angle A$.



Next we are ready to return to the original triangle *ABC*.

Because $\Delta DEF \sim \Delta ABC$, we have $\dfrac{c}{120} = \dfrac{a}{EF}$.

Then $120a = c \cdot EF$ or $120a = c \cdot$ chord of $2\angle A$

And therefore finally $a = \dfrac{c}{120} \cdot$ chord of $2\angle A$.

The Greeks constantly used the above formula for *a*. Thus, given the hypotenuse *c* and an acute angle $\angle A$, they could find the side *a* opposite that acute angle. Translating the algebra gives us the following recipe:
Double the angle $\angle A$.
Look up the chord in Ptolemy's table
Multiply by the hypotenuse.
Divide by 120.

Reminder! The original problem required us also to find $\angle B$ and the side *b* opposite $\angle B$. How do we do that?

Why the Saga Continued

Remember the last ingredient in the Greek recipe, to divide by 120? And remember that the Greeks worked in the base-60 system? Watch what happens when a base-60 number is divided by 120:

This example uses the example 37;4,55 (expressed in base-60), which is $37 + 4(1/60) + 55(1/60^2) = 37.08194444\ldots \approx 37.0819$ (expressed in our base-10).

37;4,55 divided by 120

means $$\left(37 + \frac{4}{60} + \frac{55}{60^2}\right) \div 120$$

$$= \left(37 + \frac{4}{60} + \frac{55}{60^2}\right) \times \frac{1}{120}$$

$$= \frac{37}{120} + \frac{4}{120(60)} + \frac{55}{120(60^2)}$$

$$= \frac{37}{2(60)} + \frac{4}{2(60)(60)} + \frac{55}{2(60)(60^2)}$$

$$= \frac{37}{2(60)} + \frac{4}{2(60^2)} + \frac{55}{2(60^3)}$$

$$= \left(\frac{37}{60} + \frac{4}{60^2} + \frac{55}{60^3}\right) \div 2$$

= 0;37,4,55 divided by 2 = 0;18,2,27

So, 37;4,55 divided by 120 is equal to 0;37,4,55 divided by 2. This gives a short cut when dividing a base-60 number by 120. The Greeks just had to shift the numbers one place to the right and then take half. Astronomers wondered if they could devise a chord table, which would shortcut out even the steps of doubling the angle and dividing the chord by two! Much of trigonometry was developed in seeking to create such a table. The Hindus succeeded, as recorded in the work of the Hindu astronomer Aryabhata (475-550), and they gave us our modern concept of sine as a half-chord in a circle.

Extension Exercises
Problems for Inquiry, Challenge, and Enjoyment

1. Sixtieths Column. Figure out how to the use the sixtieths column in Ptolemy's table. Note that the value in that column is one-thirtieth of the difference between the chord value in that row and the chord value in the previous row.

2. Development of Ptolemy's Table. As a project, turn to that unit in this book and complete the steps in Ptolemy's process.

The Beginnings of Trigonometry: Sine Values Found in Ptolemy's Table Teacher Notes

Students should already have worked some simple text exercises based on right triangle definitions of sin, cos, tan, sec, csc, and cot. Mention to them that the very beginnings of trigonometry did not start with that material. The definitions of the six trigonometric ratios, their names, their association with angles, and decimal notation - all did not yet exist.

Ancient trigonometry did not deal with ratios. It dealt with lengths of chords in circles. Usually, the chord is thought of as subtending an arc of a particular measure, not an angle. The concept of angle is either weak or absent. It was not until the sixteenth century that the sine and the other trigonometric functions were thought of as ratios. The word "trigonometry" first appeared in 1595 in the title of a book by Bartholomew Pitiscus (1561-1613) of Germany.

Solutions to Exercises:

1. It may be worthwhile to walk the class through this exercise, with emphasis on explaining the base-60 system. You may want to explain it as follows:

   Today we use the base-10 system. Before trying to understand base 60, let's review what base 10 means. [Write amply on the board.] As an example, look at the number 12.345. Find the decimal point. To the left is the units, so we have 2 units, and to its left are the tens, so we have 1 ten. Look at the decimal point again. To the right we have 3 tenths, then 4 hundredths, and 5 thousandths. Thus $12.345 = 1(10) + 2(1) + 3(1/10) + 4(1/10^2) + 5(1/10^3)$. Remember that the digits in the base-10 system are 0 to 9. Now look at Ptolemy's table. Take for example the first chord listed, 0;31,25. The semicolon (;) is like our decimal point. A comma separates the 31 and 25 because in base 60, the possible [quote] digits [unquote] are __?__. (Response: 0 to 59.) So a comma separates the blocks. 0;31,25 means $0(1) + 31(1/60) + 25\,(1/60^2)$, or $0(1) + 31/60 + 25/3600$. This equals __?__. (Calculator exercise. Response: 0.5236111...) Therefore 0;31,25 ≈ 0.52361.

   For a central angle of 6°, the chord is $6;16,49 = 6(1) + 16(1/60) + 49(1/60^2) = 6 + 16/60 + 49/3600 = 6.28028$.

2. Central angle 40°. Chord $= 41;2,33 = 41(1) + 2(1/60) + 33(1/60^2) = 41 + 2/60 + 33/3600 = 41.04250$

3. a. $AB = ½\,AC$. Since $\sin 20° = AB/AP$, we also have $AB = AP \sin 20° = 60 \sin 20°$.

   b. Since AC is the chord of the 40° central angle, it is equal to 41.04250. Therefore, $AB = ½\,AC = 20.52125$.

c. sin 20º = $AB/60$ = 20.52125/60 = 0.34202. The calculator gives sin 20º = 0.34202, so Ptolemy's table is accurate to at least five decimal places.

5. The chord of 36º is 37:4,55 = 37 + 4/60 + 55/3600 = 37.08194. Using the same procedure as before, we find that sin 18º is half this value divided by 60, or 37.08194/120 = 0.30902. The calculator gives sin 18º = 0.30902 as well.

6. $\sin \frac{1}{2} \angle APC = AB / r = \frac{1}{2}AC / r$. But AC is the chord of $\angle APC$ in a circle of radius r. Writing that angle as angle θ, we have $\sin(\theta/2) = \frac{1}{2}\,\text{chord}(\theta)/ r$. Another way to write this would be that $\text{chord}(\theta)=2r \sin(\theta/2)$.

How Ptolemy's Table Helped the Greeks Solve Triangles

For most of this problem, students will be working with ΔDEF whose hypotenuse is 120. If c does not equal 120, then ΔDEF is an enlargement or reduction of the original triangle. But ΔDEF is similar to ΔABC. Therefore, corresponding angles are equal and corresponding sides are proportional. Give the students comforting assurance that we will definitely return to the given value of c. Near the end of the solution we will adjust for the scale change.

$\angle EPF = 2\angle D$ because the measure of an inscribed angle is half that of its intercepted arc. The measure of a central angle equals that of its intercepted arc.

To find $\angle B$, we simply subtract the measure of $\angle A$ from 90°. We can find side b either by repeating the above process or, more simply, by applying the Pythagorean Theorem.

A second type of problem was: given the hypotenuse c and one leg a, to find the other leg b and the 2 acute angles $\angle A$ and $\angle B$. This is recognized as just the reverse of the problem worked out above.

Shadow Reckoning
(Introduction to the Tangent Function)
Teacher Notes

Description of Unit: In this unit we introduce the tangent function as the ratio of the length of a vertical pole to the length of a shadow created by that pole, due to a light source. The students will calculate tangent ratios for various locations of that light source. They will graph direct variations and use the graphs to solve proportions. The students will also calculate tangent ratios, using the solar angle as the independent variable. The unit consists of a skit, two teacher scripts for the class, with extra teacher notes, and two classroom activities. The skit may be used in any of several ways.... either acted out by two students at the start of class, left on the overhead for students to read as they enter class, acted out by the teacher and a student...you decide.

This section of the module has been targeted for a beginning Geometry class, and could be used early in the year.

Prerequisites: It is only required that students understand how to measure angles, and that they be able to use a software program such as Geometer's Sketchpad. Although it is not necessary that they know how to solve a proportion (the skill may be taught in this lesson), an acquaintance with the topic is desirable.

Materials: In order to carry out Classroom Activity #1 about 15 poles/sticks will be required. Also, rulers or measuring tapes will be needed in order to measure pole heights and shadow lengths. Graph paper will be needed in order to graph the direct variations involved. On the subsequent class session a transparency with the cumulative graph results through the day is desirable.

In order to carry out Classroom Activity #2 a computer lab setting is strongly suggested, with Geometer's Sketchpad or similar software available. A second option for this activity would be to issue protractors and do the measurements manually, but the time it would probably take to reconcile the class results makes this option awkward.

Shadow Reckoning
(Introduction to the Tangent Function)
Introductory Skit

Thales: Hey, Mom!! Can I borrow the reins to the mule?

Mom: Sure, if first you can tell me the height of the tree outside, the one the mule is tied to.

Thales: OK, I'll be right back.

(later)

Thales: Hey, Mom!! The tape measure keeps falling over before I can slide it high enough.

Mom: You're a smart kid. Go out and try it again.

(later)

Thales: Hey, Mom!! It's 47 cubits and 3 palms!!

Mom: What a good boy! How'd you do it?

Thales: Well, you're not going to like this, but I figured I'd take the mule to the store to get a longer measuring tape...and.... I forgot to untie him first . ...and......... we accidentally pulled the tree over.

Mom: Oh no! That was my favorite olive tree.

Thales: But.......... since it was down anyway, at least it made measuring it lots easier.

Mom: Thales!! I wanted to know its height, not its length!!

(The story you have just heard is false...the names have been changed to lend authenticity. What could Thales have done to measure the height of the tree without pulling it over?)

Shadow Reckoning
Teacher Script

Although trigonometry has its origins in measuring heavenly distances and paths, and true right triangle ratios are only a later distillation of this study, in some ways the right triangle has been hiding behind the scenes of these astronomical calculations, in the shadows, so to speak.

Babylonian works dating from 1700 BC give tables recording shadow lengths from a fixed vertical pole measured over a prolonged period of time. Similar tables of shadow lengths can be found in the Chinese, Egyptian, and Greek early civilizations.

A quote from the oldest known Chinese mathematical treatise, the *Zhoubi Suanjing*, shows that the Chinese were aware of the importance of the shadow/pole relationship:

> He who understands the earth is a wise man, and he who understands the heavens is a sage. Knowledge is derived from the shadow. The shadow is derived from the pole. And the combination of right angle with numbers is what guides and rules the ten thousand things.

(Joseph Needham: *Science and Civilization in China*, Vol. 3 Cambridge University Press, 1959).

The first well-known mathematician in Western/Greek culture, Thales (625-547 BC), was also known for his work with shadow reckoning. The historian, Plutarch, writes of him in his *Banquet of the Seven Wise Men*:

> Although he (the King of Egypt) admired you (Thales) for other things, yet he particularly liked the manner by which you measured the height of the pyramid without any trouble or instrument, for by merely placing your staff at the extremity of the shadow which the pyramid casts, you formed, by the impact of the sun's rays, two triangles, and so showed that the height of the pyramid was to the length of the staff in the same ratio as their respective shadows.

(David Burton: *The History of Mathematics: An Introduction*. New York: Allyn and Bacon, 1985).

These calculations and resulting tables were used for agricultural purposes, as they confirmed seasons for planting and harvesting of crops. They were also used for social and religious purposes. Egyptian and Hindu priests fixed religious rituals according to the sun's position in the sky, and they determined that position by shadow lengths. Islamic society bases three of their five prescribed times for daily prayer on shadow lengths. Shadow lengths and similar right triangles answered early questions about the nature of the universe, such as the distances between heavenly bodies.

Shadow Reckoning
Teacher Notes

1) The phrase "shadow reckoning" was still in use hundreds of years later. The Hindu mathematician Brahmagupta (625 AD) titled a chapter in his book "Measure by Shadow" and Bhaskara (1150 AD titled a chapter in his book "Determination of Shadows". One of the more famous mathematical challenges was offered by Bhaskara in his book Lilavati, and might be worth offering to your students.

> The ingenious man who tells the shadows of which the difference is measured by 19, and the difference of hypotenuses by 13, I take to be thoroughly acquainted with the whole of algebra as well as arithmetic.

The problem translated means, given two shadows from the same pole (two light sources), if one shadow is 19 longer than the other, and the corresponding hypotenuse is 13 longer than the other, to find the length of each shadow.

2) Concerning the King of Egypt's admiration for Thales for being able to determine the height of a pyramid, Plutarch's quote doesn't quite do the problem justice. Assuming a square pyramid, the ratio of shadows is not really the ratio of heights of staff and pyramid because the length of the shadow of the pyramid can only be measured up to the edge of the base of the pyramid…. Hmmm, how would you find the actual height?

If Thales waits until the length of his staff shadow is equal to the length of the staff, then both right triangles are isosceles. At that point if the pyramid shows a shadow of 20 feet, we know that the pyramid is 20 feet plus the amount of "hidden shadow" underneath the base of the pyramid.

If Thales then waits until another day, when his 6-foot staff produces a 12-foot shadow in the same direction as the previous shadow, the shadow of the staff right triangle will "have grown another staff in length". At that time the pyramid shadow will also have grown another pyramid height. So, if the original pyramid shadow was 20 feet in length, and the new measurement gave a shadow of 80 feet, then their difference, 60 feet, is the pyramid height.

This can be expressed as a system of linear equations. If h is the height of the pyramid and y is the length of the "hidden shadow" underneath the pyramid, then the two measurements yield the equations $x/(y+20) = 6/6 = 1$ and $x/(y+80) = 6/12$. By inspection the second denominator must be double the first denominator, and since increasing the denominator by 60 doubles it, the original denominator (and numerator, the pyramid height) must be 60.

3) Classroom Activity #1 is designed to give the students a concrete feeling for the concept that although shadow lengths vary according to both the height of the pole and the time of the day, that at any given time of day, the shadow length is proportional to the pole height.

Classroom Activity #1

Task: To build a table like those of the ancient civilizations. This is to be done as a single group activity.

Step 1. Hold poles of varying lengths vertically touching the ground, one pole per student or pair of students. Carefully measure the length of each pole.

Step 2. Carefully measure the length of the shadow produced by each pole.

Step 3. Make a table of pole heights versus shadow lengths.

Step 4. Graph these ordered pairs (shadow length, pole height)

Step 5. What is the ratio of the length of the pole to the shadow length?

Step 6. Write the fraction thus obtained as a ratio of relatively prime integers.

Step 7. Using your graph, answer the questions below.

a) How long a shadow will be cast by Six Foot Steve?

b) Shorty casts an 8.5-foot shadow. How tall is Shorty?

c) If 5ft 9in Fred casts an 8ft 6in shadow, is it likely to be earlier in the day or later? Why?

Teacher Notes to Classroom Activity #1

1) Check the slope (or have other classes do it) at hourly intervals throughout the day. At the next class session you should have added other lines (each through the origin), each line corresponding to one of your hourly measurements. These lines represent what are called direct variations, and you may wish to elaborate on that, depending on your students' familiarity with the topic. In addition you should now build a second table, with the time as the independent variable and the slope (ratio of pole height to shadow length) as the dependent variable. Your table should end up looking something like the one below.

time of day	8am	9am	10am	11am	12pm
ratio of pole length to shadow length	1/3	2/5	1/1	7/2	23/2

2) The question of equivalent ratios will come up, because the varying shadow lengths may make for some relatively prime denominators. You may elect to make your pole lengths be in multiples of the shortest one (i.e. 6 inches, 1ft, 1.5 ft, 2ft…) or you may wish to make the lengths random. This second approach will open the door to some elementary statistics. In order to build second table, you will need to look at the collection of data for each hour and do some elementary statistical analysis (discard outliers and discuss measures of central tendency) before deciding on your table entry. Have them consider both common denominator as well as decimal approaches, pointing out that historically decimals were not available until the 1500s.

Once the second table has been made, some typical proportion problems should be assigned to the class. These are extensions of those asked at step 7 and should be considered both by looking at the graph and by solving the appropriate proportion.

Here are some examples.

"At 11am how long will Six Foot Steve's shadow be?"
or "If Five Foot Fred casts a 12.5 foot shadow, what time is it?"

Some drawbacks to the second table thus created are:

i) Your measurement for 9 am in March will be different than one for 9 am in June. This might result in a LARGE collection of tables, one for each day, rather unwieldy for general use.

ii) It might be a rainy day, or too overcast to produce a shadow.

iii) Accurate ratios for measurements around noon may be difficult because the shadow may be very short compared to the length of the pole.

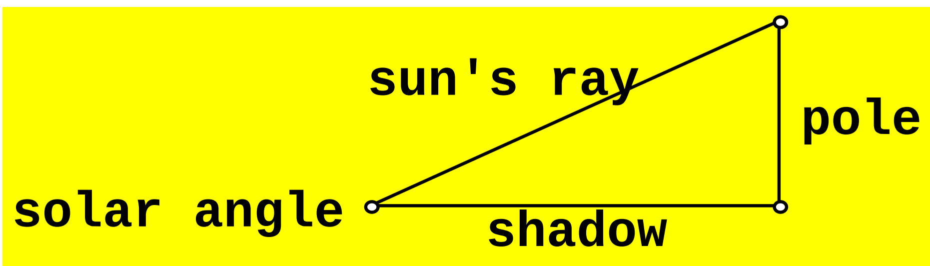
These drawbacks should be brought out in a class discussion, which then leads into how to overcome the drawbacks…. see teacher script below.

Teacher Script

Overcoming the drawbacks of the first table

To overcome the first drawback, we could choose to compare our ratios with a different quantity than time of day. Most of the major civilizations adopted the solar angle as their independent variable. This was the angle formed by the sun's ray where the shadow terminated. It is sometimes referred to as the angle of inclination for the lines graphed in class activity #1.

See the picture below, made by Geometer's Sketchpad.



Using the solar angle as the independent variable resulted in tables that looked more like the one below. This table is not dependent on the time of day or year.

solar angle	10^o	20^o	30^o	40^o	50^o
ratio of pole length to shadow length	2/11	4/11	4/7	5/6	6/5

Thanks to computer technology, we can overcome drawbacks two and three. See Classroom Activity #2.

Classroom Activity #2

To build an accurate table, using the solar angle as the independent variable. This is to be done singly or in pairs, using computer software in a lab setting.

Step 1: Open a drawing program on the computer, such as Geometer's Sketchpad.

Step 2: Construct a horizontal segment AB, as shown below.

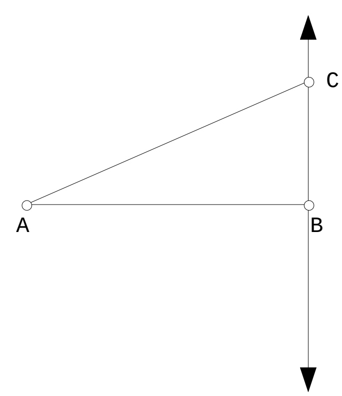
Step 3: Construct a perpendicular line through B, as shown below.

Step 4: Construct a point C on that line.

Step 5: Measure angle CAB and record it in your table.

Step 6: Measure the ratio of the segments CB and AB and record that also in your table.

Step 7: Slide C along BC, forming new angles and new ratios, and record these as well.



Teacher notes to Classroom Activity #2

1) The table developed by the students in the assignment is the tangent function. Their results can be compared to tables for tangents found in the backs of most Geometry and Trigonometry texts. They also can be compared to calculator-generated results, although the sense of "table" is lost when you utilize the calculator in this setting.

2) The assignment may be done individually, but pairs are recommended for quality control purposes, and also to provide the opportunity for observation and discussion. You may wish to have them make interpolative guesses from their tables, and then check their accuracy with the computer program.

3) You may also wish to point out to them how ratios for complementary angles are reciprocals, and ask them to explain why.

4) If your sketchpad program has a coordinate plane, you may wish to have your students make a second table, comparing the slopes of various line segments with their solar angle. This has the advantage of tying together the pole/shadow ratio with the rise/run ratio. Have each student construct a line segment whose endpoints have integer coordinates. This slope is their independent variable. Then have them measure the angle off the horizontal, and record that value as the dependent variable. Such a table would permit your students to find the angles of inclination and intersection of lines, given equations of those lines. This is a gentle way to introduce inverse functions in general to a Geometry class, and the inverse tangent function in particular.

5) The results for the 30, 45, and 60-degree angles are exactly determined in Geometry classes. It should be noted that our computer-generated ratios are the decimal approximations of these exactly known quantities. This may lead into a discussion of what other exact values can be determined. This search motivates many of the trig identities found further in this module. Since the whole concept of "decimals" was far in the future, students should be made aware that these early civilizations were drawn to consider these exact values, so that they could avoid more and more complicated integer ratios. For example, for many hundreds of years, better and better approximations of π meant more and more complicated ratios of integers, from 22/7 to 355/113. An exact value for approximations, even if expressed as a square root, cube root or other fashion, was more desirable than the more cumbersome ratios of ever-larger pairs of integers.

6) The Islamic culture developed a tool now known as the astrolabe, for measuring heavenly angles, and the relevant scales were known as the "shadow ladder", the "shadow box", and the "shadow square". The Latin phrases **umbra recta** and **umbra versa**, which refer to legs of right triangles, literally mean "straight shadow" and "turned shadow".

7) The word "tangent" comes from the Latin word **tangere**, which means, "to touch". From this word we get the words "tag", "tangible", and "attain". Even the word "tax" comes from the Latin, to touch (and keep) someone's money. You've heard of a "soft touch". The word is also used to describe lines, which touch, but don't cross, a circle or any other curve.

Development of Ptolemy's Table
Teacher Notes

Description of Unit: The exercises in this unit trace the development of a table of sines in increments of 1/2 degree. Since they are easier to work with, sines and cosines are used in the derivation instead of the chords that Ptolemy employed. The exercises could be presented as group work over three to five days depending on the background of the students. They would also be an excellent alternative or enrichment assignment. If the school has a math team, the exercises would be especially interesting to its members. Though each step is given in detail, and the mathematical background required is only basic geometry and plane trigonometry, much of the work is for students who like multi-step solutions and appreciate that not all mathematical problems are solved in less than five minutes. Many students will find the exercise in which $\sin 18°$ is calculated using the geometry of the pentagon especially challenging. After completing this unit, students should have an appreciation of the process that a mathematician goes through to find a solution to a problem. Note that calculators do not use this method of computation, but rather that of infinite series.

The following is a list of exercises:

1. Proof of Ptolemy's Theorem (for any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of opposite sides).
2. Derivation of trigonometric identities using Ptolemy's Theorem:
3. Calculation of $\sin 18°$ using the properties of a pentagon.
4. The use of the results of exercises 1-3 to estimate the sines of 1° and 1/2°.

Prerequisites: A general knowledge of the relationships between angle and arc measurement is necessary. In the derivations of the angle addition formulas for sine and cosine, it is assumed that the students know the right triangle trigonometric relationships. They should also know the values for the sine and cosine of 30º, 45º, and 60º. As stated above, the material is perhaps best appropriate for the student who loves challenges or, at least, a motivated student interested in proofs.

Materials: The student pages and exercises should be duplicated for student use. Though individual exercises may be assigned, the intent is that the student does all four, as well as read the introduction, so that the flow of mathematical thought is not lost. Answers to the exercises are supplied. A scientific calculator is needed for sine computations.

From Ptolemy's *Almagest*

In Greek, Ninth century, Claudius Ptolemy, who lived in the second century A.D., did work of enormous importance in astronomy and geography in which the Vatican Library has particularly rich holdings. The *Almagest*, written about A.D. 150, is a comprehensive treatise on all aspects of mathematical astronomy – spherical astronomy, solar, lunar, and planetary theory, eclipses, and the fixed stars. It made all of its predecessors obsolete and remained the definitive treatise on its subject for nearly fifteen hundred years. This, the most elegant of all manuscripts of the *Almagest*, is one of the oldest and best witnesses to the text, and is very rich in notes.

Source: http://metalab.unc.edu/expo/vatican.exhibit/exhibit/d-mathematics/Greek_math2.html

Ptolemy's Theorem and Trigonometric Identities Student Pages

Claudius Ptolemy (c.100-178) was a noted astronomer and geographer who lived in Alexandria, Egypt. In his *Geography* he discussed map making and projections, and gave the latitudes and longitudes of many places. He calculated the angle between the celestial equator and the ecliptic as 23º51'20", where the ecliptic is the great circle marking the apparent annual motion of the sun through the stars. For Ptolemy, the ecliptic was the actual path of the sun, because he regarded the earth as fixed at the center of the celestial sphere. In his *Mathematical Collection*, he explained the motions of the moon, sun, and planets in terms of epicycles (circles moving on circles). His model appears clumsy from a modern point of view, but it did accurately explain the observed positions of the celestial bodies. The *Mathematical Collection* quickly became the established authority in astronomy, much as had Euclid's *Elements* in geometry. In time Islamic scientists came to refer to the *Mathematical Collection* as al-magisti, meaning "the greatest," and it is now known by its derivative name, *The Almagest.*

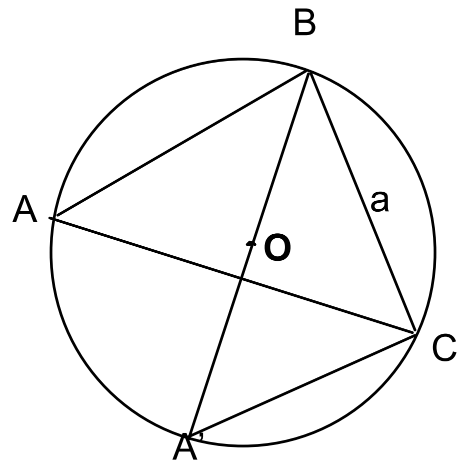
Early in *The Almagest*, Ptolemy calculated a table of chords, which is equivalent to a modern table of sines, in steps of 1/2º. We will retrace the work, using sines and cosines rather than chords. The overall plan had several steps:

1. Derive identities equivalent to those for $\sin(a+b), \cos(a+b), \sin(a-b), \cos(a-b),$ and $\sin\left(\frac{a}{2}\right)$. This had probably been done previously by Hipparchus of Bithynia (190-120 BCE), who was one of the first Greeks to use the Babylonian degree measure of angles in mathematics and who introduced celestial coordinates.
2. Calculate sin 60˚, sin 45˚, sin 30˚, and sin 18º, along with the corresponding cosine values. (We will assume known the sine and cosine of the first three angles mentioned.)
3. Use the value sin 30º and the identity for sin(a/2) to calculate the sine of 15º.
4. Use the identity for sin(a–b) to calculate sin 3º as sin(18º – 15º).
5. Use the identity for sin(a/2) to calculate, successively, the sines of (3/2)º and (3/4)º.
6. Use the above values to estimate the sines of 1º and (1/2)º.
7. Fill in the missing values by means of the known values and formulas.

This plan requires extensive calculations, but the work was carried out and produced a trigonometric table, accurate to two sexagesimal places or five decimal places. These exercises retrace the work in modern terms.

Ptolemy relied heavily on the fact, established centuries earlier in Euclid's *Elements*, that an angle inscribed in a circle equals half its intercepted arc. This implies that angles intercepting the same arc of a circle are equal.

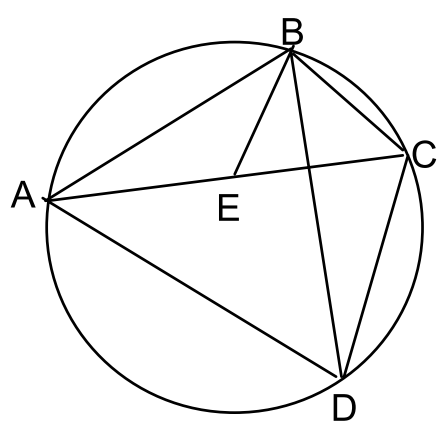
Consider the triangle ABC inscribed in a circle with diameter 1, with side a opposite angle A. Draw the diameter through B and the complete the triangle $A'BC$.



Angles A and A' are equal, since both intercept arc BC, but since $A'B$ is a diameter, the angle BCA' is a right angle. Therefore, $\sin A = \sin A' = a/A'B = a/1 = a$. A similar argument shows that the sides b and c, opposite angles B and C, respectively, are also equal to the sines of their opposite angles. In other words, $a = \sin A$, $b = \sin B$, and $c = \sin C$. This is an important fact that will be used in the exercises, which follow.

Exercise 1. Prove Ptolemy's Theorem:

For any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.



Let $ABCD$ be the given quadrilateral. Construct line BE so that $\angle ABE = \angle DBC$.
This could be done on *Geometer's Sketchpad* by copying $\angle DBC$ and letting E be the point of intersection of the newly constructed side of the angle with segment AC. Then all the length relationships that follow could be computed and confirmed with the calculator feature of *Sketchpad*.

1. Show that $\Delta ABE \sim \Delta DBC$. Remember that if two inscribed angles intercept the same (or even equal) arcs, then the angles are equal.

2. Show that $AB:DB = AE:DC$ and rewrite this as an equality of products: $AB \cdot DC = DB \cdot AE$.

3. Show that $\Delta ABD \sim \Delta EBC$.

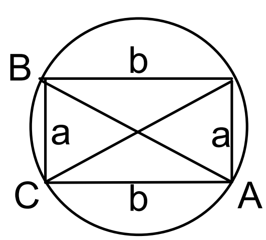
4. Show that $AD:EC = DB:BC$ and rewrite this as an equality of products: $AD \cdot BC = DB \cdot EC$.

5. Add the equalities from steps 2 and 4.

6. Factor DB on the right, and substitute for $AE + EC$ to complete the proof.

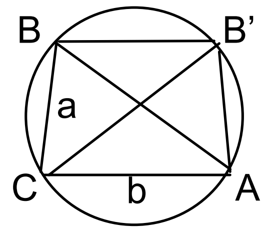
7. Here are two proofs that use Ptolemy's Theorem. Both are familiar - the Pythagorean Theorem and the Law of Cosines. For the Pythagorean theorem a rectangle is inscribed in the circle; for the Law of Cosines, a trapezoid is used.

   In the figure below, a rectangle has been inscribed in a circle. The Pythagorean Theorem follows almost immediately if Ptolemy's Theorem is applied. Try it!



c is the length of a diagonal

For the more general Law of Cosines, use the figure below. Remember that a trapezoid inscribed in a circle must be isosceles (try to prove it sometime). This proof is a lot more challenging than the previous one - but the sense of satisfaction is proportional to the effort expended. [Doesn't that sound like something a math teacher would say?]

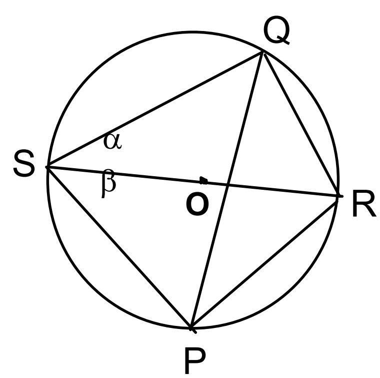


c is the length of a diagonal

If you have trouble with this proof of the Law of Cosines, here's a hint: Draw perpendiculars from A and C to the extension of BB' and show that the distance $BB' = b - 2ab\cos C$.

Exercise 2. Use Ptolemy's Theorem to Derive Trigonometric Identities.

1. Derive the formula for sin(α+β) directly from Ptolemy's theorem by means of the following diagram.

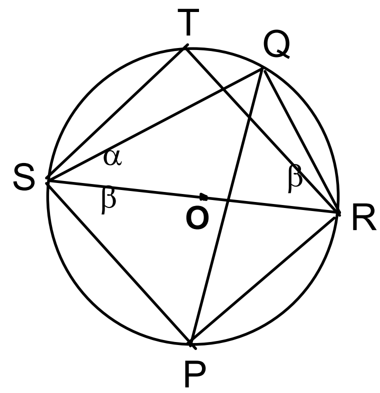


If the diameter SR=1, then lengths PQ, QR, and PR are the sines, respectively, of what angles? Remember that if circle has a diameter of 1, then the sine of any inscribed angle is equal to the length of the chord of the intercepted arc.

What do the cosines of α and β equal? (look for right triangles)

Apply Ptolemy's theorem to quadrilateral $PRQS$.

2. Derive the formula for cos(α+β) directly from Ptolemy's theorem, using the figure below, which is obtained from the earlier figure by placing the point *T* halfway around the circle from *P* and then drawing TS and TR. Recall that the diameter of the circle is 1 unit in length.

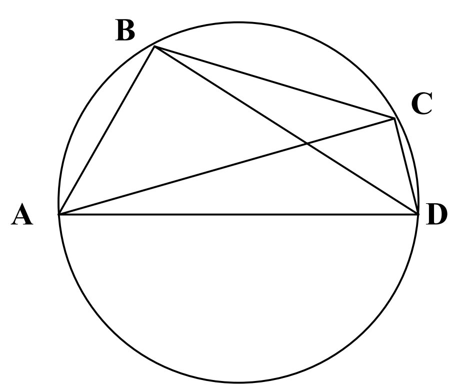


Triangles *STR* and *RPS* are congruent, and *PT* is a diameter.
As above, *PQ* = sin(α+β), and since *PT* is a diameter, Δ*PQT* (two sides of which have not been drawn, to avoid clutter) is a right triangle and the length of side *TQ* is cos(α+β).

Apply Ptolemy's theorem to quadrilateral *STQR*. Hint: *TR* = *SP* = cosβ.

The formulas for cos(α−β) and sin(α−β) may now be obtained by using –β in place of □ in the formulas for cos(α+β) and sin(α+β) and applying the rules $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = \sin\theta$.

3. An alternative way to derive the formula for sin(α−β) directly from Ptolemy's theorem is based on the figure below, in which the diameter *AD* has length 1.



$\angle BAD = \alpha$ $\angle CAD = \beta$ $\angle BAC = \alpha-\beta$

Use your knowledge of right triangles as well as the fact that, since the diameter of this circle is 1, the sine of any inscribed angle is equal to the length of the chord of the intercepted arc to determine the following lengths in terms of the sines or cosines of α, β, or $\alpha-\beta$.

a. $BD =$
b. $CD =$
c. $AB =$
d. $AC =$
e. $BC =$
f. $AD =$
g. Substitute the answers to parts a-f directly into Ptolemy's theorem. Then rearrange to get the difference formula for the sine: $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

4. Derive the half angle formulas for cosine and sine.

    a. Set $\beta = \alpha$ in the sum formula for the cosine to derive a formula for $\cos 2\alpha$.

    b. Substitute $1 - \cos^2\alpha$ for $\sin^2\alpha$ in the formula in a to get a formula for $\cos 2\alpha$ involving only the cosine.

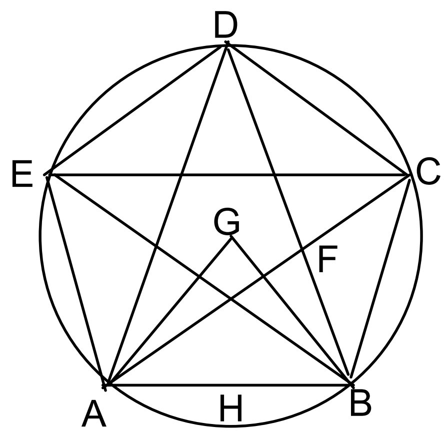
    c. Substitute θ for 2α in the formula in b (and therefore $\theta/2$ for α) to get a formula relating the cosine of $\theta/2$ to the cosine of θ. Rearrange the terms in this formula to get a formula for $\cos^2(\theta/2)$ in terms of $\cos\theta$.

    d. By substituting $1 - \sin^2(\theta/2)$ for $\cos^2(\theta/2)$ in the last formula and rearranging, obtain a formula for $\sin^2(\theta/2)$ in terms of $\cos\theta$.

    e. Solve the formulas in c and d for $\cos(\theta/2)$ and $\sin(\theta/2)$ respectively to get the half-angle formulas for the cosine and sine.

Exercise 3. Use the Geometry of the Pentagon to Calculate sin 18°.

In the diagram, *ABCDE* is a regular pentagon. That means that each of the arcs *AB, BC, CD, DE,* and *EA* are 72º.



1. Show that angle *ADB* is 36°.

2. Show that angles *DAB* and *DBA* are each 72°.

3. Show that $\Delta ABF \sim \Delta DAB$.

4. Show that triangles *ABF* and *ADF* are isosceles.

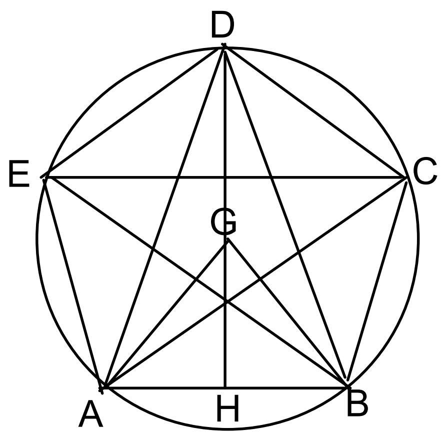
5. Let $AB = 1$ and y denote the length of *AD*. Show that $FB = y - 1$.

6. In ΔABF write the ratio of the long side to the short side in terms of y.

7. In ΔDAB write the ratio of the long side to the short side in terms of y.

8. Since the triangles in #6 and #7 are similar, the ratio of their sides must be equal. Rewrite the equality of these ratios as a quadratic equation in y and solve. Write your answer both in terms of radicals and as a decimal to as many decimal places as your calculator will handle. Remember that y must be positive.

9. Draw line DG and extend it to its intersection H with AB. Show that ΔADH is a right triangle and that the ratio AH/AD is sin 18°.



10. Use the fact that $\dfrac{AH}{AD} = \dfrac{\frac{1}{2}}{y}$ and the value of y found above to calculate sin 18° in radical form. Express your result with rational denominator.

11. Use a calculator to compute your result from #10 to as many decimal places as your calculator can handle. Compare your result with the calculator's value for sin18°. These results should agree to as many decimal places as your calculator displays.

12. Substitute your expression from #10 into the formulas for $\cos 18 = \sqrt{1 - \sin^2 18}$.

The resulting expression, $\dfrac{\sqrt{2\sqrt{5} + 10}}{4}$, is a bit awkward to calculate, but Ptolemy was able to compute its value. Calculate it to as many decimal places as your calculator can handle and compare with the calculator's value for cos 18º.

Exercise 4. Use the results derived above to estimate the sines of 1° and 1/2°

1. Use the half-angle formula for the sine, $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$, which was derived in Exercise 2, 4 and whose equivalent for chords was known to Ptolemy, to calculate sin 15°. Set $\alpha=30^o$ and use your knowledge of the cosine of 30º.

2. Find cos 15° using the half-angle formula for cosine derived in Exercise II, 4: $\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$.

3. To find sin 3°, notice that it is equal to sin(18° - 15°). Calculate this value, using the difference formula for the sine (Exercise II, 3), the values for sin 18^o and cos 18^o from Exercise III, and your answers to steps 1 and 2 above. Use as many decimal places as your calculator can handle. Compare your answer with your calculator's value for sin 3°.

4. Use the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$ to determine the value of cos 3°, again to as many decimal places as your calculator will handle. Compare this value to your calculator's value for cos 3°.

5. Since you now know the value for cos 3°, use the half-angle formula for the sine to determine sin(3/2°). Calculate cos(3/2°) by using the Pythagorean identity. Applying the half-angle formula a second time, determine sin 3/4°. Carry out those calculations to as many decimal places as your calculator can handle and compare your results with the corresponding values of the sine given directly by your calculator.

6. Clearly, the value of $\sin 1^{o}$ lies somewhere between the values of $\sin(3/4^{o})$ and $\sin(3/2^{o})$. We can now approximate $\sin 1°$ by at least two methods:

   a. First, note that sine is "almost" a linear function for very small values by looking at your calculated values for the sines of 3°, 3/2°, and 3/4°. Use linear interpolation between the two points $(3/4, \sin(3/4^{o}))$ and $(3/2, \sin(3/2^{o}))$ to find the second coordinate of $(1, \sin 1^{o})$.

   b. Second, use the idea of direct variation. Not only is the sine function almost linear, but it passes through the origin, so it is almost a direct variation. This means that $\sin\left(\frac{3}{4}\right)^{o} \approx \frac{3}{4}\sin 1^{o}$ and that $\sin\left(\frac{3}{2}\right) \approx \frac{3}{2}\sin 1^{o}$. So, now we have two approximations for $\sin 1^{o}$, namely $\frac{4}{3}\sin\left(\frac{3}{4}\right)$ and $\frac{2}{3}\sin\left(\frac{3}{2}\right)$. Using the values calculated in 5, calculate these approximations to as many decimal places as your calculator will handle. Make a guess about the value of $\sin 1^{o}$ based on the direct variation idea. You should be able to guess a value accurate to six decimal places. Check this by using your calculator to calculate $\sin 1^{o}$ directly.

7. Using the above value for the sine of 1º, calculate cos 1º. Then apply the half-angle formula to determine the value of sin (1/2º) very accurately. Also, calculate cos(1/2º).

8. In theory, since you know the sine and cosine of 1/2º and 1º, as well as the sines and cosines of some other angles, you can use the sum formula for the sine to calculate the values of the sine for every angle between 1/2º and 90º in increments of 1/2º. For example, calculate the sine of 16º.

Ptolemy, probably assisted by many human "calculators", was able to calculate his entire table using essentially the methods you have used. Of course, all the calculations, including square roots, were performed by hand. For this time period, carrying out these computations was a monumental effort. The great accuracy of Ptolemy's table and its small increment of only 1/2º made it the standard for centuries.

Answers to Student Exercises

Exercise 1:

1. $\angle ABE = \angle DBC$ construction
   $\angle BAE = \angle BDC$ angles subtend the same arc
   $\Delta ABE \sim \Delta DBC$ AA

2. $\dfrac{AB}{DB} = \dfrac{AE}{DC}$ corresponding sides of similar triangles
   $AB \cdot DC = DB \cdot AE$ product of means = product of extremes

3. $\angle ADB = \angle BCA$ angles subtend the same arc
   $\angle ABD = \angle EBC$ $\angle ABE = \angle DBC$
   $\angle ABE + \angle EBD = \angle DBC + \angle EBD$
   $\angle ABD = \angle EBC$
   $\Delta ABD \sim \Delta EBC$ AA

4. $\dfrac{AD}{EC} = \dfrac{DB}{BC}$ corresponding sides of similar triangles
   $AD \cdot BC = DB \cdot EC$ product of means = product of extremes

5. $AB \cdot DC + AD \cdot BC = DB \cdot AE + DB \cdot EC$ added answers from #2 and #4

6. $AB \cdot DC + AD \cdot BC = DB(AE + EC)$
   $AB \cdot DC + AD \cdot BC = DB(AC)$ *or*
   $DB(AC) = AB \cdot DC + AD \cdot BC$
   Since *DB* and *AC* are diagonals and *AB*, *BC*, *DC*, and *AD* are sides, the product of the diagonals of a cyclic quadrilateral equals the sum of the products of the opposite sides.

7. a. When a rectangle is inscribed in the circle, $a \cdot a + b \cdot b = c \cdot c$, thus the Pythagorean Theorem, $a^2 + b^2 = c^2$ is produced.

   b. When the figure inscribed is an isosceles trapezoid, extend *B'B* so that the perpendicular from *A* intersects the extension at *A'* and the perpendicular from *C* intersects at *C'*.
   In $\Delta A'B'A$, $\cos A'B'A = B'A'/B'A = B'A'/a$
   $\angle A'B'A = \angle B'AC$ (alternate interior angle of parallel lines *C'A'* and *CA*)
   Thus $\cos B'AC = B'A'/a$ or $B'A' = a \cdot \cos B'AC$
   $C'A' = BB' + 2B'A'$, so $B'A' = (C'A' - BB')/2$
   Substituting, $a \cdot \cos B'AC = (b - BB')/2$, or $BB' = b - 2a \cdot \cos B'AC$
   Ptolemy's Theorem tells us that $a \cdot a + b \cdot BB' = c \cdot c$

Substituting for BB', we get $a^2 + b(b - 2a\cdot\cos B'AC) = c^2$
or $a^2 + b^2 - 2ab\cdot\cos B'AC = c^2$
Since $\angle B'AC = \angle BCA$, and the latter angle is simply $\angle C$ in ΔABC, the formula simplifies to the more familiar form: $c^2 = a^2 + b^2 - 2ab\cdot\cos C$.
If $\angle C$ is a right angle, then $AB'BC$ is a rectangle and we have the Pythagorean Theorem as given in the previous exercise.

Exercise 2:

1. $\sin(\alpha + \beta) = PQ$
   $\sin\alpha = QR$
   $\sin\beta = PR$
   $\cos\alpha = SQ/SR = SQ$
   $\cos\beta = SP/SR = SP$

   $SR\cdot PQ = SQ\cdot PR + QR\cdot SP$
   $1\sin(\alpha + \beta) = \cos\alpha\cdot\sin\beta + \sin\alpha\cdot\cos\beta$
   $\sin(\alpha + \beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$

2. $SQ\cdot TR = TS\cdot QR + TQ\cdot SR$
   Since ΔSTR is congruent to ΔRPS, $TR = SP$ and $TS = RP$.
   $\cos\alpha\cdot\cos\beta = \sin\beta\cdot\sin\alpha + \cos(\alpha + \beta)\cdot 1$
   $\cos(\alpha + \beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$

   $\cos(\alpha - \beta) = \cos(\alpha + (-\beta)) = \cos\alpha\cdot\cos(-\beta) - \sin\alpha\cdot\sin(-\beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$
   $\sin(\alpha - \beta) = \sin(\alpha + (-\beta)) = \sin\alpha\cdot\cos(-\beta) + \cos\alpha\cdot\sin(-\beta) = \sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta$

3. a. $BD = \sin\alpha$
   b. $CD = \sin\beta$
   c. $AB = \cos\alpha$
   d. $AC = \cos\beta$
   e. $BC = \sin(\alpha - \beta)$
   f. $AD = 1$
   g. $BD\cdot AC = AB\cdot CD + BC\cdot AD$
      $\sin\alpha\cdot\cos\beta = \cos\alpha\cdot\sin\beta + \sin(\alpha - \beta)\cdot 1$
      $\sin(\alpha - \beta) = \sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta$

4. a. $\cos 2\alpha = \cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \cos^2\alpha - \sin^2\alpha$
   b. $\cos 2\alpha = \cos^2\alpha - (1 - \cos^2\alpha) = 2\cos^2\alpha - 1$
   c. $\cos\theta = 2\cos^2(\theta/2) - 1$; $\cos^2(\theta/2) = (1 + \cos\theta)/2$
   d. $\sin^2(\theta/2) = (1 - \cos\theta)/2$

e. $\cos\left(\frac{\theta}{2}\right)=\sqrt{\frac{1+\cos\theta}{2}}$; $\sin\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos\theta}{2}}$

Exercise 3:

1. Since *ABCDE* is a regular pentagon, arcs *AB*, *BC*, *CD*, *DE*, *EA* are all 360º/5 = 72º . Since $\angle ADB$ intercepts arc *AB*, its measure is therefore 36º.
2. $\angle DAB$ and $\angle DBA$ intercept arcs *DB* and *DA* = 144º, so each measures 72º.
3. $\angle ABF = \angle DAB = 72^\circ$; $\angle FAB = \angle ADB = 36^\circ$; so $\Delta ABF \sim \Delta DAB$ by AA.
4. Since ΔDAB is isosceles and $\Delta ABF \sim \Delta DAB$, we know that ΔABF is also isosceles; also, $\angle DAF = \angle ADF$, so $AF = DF$ and ΔADF is isosceles.
5. Since ΔABF is isosceles, $AF = AB = 1$. Since ΔADF is isosceles, $DF = AF = AB = 1$. Also $y = AD = DB = DF + FB$ and $FB = y - DF = y - 1$.
6. $\frac{long\ side}{short\ side}=\frac{1}{y-1}$
7. $\frac{long\ side}{short\ side}=\frac{y}{1}$
8. Set the right hand side of the equations in #6 and #7 equal:

$$\frac{1}{y-1}=\frac{y}{1}$$

$$1=y^2-y$$

$$y^2-y-1=0$$

$$y=\frac{1+\sqrt{5}}{2}$$ *note that this is the golden ratio*

$$y\approx 1.618033989$$

9. Since *G* is the center of the circle, *GA* = *GB*.
   Also *AD* = *DB*
   Because *D* and *G* are both equidistant from *A* and *B*, they must lie on the perpendicular bisector of *AB*. Thus *DH* is perpendicular to *AB* and ΔADH is a right triangle. Since $\angle$DAH = 72º, we know that $\angle$ADH = 18º.
   Therefore, sin18º = AH/AD

10. $$\sin 18=\frac{AH}{AD}=\frac{1/2}{y}=\frac{1/2}{\frac{1+\sqrt{5}}{2}}=\frac{1}{1+\sqrt{5}}=\frac{\sqrt{5}-1}{4}$$

11. $\frac{\sqrt{5}-1}{4}\approx .3090169944,\qquad \sin 18\approx .3090169944$

12.

$$\cos 18 = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{16-6+2\sqrt{5}}{16}} = \frac{\sqrt{2\sqrt{5}+10}}{4} \approx .9510565163$$

Exercise 4:

1. $\sin 15^o = \sqrt{\dfrac{1-\frac{\sqrt{3}}{2}}{2}} = \dfrac{1}{2}\sqrt{2-\sqrt{3}} \approx .258819045$

2. $\cos 15^o = \sqrt{\dfrac{1+\frac{\sqrt{3}}{2}}{2}} = \dfrac{1}{2}\sqrt{2+\sqrt{3}} \approx .965925826$

3. $\sin 3^o = \sin 18^o \cos 15^o - \sin 15^o \cos 18^o = \left(\dfrac{\sqrt{5}-1}{4}\right)\left(\dfrac{2+\sqrt{3}}{2}\right) - \left(\dfrac{2-\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{2\sqrt{5}+10}}{4}\right)$

$=.052335956$

4. $\cos 3^o = \sqrt{1-(\sin 3^o)^2} \approx .998629535$

5. $\sin(3/2^o) = 0.026179483$; $\cos(3/2^o) = 0.999657325$; and $\sin(3/4^o)= 0.0130895959$,

6. a. Note that $\sin(3/2^o)$ is very close to half of $\sin(3^o)$ and $\sin(3/4^o)$ is even closer to half of $\sin(3/2^o)$. This shows that the sine is "almost" linear for small values. So, since 1 is 1/3 of the way between 3/4 and 3/2 we choose the number that is 1/3 of the way between .0130895959 and .0261779483, which is .01745238.

   b. $(4/3)\sin(3/4^o) = .017452794$; $(2/3)\sin(3/2^o) = .017451299$. The average of the two values is .017452046. The values calculated in a. and b. agree to six decimal places. Thus, to that many places, $\sin 1^o = .017452$.

7. First we find $\cos 1^o = \sqrt{1-\left(\sin 1^o\right)^2} \approx \sqrt{1-.017452^2} \approx .999848$

   Next we find $\sin\left(\dfrac{1}{2}\right)^o = \sqrt{\dfrac{1-\cos 1^o}{2}} \approx \sqrt{\dfrac{1-.999848}{2}} \approx .00872$

   Finally, $\cos(1/2^o) = .99996$.

8. $\sin 16^o = \sin 15^o \cos 1^o + \cos 15^o \sin 1^o \approx (.258819)(.999848) + (.965926)(.017452)$

$\approx .275637$

The Laws of Sines and Cosines
Brahmagupta's and Heron's Theorems
Teacher Notes

Description of Unit: The activities in this unit are presented as handouts for students to complete individually or in small groups. The Laws of Sines and Cosines are introduced using historical comments and proofs often not found in the traditional mathematics textbook. The Law of Sines is derived using a circumscribed triangle instead of area. The Law of Cosines blends Euclidean proof with current notation. In each unit, there are also student exercises, which may be done, in small groups or for homework. It is anticipated that problems and applications of these theorems will be assigned from the class text.

The proof of Brahmagupta's Theorem is presented as supplemental material to review a formula that may have been previously studied in a geometry course. The more familiar Heron's formula for area of a triangle follows naturally. The exercise on these theorems are appropriate for homework, extra credit, or small group completion. Initially students are given a general outline for the proof. Advanced students might enjoy the challenge of completing their own proof. Most students will prefer to follow the steps given.

Prerequisites: The Laws of Sines and Cosines only require the knowledge of basic geometric properties and right triangle trigonometry.

Brahmagupta's and Heron's Theorem require knowledge of the Law of Cosines, the triangle area formula using sine, and basic trigonometry identities. The detailed proof of Brahmagupta's Theorem is accessible to all students with this basic knowledge.

Materials: It is suggested that students confirm the Law of Sines and Cosines by measuring angles and sides of triangles that they have drawn. Some teachers may prefer to have students use ruler and protractor while others who have access to the *Geometer's Sketchpad* or similar software may prefer using technology.

If desired, the proof of Euclid's Book II, Proposition 12 may be accomplished in a whole classroom setting by making an overhead slide of the student page.

Law of Sines
Student Pages

You have just finished a unit on how to solve right triangles. Recall that in order to solve a right triangle, two pieces of additional information about the triangle are needed. What possibilities are there for the two additional pieces?

Note that two additional angles are not sufficient to solve the triangle. Why not?

Our goal is to solve triangles that do not have a 90º angle. These are called oblique triangles. Again, we are interested in the minimum information that must be given in order to assure us that a solution exists and is unique. Recall the three triangle congruence theorems from geometry. What are they?

These three theorems each say that if we know three pieces of information about a triangle, then the triangle is completely determined. In other words, any two triangles with the same three pieces of information are congruent. What we will see is that if we in fact know those three pieces of information, we can determine the unknown sides and angles by using one or both of the Law of Sines and the Law of Cosines.

Early mathematicians were concerned with solving triangles. Originally, their intent was to learn more about the sky above them and its relationship to the earth on which they lived --- thus the triangles they studied were often spherical triangles. But they still needed to solve plane triangles, and they worked out versions of the Laws of Sines and Cosines to accomplish this. They also worked out similar laws for solving spherical triangles. These procedures for solving both plane and spherical triangles are used, for example, in Ptolemy's *Almagest*, written about 150 CE, but they were probably discovered two or three centuries earlier.

We begin with the Law of Sines, which can be used to solve triangles when two angles and one side are known. The Law of Sines states that in any triangle, the three ratios of the sides to the sines of the angles opposite are all equal. That is,

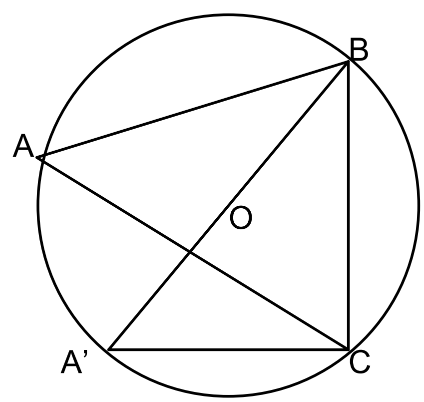
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Recall your properties of proportions, and you can see that this is equivalent to writing

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The basic nature of these ratios was known as far back as **Ptolemy** (c. 85-165) in his study of chord lengths. In his work with a circle of radius r circumscribed about a triangle, **Brahmagupta** (c. 598-660) found that $2r = a/\sin A$ which also equaled $b/\sin B$ and $c/\sin C$. **Abū l-Wafā** (940-998) systematized much of the existing trigonometry knowledge and developed the Law of Sines. **Al-Bīrūnī** (973-1048) actually wrote the sine law for plane triangles, and translators later brought it to Europe. If you were to read any of the works of these mathematicians, you might not recognize the Law of Sines. Remember that theorems were written out in words, and the symbols we use today came later in the history of mathematics.

First, let us consider a proof of the Law of Sines that Brahmagupta might have produced over 1300 years ago. We ask you to fill in the missing reasons for the various steps in the proof.



1. Given acute triangle ABC, draw the circumscribed circle.
   Euclid would have preferred us construct the circle - how could we have done that using only compass and straightedge?

2. Draw a diameter from B intersecting line segment AC. Name the other endpoint of the diameter A'.

3. What type of angle is BCA'? Why?

4. What angle has the same measure as A' and why?

5. Therefore, $\sin A = \sin A'$.

6. Using the right triangle $A'BC$, $\sin A'$ is the ratio of which two sides?

7. Show that your answer to 6 can be rewritten in the form $\sin A = a$ /diameter of the circle.

8. Conclude that $a/\sin A = 2r$, where r is the radius of the circle.

9. The same reasoning can be used to show $2r = b/\sin B$ and $c/\sin C$ by drawing a diameter from each of the other two vertices of the triangle.

10. Thus we have proved the law of sines: $a/\sin A = b/\sin B = c/\sin C$

Exercises:

1. Most textbooks do not use this proof of the Law of Sines. More commonly, the proof involves the area of a triangle. Look in your textbook and be prepared to explain that proof when you come to class tomorrow.

2. Do you really believe in this property of sines? - or are you obeying because you are a law-abiding citizen? Draw three triangles - one obtuse, one right, and one acute. Using your ruler and protractor, measure the three angles and the three sides of each triangle and compare the ratio of length of side divided by the sine of the opposite angle. Though the ratio will be different for each triangle, the three ratios in each triangle would be approximately equal. **Hint:** use centimeters and measure <u>very</u> carefully.

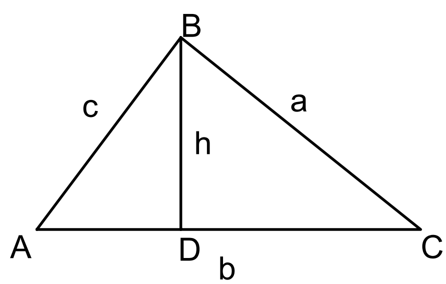
Law of Sines
Solutions and Teacher Notes

Proof of the Law of Sines:

1. To construct the center of the circle, construct the perpendicular bisectors of two of the sides of the triangle; their intersection will be the center of the circumscribed circle.
3. Angle BCA' is a right angle because it is an angle inscribed in a semicircle.
4. Angles A and A' have the same measure since they intercept the same arc BC.
6. $\sin A' = BC/A'B$, or side a divided by the diameter of the circle.

Notes on the homework assignment:

1. In case your text does not include the area proof, one is given below. One of the advantages of this proof is that the original triangle can be obtuse, and a most worthwhile formula for area is also derived (area = half of the product of two sides and the sine of the angle between them), a formula first discussed in the writings of Regiomontanus (1464). His work *De triangulis omnimodis* was printed in 1533 and it was distributed throughout Europe.



BD is an altitude (h) of triangle ABC; $\sin A = h/c$ and $h = c \sin A$
The area of a triangle is one-half base times height, which in this case is $\frac{1}{2} bc \sin A$
Using the same reasoning: $\sin C = h/a$, $h = a \sin C$, the area being $\frac{1}{2} ba \sin C$
These two expressions for area must be equal:

$$\frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C$$
$$c \sin A = a \sin C$$
$$a/\sin A = c/\sin C$$

Similar reasoning using a different altitude shows these ratios also equal to $b/\sin B$.

2. An alternate assignment would be for students to use *Geometer's Sketchpad* or similar software to verify the ratios for a triangle. By dragging a vertex, many triangles can be observed.

Further ideas: Assign problems from the text that students will solve using the Law of Sines. These will be triangles with two angles and one side given, though some texts do have problems involving two sides and a nonincluded angle before the ambiguous case is introduced. The ambiguous case could be introduced before the assignment is made.

Law of Cosines
Student Pages

As you have seen, the Law of Sines enables us to solve triangles given two angles and a side. We could solve triangles given two sides and the included angle or given three sides by drawing an altitude to divide the triangle into two right triangles and then using right triangle relationships. Can you explain how? (Try this with a triangle with sides 10 and 2 and included angle 40º and then with a triangle with sides 5, 6, and 7.)

However, wouldn't it be nice to have a formula to express these quantities in one equation and thus speed our computation? Thus the Law of Cosines becomes our next quest. As was the case with the Law of Sines, Ptolemy indeed was able to solve oblique triangles using procedures that are equivalent to the Law of Cosines. They are illustrated throughout the *Almagest*. Al-Bāttānī, one of many Islamic mathematicians involved in trigonometry, demonstrated the Law of Cosines for oblique spherical triangles around 920. Later it was written for plane triangles. Many scholars believe that Viete (1540-1603) first wrote the Law, as we know it today.

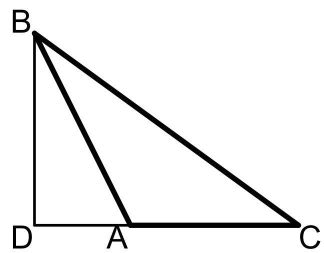
We also know that both the Law of Sines and the Law of Cosines were being used throughout Europe and Asia by the seventeenth century. In China, *A Treatise on (Astronomy and) Calendrical Science* was published in 1631 and contained chapters dealing with surveying and with trigonometric functions. Trigonometric identities as well as the Law of Sines and Law of Cosines were stated - all this "according to the new Western Methods."

Surprisingly, a version of the Law of Cosines appears in Euclid's *Elements* Book II, Propositions 12 (obtuse triangle) and 13 (acute triangle). As you will see, the proof depends on the Pythagorean Theorem and does not mention the word cosine. Recall that the Pythagorean Theorem states that in a right triangle, the square on the hypotenuse is equal to the sum of the squares on the legs. In stating his theorem and giving his proof, Euclid was dealing with actual geometric squares constructed on the two legs and the hypotenuse. Similarly, in what follows, you need to draw appropriate squares and rectangles. You should also fill in the reasons for each step in the space given beside the step. It may be easier if you write out the steps in algebraic form.

Euclid's Elements, Book II, Proposition 12: *In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.*

Let *ABC* be an obtuse-angled triangle having the angle *BAC* obtuse, and let *BD* be drawn from the point *B* perpendicular to *CA* produced.

I say that the square on *BC* is greater than the squares on *BA,* AC by twice the rectangle contained by CA*, AD*.



For, since the straight line *CD* has been cut at random at the point *A*, the square on DC is equal to the squares on *CA, AD* and twice the rectangle contained by *CA, AD*.

Let the square on *DB* be added to each; therefore the squares on *CD, DB* are equal to the squares on *CA, AD, DB,* and twice the rectangle *CA, AD*.

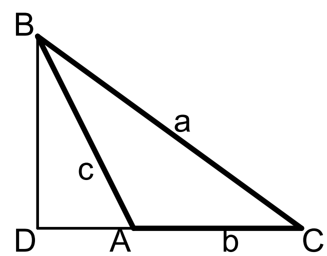
But the square on *CB* is equal to the squares on *CD, DB*, for the angle at *D* is right; and the square on *AB* is equal to the squares on *AD, DB*;

Therefore the square on *CB* is equal to the squares on *CA, AB* and twice the rectangle contained by *CA, AD* ;
So that the square on *CB* is greater than the squares on *CA, AB* by twice the rectangle contained by *CA, AD*

Q.E.D.

But what happened to the cosine? We are studying the Law of Cosines and the word is not even used in the statement of the proposition. Besides, the formula we will learn has a subtraction sign, not an addition sign. Let us look again at what we have done:

Redraw the triangle and label the sides with the lower case letters of the angles opposite them (just like we did with the Pythagorean Theorem).



$$CB^2 = CA^2 + AB^2 + 2CA \cdot AD \text{ becomes } a^2 = b^2 + c^2 + 2b \cdot AD$$

Now Euclid did not use trigonometric functions, but we can. Look at triangle ADB.
What is $\cos\angle DAB$?

We can rewrite that statement as AD = $c \cos\angle DAB$.
Now, $\angle$DAB is the supplement of $\angle$CAB, and the cosines of supplementary angles are opposite in sign,
Therefore, $\cos\angle DAB = -\cos\angle CAB$.
Substituting, we get that $AD = -c \cos\angle CAB$. We conclude that
$a^2 = b^2 + c^2 - 2bc\cos\angle CAB$ or, ignoring triangle ADB, $a^2 = b^2 + c^2 - 2bc\cos A$.
How would you write the formula if C were the obtuse angle?

What about if B were the obtuse angle?

What happens to the Law of Cosines if $\angle$C is a right angle?

In the Book II, Proposition 13 Euclid gave an analogous result for acute-angled triangles:

In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular fall, and the straight line cut off within by the perpendicular towards to the acute angle.

Give a proof of this result modeled on the proof of Proposition 12. Then translate this result into a result using cosines similar to the results given above for Proposition 12.

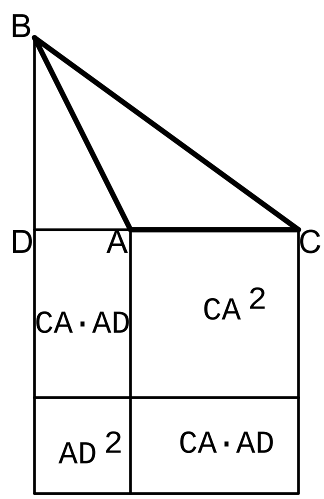
Law of Cosines
Solutions and Teacher Notes

Here is the complete proof of Proposition II, 12, with completed diagram and algebraic equivalents:

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Let *ABC* be an obtuse-angled triangle having the angle *BAC* obtuse, and let *BD* be drawn from the point *B* perpendicular to *CA* produced.

I say that the square on *BC* is greater than the squares on *BA,* AC by twice the rectangle contained by *CA, AD*.



For, since the straight line *CD* has been cut at random at the point *A*, the square on *DC* is equal to the squares on *CA, AD* and twice the rectangle contained by *CA, AD* .

$$CD^2 = CA^2 + AD^2 + 2\ CA \bullet AD$$

[this is algebraically equivalent to $(a+b)^2 = a^2 + 2ab + b^2$]

Let the square on *DB* be added to each; therefore the squares on *CD, DB* are equal to the squares on *CA, AD, DB,* and twice the rectangle *CA, AD*.

$$CD^2 + DB^2 = CA^2 + AD^2 + 2CA \bullet AD + DB^2$$

[note: DB^2 is the square on the other leg of right triangle *BDC*]

But the square on *CB* is equal to the squares on *CD, DB*, for the angle at *D* is right; and the square on *AB* is equal to the squares on *AD, DB*;

$$CB^2 = CD^2 + DB^2$$
$$AB^2 = AD^2 + DB^2$$

[Pythagorean Theorem using triangles *BDC* and *ADB*]

Therefore the square on *CB* is equal to the squares on *CA, AB* and twice the rectangle contained by *CA, AD*;

$$CB^2 = CA^2 + AB^2 + 2CA \bullet AD$$

[substitution]

So that the square on *CB* is greater than the squares on *CA, AB* by twice the rectangle contained by *CA, AD*

Q.E.D.

Answers to the discussion about cosine:

If *C* is the obtuse angle, then the formula becomes $c^2 = a^2 + b^2 - 2ab\cos C$.
If *B* is the obtuse angle, then the formula becomes $b^2 = a^2 + c^2 - 2ac\cos B$.
If *C* is a right angle, the law of cosines reduces to the Pythagorean Theorem.

A complete proof of Book II, Proposition 13 is given below:

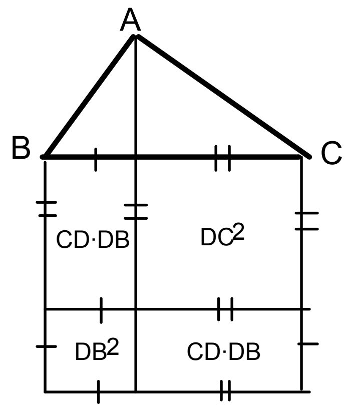
In acute angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Let *ABC* be an acute-angled triangle having the angle at *B* acute, and let *AD* be drawn from the point *A* perpendicular to *BC* ;

I say that the square on *AC* is less than the squares on *CB, BA* by twice the rectangle contained by *CB, BD* .

For, since the straight line *CB* has been cut at random at *D* , the squares on *CB, BD* are equal to twice the rectangle contained by *CB, BD* and the square on *DC* .

$$CB^2 + DB^2 = 2CB \bullet BD + DC^2$$



It helps to label congruent segments to see that CB^2 is composed of $DB^2 + 2CD{\bullet}DB + CD^2$:
a large square, a small square, and two congruent rectangles. When another large square (DC^2) is added, there are now two large squares that, when added to the two rectangles, produce two rectangles *CB* by *DB*. Small square DC^2 remains.

Let the square on *DA* be added to each;
therefore the squares on *CB, BD, DA* are equal to twice the rectangle contained by *CB, BD* and the squares on *AD, DC* .

$$DA^2 + CB^2 + DB^2 = 2\, CB{\bullet}BD + DC^2 + AD^2$$

But the square on *AB* is equal to the squares on *BD, DA* , for the angle at *D* is right;
and the square on *AC* is equal to the squares on *AD, DC* ;

$$AB^2 = BD^2 + DA^2 \qquad \text{and} \qquad AC^2 = AD^2 + DC^2$$

therefore the squares on *CB, BA* are equal to the square on *AC* and twice the rectangle *CB, BD* ,

$$CB^2 + BA^2 = AC^2 + 2\, CB{\bullet}BD$$

so that the square on *AC* alone is less than the squares on *CB, BA* by twice the rectangle contained by *CB, BD* .

$$AC^2 = CB^2 + BA^2 - 2\, CB{\bullet}BD$$

Q.E.D.

Again, moving to modern notation, using lower case letters for the sides of the triangle, the above conclusion could be rewritten as

$$b^2 = a^2 + c^2 - 2a\,BD$$

$$\cos B = BD/c \text{ and therefore } BD = c \cos B$$

$$\text{By substitution, } b^2 = a^2 + c^2 - 2ac\cos B .$$

Analogous expressions can be obtained for the squares on the other two sides.

If you (or your school) do not have access to a copy of *The Elements*, it is recommended that you purchase the translation, with commentary, by Thomas Heath. Some editions do not have Heath's explanations of Euclid's proofs. The discussion following each proposition helps the reader understand some of Euclid's statements and make reference to the works of other mathematicians.

As with the Law of Sines, there is probably a different proof in your text that you will want students to review either in class or for homework. Again, they can verify the Law of Cosines by drawing triangles and measuring or by using *Geometer's Sketchpad*. There is an interesting internet site that uses animation and area to illustrate this theorem. The address is:
http:www.edc.org/LTT/ConnGeo/cosines.html.

Brahmagupta's and Heron's Theorems
Student Pages

In your geometry class, you may have learned Heron's formula (also referred to as Hero's formula) for calculating the area of a triangle from the lengths of its sides (Heron of Alexandria lived in the first century). There is a similar formula due to for calculating the area of a cyclic quadrilateral in terms of its sides. You may recognize Brahmagupta's name (born in India in 598) from the discussion of the Law of Sines and other work he did in trigonometry. Like other mathematicians of India he left no proof of his work, only algorithms for solving problems. The proof outlined below, one of many possible, is for you to complete as an exercise. You will apply important trigonometry – the Law of Cosines, Law of Sines, and several trigonometric identities. You will also use quite a bit of algebra, which may seem tedious at times, but then mathematics sometimes does require detail, exactness, and care.

Though he probably discovered his area property independently of Heron, Brahmagupta's Theorem has been called a generalization of Heron's Theorem. However, Heron's Theorem has also been called a special case of Brahmagupta's Theorem. Regardless, it is a gem. The appeal of beauty is what drives certain people deeper into mathematical activity. The thrill of discovering new relationships, which connect algebra and geometry, is exciting. Brahmagupta seems to have played with a great deal of beautiful mathematics that did not directly apply to his regular job as an astronomer. This should be no surprise since he titled his astronomical work *The Opening of the Universe* and wrote it in poetic stanzas.

The following exercise leads you step by step through the proof of Brahmagupta's formula for the area of a cyclic quadrilateral. Some of you may prefer to generate the proof on your own. A general outline is given above the diagram. These comments will also help you see the "big picture" rather than get caught up in details. See how far you can go on your own before looking at the numbered steps. If you prefer, follow the more specific procedure given below the diagram.

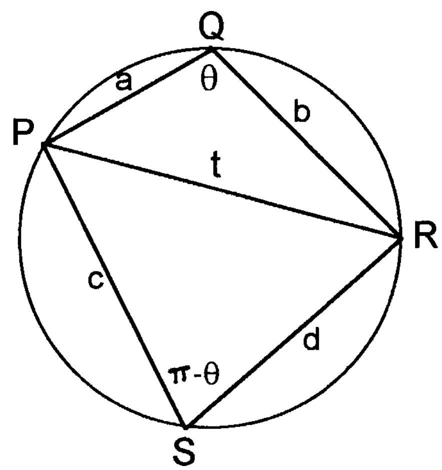
Keep in mind that the formula is not valid for any quadrilateral - only those that are cyclic. The vertices of the quadrilateral must be shown to lie on the circumference of a circle. One method is to show that opposite angles are supplementary.

Enjoy your excursion through the proof of Brahmagupta's Theorem.

Brahmagupta's Theorem:

Let P, Q, R, S be points on a circle, lettered in clockwise order, and let a, b, c, and d be the lengths of the sides of quadrilateral PQRS as shown. Then the area A of quadrilateral PQRS is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ *, where s is the semi perimeter (a+b+c+d)/2.*

General Outline of Proof:



Find an equation relating *a, b, c, d*, θ, and π – θ by first using the Law of Cosines on triangles *PQR* and *RSP*.

To find the area of quadrilateral *PQRS*, find the areas of triangles *PQR* and *RSP* and add them.

Find a substitution that will enable the area equation to be written in terms of cosine rather than sine (since you already have an equation for cos θ in terms of *a, b, c, d*)

Substitute your formula for cos θ then do lots of algebra to cajole the equation into the form Brahmagupta used.

Proof:

Begin with linking triangles *PQR* and *RSP* through trigonometry. First, since *PR = t* is the common side, apply the Law of Cosines to express t^2 two different ways. Simplify cos(π - θ) in terms of cos θ. Soon your proof will need a substitution for cos θ. Right now is a good time to set the two t^2 expressions equal to each other and solve for cos θ. [Steps 1 - 4]

If $\angle PQR = \theta$, then, as pictured, $\angle RSP = \pi - \theta$ since an angle inscribed in a circle equals half its intercepted arc and arcs *PQR* and *RSP* total 2π.

1. Apply the Law of Cosines to triangle *PQR* to express t^2 ($t = PR$) in terms of a, b, and $\cos\theta$.

2. Apply the Law of Cosines to triangle *RSP* to express t^2 in terms of c, d, and $\cos(\pi - \theta)$.

3. Rewrite your answer to #2, using the identity $\cos\theta = -\cos(\pi - \theta)$

4. Since the expressions in #1 and #2 are both equal to t^2, set them equal to each other and then solve for $\cos\theta$ in terms of a, b, c, and d.

The second link between the triangles is that together they make up the cyclic quadrilateral. Express each area is terms of sine; simplify $\sin(\pi - \theta)$ in terms of $\sin\theta$. Recall that A represents the area of quadrilateral *PQRS*. Write *A* as a sum of the two triangular areas. [Step 5]

5. The area of triangle *PQR* is ½ $ab\sin\theta$. Why? The area of triangle *RSP* is ½ $cd\sin(\pi - \theta)$. Add the two areas, use the fact that $\sin\theta = \sin(\pi - \theta)$ to express *A*, the area of *PQRS*, in terms of a, b, c, d, and $\sin\theta$.

Naturally you are wondering why the statement of Brahmagupta's Theorem contains no trigonometric functions. Here is where the cosine and sine functions are eliminated. [Steps 6 - 8]

6. Square your answer from #5 but do not carry out the multiplication on the right side of the equation

7. Substitute $(1 - \cos^2 \theta)$ for $\sin^2 \theta$.

8. In your answer to #7, substitute your expression for $\cos \theta$ from #4.

In algebra you learned how to simplify a compound fraction. Do it now. You will end up with an equation having $16A^2$ on one side (the left side, please). After all, the purpose here is to eliminate fractions. The right side is special in that it contains several expressions that are the difference of two squares. [Steps 9-14]

9. Fill in the blanks, using a common denominator in the square bracket:

$$A^2 = \frac{(ab+cd)^2}{4}\left[\frac{4(\quad + \quad)^2 - (\quad + \quad - \quad - \quad)^2}{4(\quad + \quad)^2}\right]$$

10. Multiply by 16 and cancel factors of $(ab+cd)^2$. Fill in the blanks with your answers:

$$16A^2 = 4(\quad + \quad)^2 - (\quad + \quad - \quad - \quad)^2$$

11. Factor the expression on the right hand side in #10 as a difference of squares, using the form $x^2 - y^2 = (x-y)(x+y)$ to fill in the blanks.

$$16A^2 = [2(\quad + \quad) + (\quad + \quad - \quad - \quad)]\,[2(\quad + \quad) - (\quad + \quad - \quad - \quad)]$$

12. In each square bracket, group the terms with a and **b** in one set of parentheses and those with c and d in the other.

$$16A^2 = [(\quad + \quad + \quad) - (\quad - \quad + \quad)]\,[(\quad + \quad + \quad) - (\quad - \quad + \quad)]$$

13. Rewrite your answer to #12, expressing the terms in each set of parentheses as the square of a binomial.

$$16A^2 = [(\quad + \quad)^2 - (\quad - \quad)^2]\,[(\quad + \quad)^2 - (\quad - \quad)^2]$$

14. Factor the expression in each square bracket from #13 as a difference of squares using the form $x^2 - y^2 = (x - y)(x + y)$ to fill in the blanks.

$$16A^2 = [(\quad + \quad) + (\quad - \quad)]\,[(\quad + \quad) - (\quad - \quad)]\,[(\quad + \quad) + (\quad - \quad)]\,[(\quad + \quad) - (\quad - \quad)]$$

Now we figure out what all this has to do with the semiperimeter. [Steps 15 - 17]

15. Note that $(a + b + c + d)$ is the perimeter of *PQRS*. However each expression inside a bracket in #14 unfortunately has a single negative term. What can be done to turn $(a + b + c - d)$ into $(a + b + c + d)$? Rewrite $(a + b + c - d)$ as $(a + b + c + d - 2d)$ = perimeter – $2d$. Now you write the remaining three expression (from the brackets in #14) so that each contains the perimeter of quadrilateral *PQRS*:

$$a + b - c + d = \text{perimeter } - ?$$
$$a - b + c + d = \text{perimeter } - ?$$
$$-a + b + c + d = \text{perimeter } - ?$$

16. Brahmagupta's Formula uses the semiperimeter (s) rather than the perimeter, so replace the perimeter with $2s$:

perimeter – $2d$ = $2s - 2d$ = $2(s - d)$
perimeter – $2c$ =
perimeter – $2b$ =
perimeter – $2a$ =

17. Substitute your final expressions from #16 into the equation from #14.

The final step !

18. Divide by 16 and take the positive square root.

AHA!! Brahmagupta's Formula appears!

Now, how about Heron's Formula?

Heron's formula is for the area of a triangle; Brahmagupta's for a quadrilateral.
A quadrilateral has four sides, a triangle only three. So simply let $d = 0$.
Presto, a triangle with area =

Heron included this formula in *Metrika*, one of several practical mathematics books he wrote in the first century C.E. The astronomer/mathematician al-Bīrūnī (973-1055) wrote that Archimedes knew of Heron's formula some three centuries before Heron. However, much of Archimedes' work is lost, so there is no evidence at this time to support or refute al-Bīrūnī's claim.

Answers to the Proof of Brahmagupta's Theorem

1. $t^2 = a^2 + b^2 - 2ab\cos\theta$

2. $t^2 = c^2 + d^2 - 2cd\cos(\pi - \theta)$

3. $t^2 = c^2 + d^2 + 2cd\cos\theta$

4. $a^2 + b^2 - 2ab\cos\theta \;=\; c^2 + d^2 + 2cd\cos\theta$

   $$\cos\theta = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

5. $A \;=\; \dfrac{ab\sin\theta}{2} + \dfrac{cd\sin\theta}{2} \;=\; \dfrac{(ab + cd)}{2}\cdot\sin\theta$

6. $A^2 \;=\; \dfrac{(ab + cd)^2}{4}\cdot\sin^2\theta$

7. $A^2 \;=\; \dfrac{(ab + cd)^2}{4}\cdot(1 - \cos^2\theta)$

8. $A^2 \;=\; \dfrac{(ab + cd)^2}{4}\cdot\left\lfloor 1 - \left(\dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}\right)^2 \right\rfloor$

9. $A^2 \;=\; \dfrac{(ab + cd)^2}{4}\cdot\left\lfloor \dfrac{4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2} \right\rfloor$

10. $16A^2 = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$

11. $16A^2 = \left[2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\right]\left[2(ab + cd) - (a^2 + b^2 - c^2 - d^2\right]$

12. $16A^2 = \left[\left(a^2 + 2ab + b^2\right) - \left(c^2 - 2cd + d^2\right)\right]\left[\left(c^2 + 2cd + d^2\right) - \left(a^2 - 2ab + b^2\right)\right]$

13. $16A^2 = \left[(a + b)^2 - (c - d)^2\right]\left[(c + d)^2 - (a - b)^2\right]$

14. $16A^2 = \left[(a + b) + (c - d)\right]\left[(a + b) - (c - d)\right]\left[(c + d) + (a - b)\right]\left[(c + d) - (a - b)\right]$

15. perimeter – $2c$
perimeter – $2b$
perimeter – $2a$

16. $2s - 2c = 2(s - c)$
$2s - 2d = 2(s - b)$
$2s - 2a = 2(s - a)$

17. $16A^2 = [2(s - d)][2(s - c)][2(s - b)][2(s - a)]$

18. $A = \sqrt{(s - a)(s - b)(s - c)(s - d)}$

Charting the Heavens
Teacher Notes

Description of Unit: In this unit we demonstrate how the early astronomers used simple plane trigonometry to determine distances, times and orbits of the major bodies in our solar system.

The highlights of their efforts are:

Finding the distance from the moon to the earth.
Finding the distance from the sun to the earth.
Finding the circumference of the earth.
Finding the radii of the moon and the sun.
Finding the distances from Venus and Mercury to the sun.
Finding the distances from Mars and the outer planets to the sun.
Finding the length of time of a Martian year.

Each highlight is written as a student page with historical comments and directions for duplicating the calculations. Accompanying each of the student pages is a page of notes for the teacher, with additional historical information, relevant additional sources, and the completed calculations.

Prerequisites: Many of the student pages involve nothing more difficult than right triangle trigonometry and can be used as soon as that topic appears in the curriculum. It has been successfully tested with second semester Geometry students as well as first semester Trigonometry students. The only section, which requires the Law of Sines or Cosines is the one, titled *The Distance from Mars to the Sun.*

Materials:

1. A scientific calculator.

Some optional materials might include

2. A flashlight and a few spheres, to illustrate the sun/moon/earth right triangle.
3. *Geometer's Sketchpad* to illustrate the point that the orbit of Venus produces a right angle at Venus, when that planet appears to be farthest from the sun.
4. *Geometer's Sketchpad* to help visualize the simultaneous orbiting of Mars and the earth, in the section on determining the length of a Martian year.
5. An almanac or similar resource to independently verify student work.

Charting the Heavens
Student Pages

A single person did not develop trigonometry, nor by a single civilization, and contrary to the implication of many modern textbooks, it was not developed primarily to find heights of mountains, distances to shores, or boundaries of nations. The primary driving force behind the creation of this beautiful piece of mathematics was to conquer or tame the heavens.

Regardless of its location on the earth, each civilization was able to observe the sun, the moon, and various constellations, all moving in an intricate dance in the sky. People were able to determine that the movements of the sun and moon were related to a wide variety of terrestrial phenomena, such as tides, hot or cold weather, planting seasons, menstrual cycles, availability of wild animals for food, and number of daylight hours in a 24 hour period. To understand and be able to predict the motions of the heavenly bodies meant to be able to make predictions about those corresponding matters on earth, which enabled people to be able to take precautions and make preparations to better insure survival. Seen in this way, charting the heavens was a life-or-death issue.

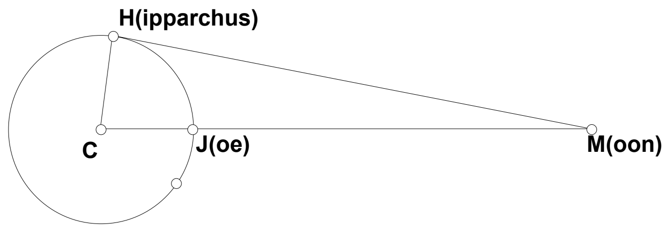
A particular puzzle for those societies was the path of "wandering stars," now known as planets. The sun's trip was relatively predictable. It rose in almost the same place each morning, set in almost the same place each evening, and traveled almost the same path each day. The moon's trip was only slightly more difficult to chart. Most of the stars exhibited only a rotation about some fixed place in the heavens. But <u>some</u> stars exhibited an erratic behavior. They would seem to stop and reverse direction mid-path. Their journey was out of character with the rest of the heavens and naturally piqued the interest of the ancient mathematicians. Note the absence of the phrase "ancient astronomers." Someone with training in the mathematical arts of that society was naturally expected to bend those talents to this most serious endeavor, that of making sense of the heavenly movements. Mathematicians <u>were</u> astronomers. In this unit we look at seven highlights of their efforts.

The distance from the moon to the earth
The distance from the sun to the earth
The circumference of the earth
The radii of the moon and the sun
The distance from Venus and Mercury to the sun
The distance from Mars and the outer planets to the sun
The length in days of the Martian year

For a complete index of mathematicians and a further overview of astronomy, here is a good website.

www-history.mcs.st-and.ac.uk/history/
Accessed June 27, 2001

The Distance from the Moon to the Earth
Hipparchus (190 - 120 B.C.)



Suppose Hipparchus and Joe are standing at two spots on the earth from which the moon is visible at the same time, Hipparchus at a location in which the moon is on the horizon and Joe at a place where the moon is directly overhead.

Imagine a triangle connecting Hipparchus, the moon, and the center of the earth. This triangle will be a right triangle, whose hypotenuse contains Joe. If the distance between Joe and Hipparchus is found (length of arc *JH*, suppose it to be 6218 miles) and the radius of the earth be known (it is 4000 miles), then the measure of the arc *JH* (and the central angle) may be determined. Find this value.

Arc *JH* = _________

The central angle *HCJ* cut off by arc *JH* is an angle of the right triangle *CHM*, and since the short leg of the triangle (*CH*, radius of earth) is known, the length of the hypotenuse may be determined using the cosine of $\angle HCJ$. Find that length.

Length of Segment *CM*= _______

The distance from the moon to the earth is that length minus the radius *CJ* of the earth. What is your calculation for the distance between the earth and the moon?

Distance = _____________

A minor problem to consider. Since the moon is constantly in motion, if the moon is determined to be on the horizon for Hipparchus and directly above for Joe, but at different times, we have different triangles. It was required that these observations be made at the same time. In our present age, we could simply arrange for simultaneous measurements by either synchronizing our watches or conversing by cell phone. However neither of those options was available to Hipparchus. Instead, he utilized a time when the moon would make a grand simultaneous signal to everyone on the earth….an eclipse. Pretty clever, huh?

We now know the distance to the moon to within 5 cm because Apollos 11, 14, and 15 dropped off mirrors on the surface. Now we just shoot a laser to the mirrors and measure the time it takes to return. See http://nssdc.gsfc.nasa.gov/cgi-bin/database/www-nmc?69-059C-04. Accessed June 27, 2001. Also, a nice website on Hipparchus' work finding the distances to the moon and to the sun can be found at http://spaceboy.nasda.go.jp/note/shikumi/e/Shi08_e.html Accessed June 27, 2001.

Distance from Moon to Earth
Teacher Notes

Answers to the exercises are:

Arc *JH* = 89.066^o
Length of Segment *CM*= 245389
Distance= 241389

An alternative way to discover the distance would be to break the students into groups and assign each group a different value for the length of arc *HJ*.

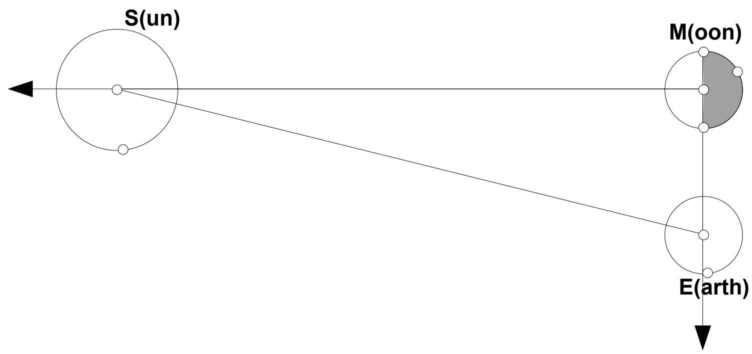
Here is a table of the corresponding values of *CM* for various *HJ*.

HJ	6000	6050	6100	6150	6200	6250	6300
Angle *C*	85.94	86.66	87.38	88.09	88.81	89.52	90.24
CM	56.547	68.653	87.373	120,155	192,355	482,146	?
JM	52,547	64,653	83,373	116,155	188,355	478,146	?

From the table above, you can see that in order to have determined that the distance from the earth to the moon is 240,000 miles, the length of arc *HJ* must be somewhere between 6200 miles and 6250 miles. This might be an appropriate time to ask the class to determine the value of arc *HJ* which will produce the correct distance.

Distance from the Sun to the Earth
Aristarchus (310- 230 B.C.)
Student Page

At certain times of the month, the moon appears to be neither a full moon nor a slim crescent, but to be half illuminated and half in shadow. At such a time, its center forms the vertex of a right angle whose rays extend through the centers of the sun and earth.



If, at such a time, the sun is also visible from the earth, then the angle *SEM* may be measured. Suppose it to be 87 degrees, the value stated by Aristarchus. Aristarchus could then determine by a geometrical method the ratio of the distance of the sun to the earth to the distance of the moon to the earth. Calculate this ratio via trigonometry, a method not yet invented in the time of Aristarchus.

Ratio of *SE* to *ME*______________________

If the distance from the earth to the moon is known (about 240,000 miles), then you can calculate the actual distance from the earth to the sun. Find that distance.

Distance from earth to sun= ___________

Some minor problems to consider. First, the angle *SEM* will be very nearly 90 degrees. (Aristarchus claimed it was 87 degrees.) Consequently, minor errors in its measurement will have severe repercussions in the distances determined, since the tangents of angles grow rapidly as the angles approach 90 degrees. Also, if the sun is close to the horizon, the path of light, *SE*, will be bent, making the apparent measurement inaccurate. Determining exactly WHEN the moon is exactly half illuminated is also subject to error, leading to more potential inaccuracies.

Note that in order to find the distance to the sun, it is required that we know the distance to the moon. Earlier we noted that to find the distance to the moon it was required to know the radius of the earth.   Since knowing the radius of the earth was critical to determining the distances to the sum and moon, that became an even more significant question, one that was first solved by Eratosthenes in about 200 BCE.

Distance from the Sun to the Earth
Teacher Notes

Aristarchus asserted that angle *SEM* was 87 degrees. It follows that the ratio of *SE* to *ME* is the secant of 87 degrees, or about 19.1. If we then know that the distance from the earth to the moon is 240,000 miles, it follows that *SE*, the distance from the sun to the earth, is approximately 4.6 million miles.

Angle *SEM* is in fact about 89.85 degrees. This gives a strikingly larger answer for the distance from the sun to the earth. This indicates the tremendous importance that accurate measurement played in the resulting theories about the heavens.

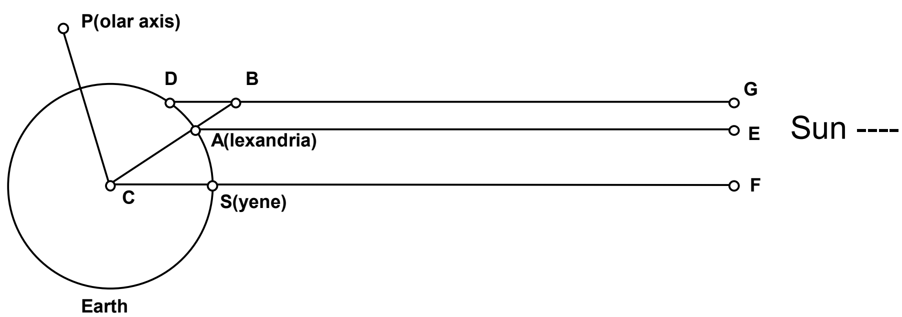
$SE:ME = \sec(89.85^\circ) = 381.97.$ ……… $SE = 91.7$ million miles

Note that neither of these results depends on knowing whether the earth revolves about the sun or the sun about the earth. The Greek world was split over this issue. Ptolemy and Aristotle led the geocentric (earth center) approach, while Hipparchus, Aristarchus, and Eratosthenes all favored the heliocentric (sun center) approach. The public popularity of Ptolemy and Aristotle, together with the lack of any evidence of the earth's motion, persuaded much of the scientific community to side with the geocentric approach. It was not until the seventeenth century that Kepler and Galileo were able to persuade astronomers, and ultimately the public, that the earth traveled around the sun, although Copernicus asserted this in a major astronomical work in 1545.

Both of the Greek factions were aware, however, that the earth was spherical and not flat. Texts that suggest that the flat earth theory was still in vogue among the scientific community at the time of Columbus are in error.

The Circumference of the Earth
Eratosthenes (280 – 195 B.C.)
Student Page

The Greeks had observed that in the town of Syene in Egypt, water at the bottom of a very deep well within the city limits reflected the sun at noon on the longest day of the year. That meant that the sun was directly overhead at that time. In the diagram below, this means that C, S, and F are collinear. Eratosthenes exploited this occurrence to determine the circumference of the earth. In Alexandria, where he lived, he erected a pole (*AB*) perpendicular to the earth's surface on that same longest day. He then measured the lengths of the shadows at various times of the day (*AD*). Since he had no wristwatch to tell him when noon was, he determined this by finding the time of the shortest shadow.



By knowing the length of that shortest shadow and the height of his pole, he was able to determine that the angle *BAE* (= angle *DBA*) was equal to 1/50 of 360 degrees, namely 7 1/5 degrees. It followed that angle *ACS* was also 7 1/5 degrees. Thus arc *AS* on the sphere of the earth had the same measure. By knowing that the distance between Syene and Alexandria was 5000 stades, Eratosthenes was able to determine the circumference of the earth. What value did he get?

Circumference of the earth=______________

There is one minor problem to consider, from our modern perspective. How long is a stade? Unfortunately, there are several different stades from ancient times. We do not know which definition Eratosthenes used. However, we do know the modern distance between Alexandria and Syene, which is now underwater as part of the Aswan Dam Project. This distance is 493 miles, so we can rework the problem using these modern units of measurement. What value do you get this way?

____

For a great website on this experiment, including information about subsequent attempts to refine Eratosthenes' answer, go to the following website:
www.eso.org/outreach/spec-prog/aol/market/collaboration/erathostenes/
Accessed June 27, 2001.

The Circumference of the Earth
Teacher Notes

To determine the circumference in the last step, the student must solve the proportion below.

$$\frac{7\frac{1}{5}}{360} = \frac{5000\,\text{stades}}{\text{circumference}}$$

This yields an estimated circumference of 250,000 stades. Although we do not know for sure which of the many types of stades was used for this measurement, one of the common choices for that length corresponds to about one tenth of our mile. This choice would provide us with a circumference of 25,000 miles and therefore with the radius of the earth about 3980 miles. The student problem using the value of 493 miles yields a polar circumference of 24,650 miles and therefore a polar radius of about 7846 miles. (We use the phrase "polar" as a reminder that the shape of the earth is not truly spherical, but flattened at the poles).

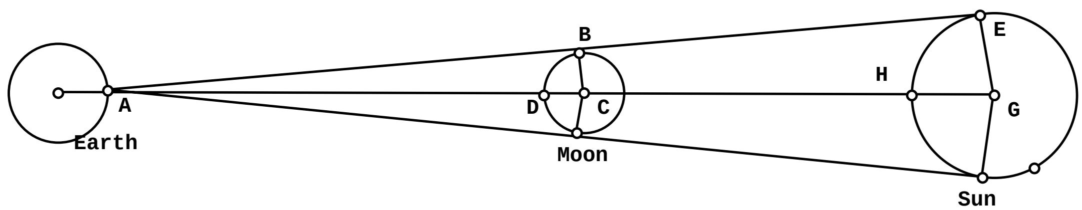
These values for the radius and circumference of the earth are quite accurate, and it is unfortunate and historically significant that a second value, about 25% less, pre-empted the value found here by Eratosthenes. Poseidonius (130 - 51 BCE) used a similar technique, measuring the change in a star's elevation above the horizon at both Rhodes and Alexandria. Although the angle measured was of reasonable accuracy, the distance between the cities was taken to be about 375 miles by a reporting geographer, Strabo. The distance is actually about 500 miles. Since this incorrect distance is about 25% less than the true distance, it yielded a circumference that is also about 25% less than the correct one. The historical significance of this popularly accepted but erroneous value is that Columbus believed Strabo's calculations were accurate. He used this smaller circumference value, as well as an erroneous notion of the extent to the Asian continent, to contend that sailing west from Spain to the Indies was feasible. How could such a large error have been made in the distance between Rhodes and Alexandria? They were separated by the Mediterranean rather than by land, and distances across water were extremely difficult to measure accurately at this time.

Today, we usually use the rough version of 4000 miles for the radius of the earth, remembering that not all earth radii are alike. The shape of the earth is called "an oblate spheroid", indicating that the great circle at the equator is larger than the great circles through the poles. How did people verify that the shape was not spherical? A surveying crew went out and measured one degree of longitude at two different latitudes, and reached statistically significantly different results. More about that can be found in the website below:
http://www-history.mcs.st-and.ac.uk/history/ Accessed June 27, 2001

The Radii of the Moon and Sun
Aristarchus (310-230 B.C.) Hipparchus (190-120 B.C.)
Student Page

It is fortuitous for mathematicians that during a total eclipse of the sun, the moon completely covers the solar disk, but just barely. The angle from the top of the disk to the eye to the bottom of the disk is about 1/2 degree (so angle *BAC* is about 1/4 of a degree), although Aristarchus significantly overestimated it to be about 2 degrees. Since the sun and moon *look* the same size from the earth, the ratio of their actual diameters must be the same as the ratio of their distances from the earth. This is because triangles *ABC* and *AEG* are similar, so that the ratio of *BC* to *EG* equals the ratio of *AC* to *AG*.

This was the contribution of Aristarchus, tying the radius of each sphere to its distance from the earth in a direct proportion.



To find the radius of the moon, consider the right triangle *ABC*. Let's amend Aristarchus's measurement of angle *BAC* to be 1/4 of a degree. If the distance from the earth to the moon is known, which Hipparchus was finally able to do in about 140 BC (see the handout on distance from the moon to the earth), enough information about the right triangle is known to find the missing sides and angles. Use the value of 240,000 miles for the distance from the earth to the moon and determine the radius *BC* of the moon.

Radius of the Moon=_______________

There is a minor problem. Is the distance from the earth to the moon given by *AD* or by *AC*? According to Hipparchus, who used the eclipse of the moon to signal when to make the measurements, the distance should be *AD*. This makes for a slightly more complicated algebra problem. Use the value you did not use in the first part to recalculate the radius of the moon.

New value for Radius of Moon__________

Good news. Did you discover that the two answers for the radius of the moon differ by less than 1/2% ? Since Hipparchus was not able to obtain nearly that accuracy in his distance from the earth to the moon, hindsight says that we could have chosen the easier equation. Nowadays, we know the distance to the surface of the moon to within 5 cm, because one of the Apollo missions dropped off a mirror on the surface, and we just shoot a laser beam to the mirror and record the time it takes to reflect back. See the handout on distance to the moon for a great website on the subject.

The distance from the earth to the sun is about 93,000,000 miles. What is the radius of the sun? _______________________

The Radii of the Moon and the Sun
Teacher Notes

Concerning which distance to the moon we should use, *AC* or *AD*.

Let us use 240,000 miles for the distance *AD* and see what value that gives for the radius of the moon.

First solve the equation $\frac{r}{r+240{,}000} = \sin(\frac{1}{4})$ where the angle is in degrees, to find the radius of the moon. This gives a radius of about 1051 miles.

If we consider 240,000 to be the distance *AC*, the slightly easier equation to solve is $\frac{r}{240{,}000} = \sin\frac{1}{4}$. This gives a radius of about 1047 miles.

Pretty close. Let's call it 1050 miles. Current measurements are about 1090 miles for the radius of the moon. Isn't it amazing how close we can get with elementary trigonometry?

To find the radius of the sun, let's use *AG* to represent the distance from the earth to the sun, since the equation is so much easier than if we use *AH*, and the results will be so similar.

$$\frac{R}{93{,}000{,}000} = \sin\frac{1}{4}$$

Solving this gives a radius of slightly over 400,000 miles. Current measurements for the radius of the sun are about 436,000 miles.

In 1688, the famous British mathematician, John Wallis, was able to translate the Greek text of Aristarchus into Latin, so that the current mathematics world could understand it. The next two pages are from Aristarchus's book. The first is the frontispiece of Wallis' translation. Your students should be able to decipher where it says *De Magnitudinibus & Distantiis Solis & Lunae*, as well as the author's name, Aristarchus of Samos. The second is the page from a Greek manuscript of the text whose diagram most closely resembles the diagram on the student page. A transparency of either page is a definite attention getter.

ΑΡΙΣΤΑΡΧΟΥ ΣΑΜΙΟΥ

Περὶ μεγεθῶν καὶ ἀποστημάτων Ἡλίου καὶ Σελήνης,

ΒΙΒΛΙΟΝ.

ΠΑΠΠΟΥ ΑΛΕΞΑΝΔΡΕΩΣ

Τοῦ τῆς Συναγωγῆς ΒΙΒΛΙΟΥ Β′

Ἀπόσπασμα.

ARISTARCHI SAMII

De Magnitudinibus & Diſtantiis Solis & Lunæ,

LIBER.

Nunc primum Græce editus cum Federici Commandini *verſione Latina, notiſq; illius & Editoris.*

PAPPI ALEXANDRINI

SECUNDI LIBRI

MATHEMATICÆ COLLECTIONIS,

Fragmentum,

Hactenus Deſideratum.

E Codice MS. edidit, Latinum fecit, Notiſque illuſtravit

JOHANNES WALLIS, S.T.D. Geometriæ Profeſſor Savilianus; & *Regalis Societatis* Londini, Sodalis.

OXONIÆ,

E THEATRO SHELDONIANO,

1688.

From Aristarchus'
On the Distances and Sizes of the Sun and Moon

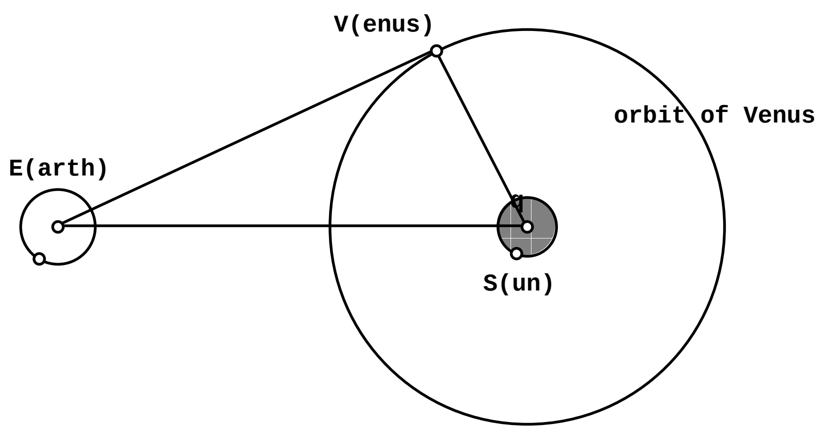
Shown here is Proposition 13, with many scholia, concerned with the ratio to the diameters of the moon and sun of the line subtending the arc dividing the light and dark portions of the moon in a lunar eclipse.

Source: http://metalab.unc.edu/expo/vartican.exhibit/exhibit/d-mathematics/Greek_math2.html

The Distance from Venus to the Sun
Student Page

The orbital radius of Venus is less than the orbital radius of the Earth. As we chart the path of Venus about the sun, we observe that the angle *V*(enus)*E*(arth)*S*(un) reaches a maximum at two places. At each position of maximum angular separation, the angle *EVS* is a right angle.

When the sun was just below the horizon, and Venus was at this position of maximum angular separation, it was most easily visible from the Earth. Depending on whether the sun was rising or setting, this planet was known as the Morning Star or the Evening Star.



Someone who knows the distance from the earth to the sun (93 million miles), and can measure this maximum angle (it is about 46 degrees), can determine the radius of the orbit of Venus. Calculate the radius of the orbit of Venus.

Radius of the orbit of Venus= _______________

It wasn't until the advent of strong telescopes that a similar plan could be used to find the radius of the orbit of Mercury (why were telescopes needed for this planet?). Its maximum angular separation is only 23 degrees. Calculate the radius of the orbit of Mercury.

Radius of the orbit of Mercury? ___________________

Why can't we use an identical plan for the orbit of Mars?

The Distance from Venus to the Sun
Teacher Notes

The radius of Venus may be found from the equation

$$\frac{r}{93{,}000{,}000} = \sin 46^{o}$$ which yields a value of about 67 million miles.

The reasons that stronger telescopes were needed before we could do the identical calculations on Mercury are:

1. It is much smaller in radius, and hence harder to see
2. It is further away from the earth, also making it harder to see.
3. It is much closer to the sun, so it is more frequently invisible (lost in the brightness of the sun's rays).

The radius of the orbit of Mercury may be found from the equation

$$\frac{r}{93{,}000{,}000} = \sin 23^{o}$$ which yields a value of a little over 36 million miles.

This right triangle drawing and simple equation work because the orbits of Venus and of Mercury lie within the orbit of Earth. Mars and the other planets lie outside our orbit and so a different strategy is required to determine their radii of orbit. You might ask the class if they can determine a method for finding the radius of orbits for those planets whose paths lie outside that of the earth. An explanation of the method that Johann Kepler developed can be found in the section on the distance from Mars to the Sun.

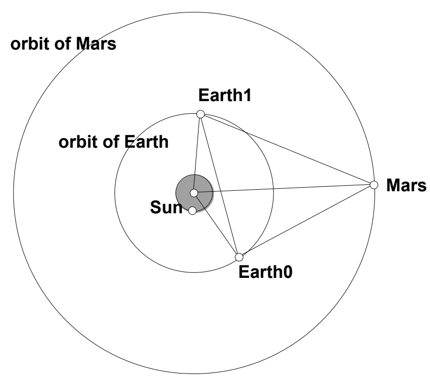
A true and perhaps useful anecdote here……

The author of this section of the module frequently walks to school in the morning, and his school is directly east of his home. During a portion of the school year, he sees the sun rise directly in his path during his walk, and for a period of several days was able to see the Morning Star as well. In fact, he observed that its elevation at the time of sunrise appeared to increase for a number of days, and then level off. So…one morning, at this maximal elevation, he measured the angle of elevation with his thumb and forefinger, noting that it looked like about 45 degrees. Since he knew the distance from earth to sun (93 million miles) and because he knew the relationship among the sides of a 45-45-90 triangle, he knew that the radius of the orbit of Venus would be 93 million miles divided by $\sqrt{2}$. The remainder of his walk to school was spent approximating this quantity, which he determined to be about 65 million miles. Upon arriving at school he checked, through the Internet, for the official value of that radius and found it to be 67 million miles……. There's something empowering about the subject of mathematics, to permit one to determine the radius of the orbit of a planet while simply walking to school and making some innocent observations!!!

The Distance from Mars to the Sun
Johannes Kepler (1571 - 1630 A.D.)
Student Pages

Finding the radius of the orbit of Mars is a more difficult problem than finding the radius of the orbit of Venus, because our vantage point in the solar system doesn't permit us to keep an eye on the sun and the entire orbit of the planet. Things are happening "behind" us. To determine this distance required an exceptional geometrician, Johannes Kepler, who inherited a HUGE amount of data from the astronomer, Tycho Brahe (1546 - 1601). It was Kepler's ambition to describe the orbit of Mars, and, in doing so, he completely demolished the geocentric (earth centered) notion of the solar system. Brahe's data and Kepler's interpretation of the data led to the conclusion that the true path of Mars was not a circle, but an ellipse with one focus at the sun.

Nonetheless, his first approximation of that orbit was a circle, and we faithfully follow his measurements and calculations below based on that premise, a good first approximation.



The crux of the matter hinges on knowing the length of the Mars year; that is, how many earth years it takes Mars to complete a single orbit. Let's assume we know that (see handout on Length of Mars Year for details). Since we're into the 1600's, let us make use of variables and say that Mars completes one orbit in *k* earth years. The value of *k* is not an integer (you will see that at the end), which means that the earth will not be in the same position relative to the sun after Mars makes each orbit.

Suppose that at an initial time 0, the earth is located at *E*(0), marked Earth0 on the diagram. At that time we measure the angle *S*(un)-*E*(arth)-*M*(ars), and then we wait *k* years, at which time Mars will be back at its original place and the earth will have moved to a position we call E(1), marked Earth1. We measure the angle *S*-*E*(1)-*M*. Now we do trigonometry.

We know that *SE* = 93 million miles. We are also assuming we know the value of *k*.

This means we can determine the angle *E*(0)-*S*-*E*(1). How? Suppose that *k*=3.4 . That means that the earth has completed 3 revolutions and an additional important 0.4 of a revolution, which corresponds to 0.4 times 360 = 144 degrees. That would be the measure of angle *E*(0)-*S*-

$E(1)$. We can now find the length of the segment connecting $E(0)$ and $E(1)$ by using the Law of Cosines, as well as the measures of the two base angles of the isosceles triangle, $E(1)$-S-$E(0)$.

Since we had Brahe's excellent measurements of angles S-$E(0)$-M and S-$E(1)$-M, knowledge of those base angles leads us to find the measures of angles M-$E(0)$-$E(1)$ and M-$E(1)$-$E(0)$. The Law of Sines gives us the distance between earth and Mars in each of the two settings, and the Law of Cosines then gives us the length of SM.

We are done, if only we can find the number k of earth-years it takes Mars to complete one orbit. Remember that we needed the value of k to determine the measure of the angle $E(0)$-S-$E(1)$, which got us started. That value is 1 year and 322 days. (For information on how this was determined, see the handout titled The Length of a Mars Year.)

1. First find k as a decimal ___________
2. Use the fractional part of k to find the measure of angle $E(0)$-S-$E(1)$ in degrees. ________
3. That's the major arc, so subtract from 360 to find the desired angle _______
4. Find the length of the segment $E(0)E(1)$. ____________
5. Find the measure of angles S-$E(0)$-$E(1)$=S-$E(1)$-$E(0)$= __________
6. Use measurements from Tycho Brahe of S-$E(0)$-M = 132.1 degrees and S-$E(1)$-M= 131.4 degrees, to determine the values of angles $E(0)$-$E(1)$-M=________________ and $E(1)$-$E(0)$-M=________________
7. Determine the value of angle $E(1)$-M-$E(0)$=___________
8. Find the length of segments $E(0)M$ = ______________ and $E(1)M$=__________________
9. Last step, use the law of cosines to find the length of segment SM=____________

There, that wasn't so bad, was it?

Note: The given measurements due to Tycho Brahe were those taken on March 10, 1585 and on January 26, 1587. You can check that those dates are 1 year, 322 days apart.

The Distance from Mars to the Sun
Teacher Notes

The problem of measuring angles becomes trickier when the orbits lie outside that of the earth. How does one measure the frequently obtuse angle *S*(un)-*E*(arth)-*M*(ars)? Consider the ray from Sun to Earth, passing through Earth and heading out into space. The angles *S*(un)-*E*(arth)-*M*(ars) and *M*(ars)-*E*(arth)-*S*(pace) form a supplementary pair. If we take our angle measurement at true midnight (halfway between sunset and sunrise), and measure the angle then between Mars and the meridian, its supplement is the angle we desire. This is because at midnight, the sun is on the meridian (although we cannot see it).

The length of time that Mars takes to complete one revolution about the sun is 687 days. Here are the answers for the questions asked in the student page.

1. Since $k = 687/365$, the decimal version of $k = 1.8822$. This means the earth has completed 1 revolution and part of the next in one Martian orbit.

2. The extra part of the orbit, in degrees, is given by .8822 x 360 = 317.6 degrees.

3. This latter value is the major arc. We want 360 minus that, which is 42.4 degrees. This is angle $E(0)$-S-$E(1)$.

4. Now use the Law of Cosines on triangle $E(0)$-S-$E(1)$, where the legs of the isosceles triangle are each 93 million miles. This yields a value of about 67.3 million miles for the distance $E(0)$-$E(1)$ between the two earth positions.

5. The sum of the two angles is 180 – 42.4 = 137.6, so each angle is 68.8 degrees.

6. $E(0)$-$E(1)$-M = 131.4 – 68.8 = 62.6 degrees. $E(1)$-$E(0)$-M = 132.1 – 68.8 = 63.3 degrees.

7. $E(1)$-M-$E(0)$ = 180 – (62.6 + 63.3) = 54.1 degrees

8. 67.3 million miles / sin 54.1 = $E(0)M$ / sin 62.6. This gives $E(0)M$ = 73.8 million miles
   67.3 million miles / sin 54.1 = $E(1)M$ / sin 63.3. This gives $E(1)M$ = 74.2 million miles

9. Using triangle S-$E(1)$-M, we have, using the Law of Cosines,
   $SM^2 = (93 \text{ million})^2 + (74.2 \text{ million})^2 - 2(93 \text{ million})(74.2 \text{ million}) \cos 131.4$
   This gives SM = 152.6 million miles for our calculations.

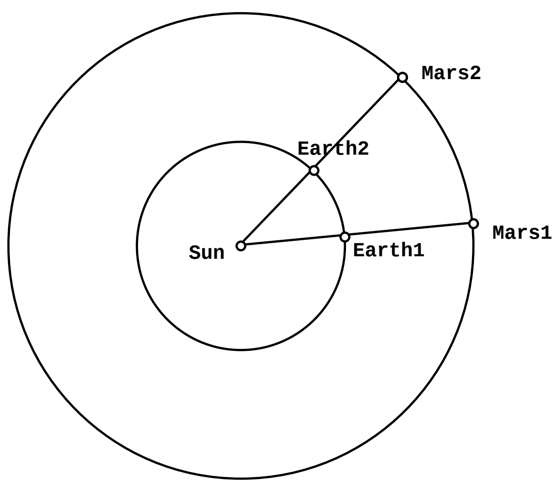
Since the orbit of Mars is an ellipse, the actual distance from Mars to the sun ranges from about 129 million miles to 155 million miles. Since our value is close to the maximum that must mean that Tycho's two measurements were taken when Mars was at close to its maximum distance from the sun. If you took the two measurements at another time, you would get a smaller answer. In any case, the value we have obtained from fairly simple geometrical considerations is a very good approximation to the truth.

The Length of a Mars Year
(Johannes Kepler 1571 - 1630 A.D.)
Student Pages

The key to determining the radius of the orbit of Mars lies in finding out how long it takes Mars to complete an orbit around the sun. Kepler was fortunate that the orbit of the earth and Mars lie in the same plane, and he exploited that fact, with some measurements from Tycho Brahe (1546 - 1601) and some basic algebra.

Since the orbits are in the same plane, there must exist a time at which the earth is between and collinear with the sun and Mars. At this time the sun and Mars are said to be in opposition, and Brahe had documented the dates of those occasions.

How can that be done? When they are observed from opposite sides of the earth, how can one see both of them to determine that the angle is truly 180 degrees? The key is to realize that the sun is directly opposite a spot on the earth at midnight. A point of order must be made here. He needed "real" midnight instead of the "practically" midnight that time zones and Daylight Savings Time have artificially created. With that as a *caveat*, Brahe simply had to watch for when Mars was directly on the North/South meridian at midnight. That was the "day of opposition".



Suppose, for example, that consecutive occurrences of opposition are 1 and 2/9 earth years apart. That would mean that the earth had traveled 1 orbit and 2/9 of the next orbit, while Mars had traveled only 2/9 of its orbit.

The next occurrence of opposition would be in another 1 2/9 earth years, at which time the earth would have made 2 and 4/9 orbits, while Mars would have made 4/9 of an orbit. In the next two occurrences of opposition, Mars will have made 6/9 and 8/9 of an orbit. After four and one half occurrences of opposition, Mars will have made 9/9 of an orbit. That's an entire orbit, and it took 4.5 occurrences of opposition, each of which took 1 and 2/9 earth years….and that product yields 5 1/2 earth years, for Mars to complete one orbit.

Let's generalize this. Suppose occurrences of opposition occur every $1 + a/b$ earth years. Then the earth has gained a/b of a revolution on Mars, and when b/a such occurrences have transpired, Mars will have completed a revolution. The product of b/a and $(1 + a/b)$, which is $1 + b/a$, is the length of a Martian year in terms of earth years. So, for occurrences of opposition every $1 + a/b$ earth years, the length of the Martian year will be $1 + b/a$ earth years.

The actual length of time between occurrences of opposition, as measured by Tycho Brahe, was 2 years and 48 days. Set this equal to $1 + a/b$, and see how long he determined the Martian year to be.

Determine the value of a/b = ________
Now determine the value of b/a = ________
Now the value of $1 + b/a$ (length of Martian year) =______

Did you get 687 days? Great!!!! This is the piece of information Kepler was missing in order to be able to find out the radius of the orbit of Mars about the Sun.

A good and relevant website is
http://csep10.phys.utk.edu/astr161/lect/retrograde/copernican.html
Accessed June 27, 2001

The Length of a Mars Year
Teacher Notes

Tycho Brahe measured consecutive occurrences of opposition at 2yrs 48 days. That is the number 1 + *a*/*b*, which means that the fraction *a*/*b* is equal to 1yr 48 days...or 1 + 48/365 years...or 413/365 years. That means that the fraction *b*/*a* equals 365/413 and the expression 1 + *b*/*a* equals 1 + 365/413 which equals 778/413 which is the length of the Martian year in Earth years. In other words, the Martian year is 1.88 earth years, or approximately 687 days.

Triangle Applications
Teacher Notes

Description of Unit: The two parts Right Triangle Problems and General Triangle Problems complement the class textbook and do not replace it. They can be used independently in case the textbook places the topics in two different chapters, or by teacher choice. Each part has a student handout consisting of exercises linked by a chronological narrative, and a "teacher's manual" titled Teacher's Notes, Answers, and Solutions. The exercises are either historically significant or represent a historically significant type of work. Methods of solution are outlined in steps for the student; they recreate in modern terms the actual historical methods.

Time allowed could be 2 to 4 days depending on time constraints, assignment in class and homework, your selection of exercises, investigating historical setting, and the level of student background, ability, and interest.

Connections apply to the units Shadow Reckoning, Charting the Heavens, and the Law of Sines and Cosines. Mention that for actually navigating and surveying large areas the curvature of the Earth must be considered. More challenging concepts may be explored in the unit Spherical Trigonometry.

Prerequisites: For the Right Triangle Problems: The trigonometry needed is an understanding of the 6 ratios of an acute angle in a right triangle. This can be done in a Geometry class. A couple of the problems involve two algebra skills of solving equations: proportion ($a/b = c/d$) and like terms ($ax = c + bx$).

For the General Triangle Problems: Students should have learned the statements of the Law of Sines and the Law of Cosines and have done at least some abstract textbook exercises.

Materials: Scientific calculator.

The second Right Triangle problem, "Navigation: Miles of Longitude and Pedro Nunes' Quadrant", calls for a reference such as dictionary or almanac in order to convert leagues to miles and to determine the Earth's circumference. Recommended in addition: a globe with the usual lines of longitude and latitude, and the Mercator map (try your school library or social studies department).

Optional, depending on need in student visualization: a student inclinometer (instrument for measuring vertical angles), and a student transit (instrument for measuring horizontal angles). Both functions could conceivably be in a single tool. Such could be made by a student (protractor, string, weight, straw – directions can be found) or bought inexpensively from math/science supply houses.

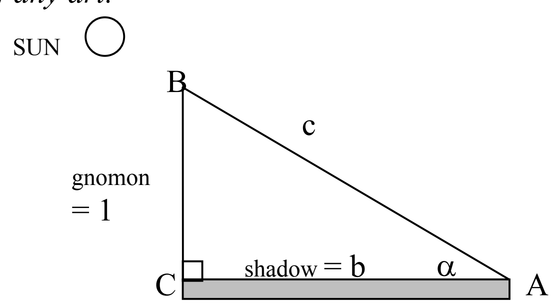
Triangle Applications
Right Triangle Problems
Student Pages

The first book to define trigonometric ratios for an acute angle in a right triangle was published in 1551 in Leipzig, Germany, the *Canon doctrinae triangulorum* by Georg Joachim Rhaeticus (1514–1576). Previously the trigonometric functions had been defined as functions of an arc in a circle. The *Canon* was also the first book to contain all six trigonometric functions. Rhaeticus was the preeminent German astronomer in the first half of the sixteenth century. He became an associate of Copernicus (1473–1543), helped him with mathematics, and persuaded him to publish the heliocentric (sun centered) model of the universe.

Sun's Angle by Shadow Reckoning

Astronomers in Greece, India, and the Arab world calculated the sides and angles of spherical and plane triangles for the purposes of keeping time, making calendars, and knowing directions. The way to face during prayer is explored in the unit on spherical trigonometry.

In his *Exhaustive Treatise on Shadows*, the Islamic astronomer Al-Biruni (973–1055) found the angle of elevation of the sun, given the length of a stick ("gnomon") and its shadow on the ground. Al-Biruni's method was equivalent to finding the hypotenuse (cosecant) by a Pythagorean identity, converting it to sine, then looking up the angle in a sine table. In France late in the twelfth century, the same question and basic solution appeared in an anonymous text named *The perfection of any art.*



Exercise:

a) Express the hypotenuse in terms of the given gnomon length 1 unit and shadow length b.

b) Compute and compare csc α and sin α. How would you find the angle α?

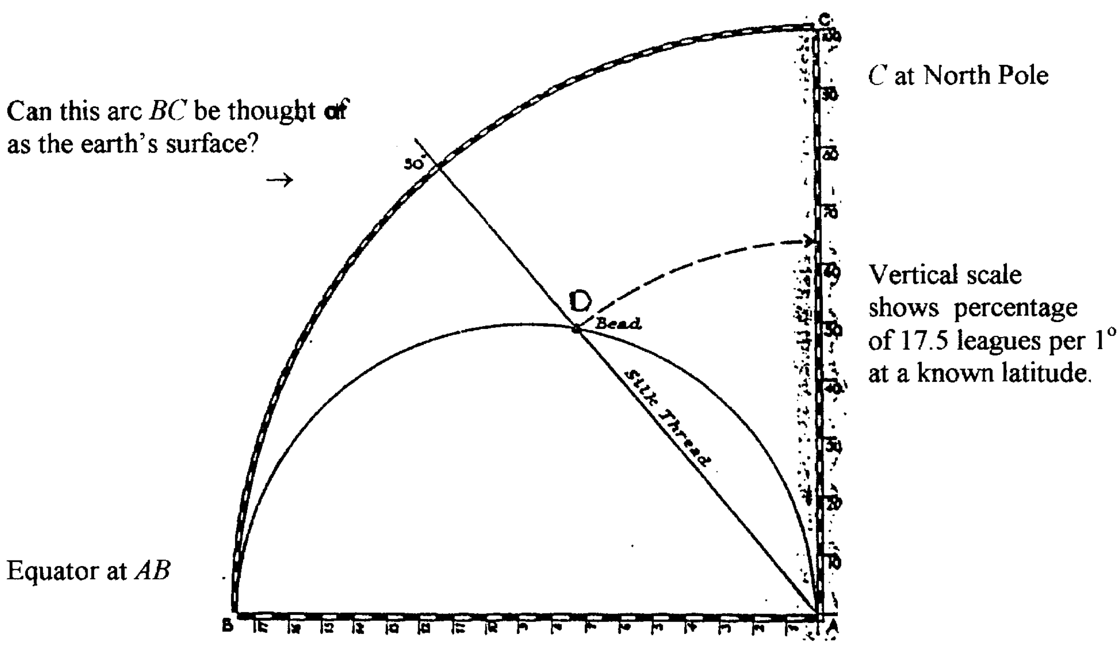
c) Only a sine table was available at the time. With today's knowledge of all six trigonometric ratios, describe a simpler way to find the angle of elevation of the sun.

Navigation: Miles of Longitude, and Pedro Nunez's Quadrant

During the sixteenth and seventeenth centuries, navigation for trade, exploration, and settlement of new lands took a mathematical approach. Navigators adopted tables, formulas, and instruments based on trigonometry and astronomy in order to reckon position accurately when they were out of sight of land. Spherical trigonometry applied on the globe, but for short distances, plane trigonometry served. Latitude (north–south coordinate) was readily found by the altitude of the sun or a star, but finding longitude (east–west coordinate) remained a problem.

Exercise: Sailors valued knowing the length of a degree of longitude, at their given latitude, in order to really know where they were. Pedro Nunes (Portugal 1502–1578) designed a quadrant instrument for determining this length. Nunes specialized in mathematical navigation as a professor in a country whose riches chiefly derived from sea trade. He was the first to point out how a rhumb line differed from a great circle. Nunes held a medical degree, wrote poetry, and was Chief Royal Cosmographer as well as the greatest Portuguese mathematician of his time.

The quadrant's scale on the line *AB* meant there were 17 ½ leagues in 1° at the equator; the scale on *AC* represented percentage. The scale along the arc BC (not numbered) represented latitude, with each of the alternating black and write boxes representing 1°. The semicircle provided had diameter *AB*. The silk thread came attached at *A*, with a movable bead on it. First the sailor put the thread along the arc *BC* (*B* = 0°, *C* = 90°) to his latitude (in this example 50°). He placed the bead on the intersection of the thread with the semicircle (here at *D*). Using *AD* as a radius he swung an arc to intersect the AC scale (here at 64%). Thus his ship would cover (0.64)(17.5) = 11.2 leagues per 1° of longitude.



Horizontal scale shows leagues per 1° at the Equator.

Source of quadrant: *The Haven-Finding Art: A History of Navigation from Odysseus to Captain Cook*, by E. G. R. Taylor (New York: Abelard-Schuman, 1957).

a) Show how the sailor could find the number of leagues per degree directly without multiplying.

b) On Nunes's quadrant find the number of leagues per 1° of longitude at your latitude. Look up the definition of "league" and convert your distance to miles. Compare this with a modern determination of the number of miles per degree of longitude at your latitude.

c) Use trigonometry to show why Nunes's quadrant worked. Hint: Connect *BD* and consider the right triangle *ABD*.

The familiar map (1569) by Gerardus Mercator (Belgium, 1512–1594) displays higher latitudes stretched wider by the factor of (circumference of earth at equator) / (circumference of earth at latitude L°) = (2 π radius r of Earth) / (2 π r cos L°) = sec L°. To keep directions correct, Mercator also stretched the map in the north–south direction by the same factor. Navigators could then plan and sail in a constant and correct direction. In 1599 Edward Wright (England) wrote out the trigonometric principles behind Mercator's "chart".

The Tower Problem of Pitiscus

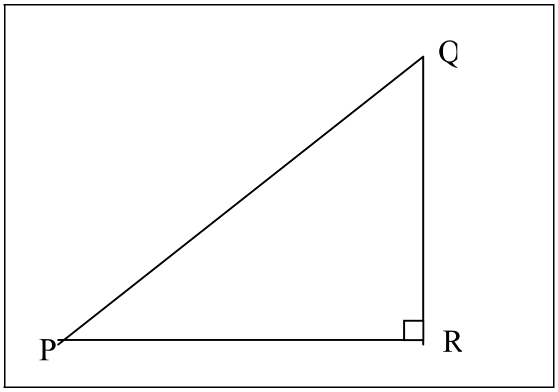
In medieval Europe, up into the fourteenth century, astronomy remained the primary focus of plane and spherical trigonometry. This is seen in popular texts written by Richard of Wallingford (1291–1336) in England and Levi ben Gershon (1288–1344) in France.

An unknown side of a triangle, such as the height of an obelisk or the distance to an enemy fort, was found by means of geometry and similar triangles. A surveyor would measure off an accessible distance and then design a triangle having a side of that length. He could sight an angle on an instrument called a quadrant. This tool is similar to the inclinometer that you may have used to sight the top of the school flagpole in order to calculate its height. The medieval surveyor often did not need to take angle measures because some types of quadrant had scales, labeled *umbra recta* and *umbra versa*, from which he could merely read an appropriate ratio.

In a now famous textbook exercise, Bartholomew Pitiscus (Germany, 1561–1613) posed the task of indirectly measuring the height of a tower not by similar triangles but by trigonometric functions. This problem, among many others, appeared in his *Trigonometriae* (1595), a book which was the first to introduce the word "trigonometry."

(Source of the problem: *A History of Mathematics*, 2nd ed. by Victor J. Katz, publ. Addison–Wesley 1998.)

In the problem, $\angle Q$ is measured with a quadrant to be $60° 20'$, and *PR* is measured off at 200 feet. The student was to determine the height *QR*.



Undoubtedly you find it strange that $\angle Q$ is specified, but that is indeed the measure Pitiscus presented. It is definitely not the angle of elevation $\angle P$ that you have learned to observe from the ground (to the top of the flagpole), nor is it the congruent angle of depression sighted from *Q* to *P* (by a prisoner in the enemy's tower who happened to have a quadrant in his back pocket). Pitiscus's choice of $\angle Q$ would make better sense however if we look at what was going on in the math world of his own time. Due to lack of detailed records we cannot of course know for certain, but as a mathematics scholar Pitiscus probably would have been familiar with the real–world mathematical applications of his time, such as gunnery, as illustrated in this page from *La nova scientia* (Italy 1537) by the influential Niccolo Tartaglia (1499–1557). In this treatise Tartaglia explains surveying for range–finding as well as the use of a gunner's quadrant. You should notice that when the muzzle is elevated, the weighted string intercepts the "protractor" scale in a plausibly similar position and fashion as the $\angle Q$ in Pitiscus's tower problem.

(Source of picture: http://www.mhs.ox.ac.uk/geometry/cat1.htm)

Exercise: Equipped with the usual sine and tangent tables of his day, Pitiscus gave two solutions, one based on sine and the other on tangent. The following solution uses tangent; the solution with sine is below.

a) Evaluate $\angle P$, the complement of $\angle Q$.
Write an expression for tan $\angle P$ in right triangle *PQR*.
Express *QR* in terms of tan $\angle P$.

b) Evaluate *QR* to determine the height of the tower.

To see why Pitiscus probably chose to express *QR* in terms of the tangent of angle *P* rather than angle *Q*, do the following:

c) Write an expression for tan $\angle Q$ in right triangle *PQR*.
Express *QR* in terms of tan $\angle Q$.
Compare with *QR* in terms of tan $\angle P$.

Before electronic calculators, multiplying by the tangent of an angle was easier than dividing by it because the tangent value contains many digits.

Pitiscus actually set up the proportion *PR* / 100,000 = *QR* / tan 29° 40′ to solve the problem. The 100,000 came from the fact that people of his time based tables upon a circle of radius 100,000 (where today we have the radius = the unit 1). Thus in his table, the tangent of 29° 40′ was expressed as the integer 56962.

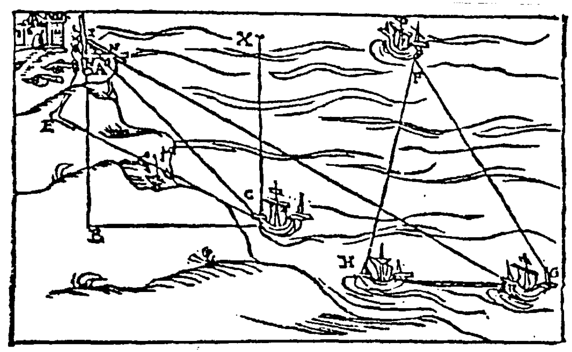
This was only one of two solutions by Pitiscus. He also used a sine table and presented a solution using sine. In any triangle, there is a constant ratio between the sine of an angle and the side opposite the angle. You will learn this amazing property later formally by the name of the Law of Sines when you study general triangles (including oblique, that is, non–right triangles). Pitiscus and others applied the Law of Sines to right triangles quite readily. This is Pitiscus' solution using sines.

d) In triangle *PQR* , you have the proportion $\dfrac{\sin\angle Q}{PR} = \dfrac{\sin\angle P}{QR}$. Substitute the known values of $\angle Q$, $\angle P$, and *PR* and solve for *QR*.

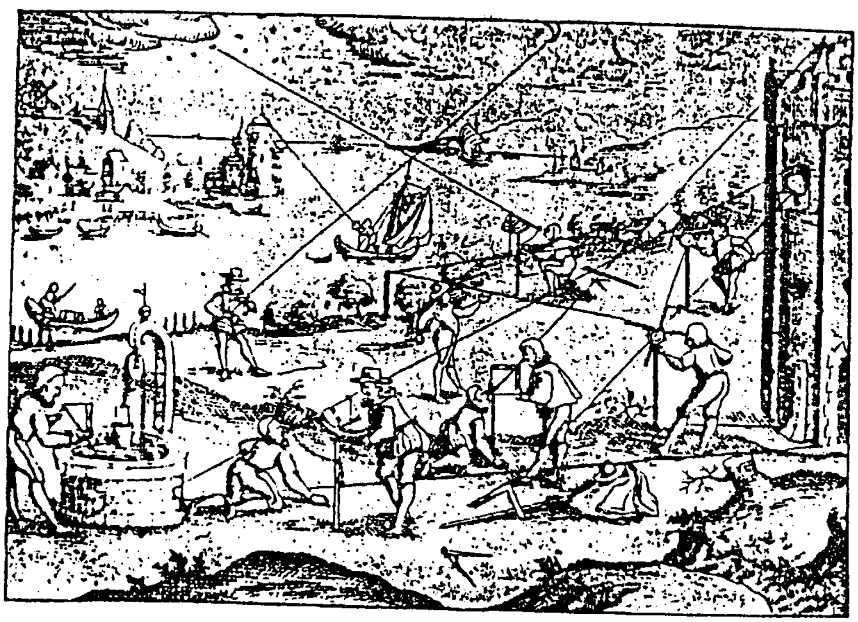
e) Pitiscus, using his sine table based on a circle of radius 100,000, had sin 60° 20′ = 86892 and sin 29° 40′ = 49495. He gave the answer $QR = 113\dfrac{80204}{86892}$ feet. Compare your answer with his.

Finding a Height if You Know Its Distance and the Angle of Elevation

During the medieval period, waging battle involved studying the motion of cannonballs, muzzle angle in gunnery, distance to enemies, height of their fortifications, distance between two separated outposts, and the depth of ditches. In a field manual (1590) for his "familiar staffe" instrument, John Blagrave wrote about altitude: "If a Wall or Tower were to be scaled ... How ...to get the height thereof, thereby to make your scaling ladders accordingly." Concerning profundity: "If a man were prisoner with the enemie, how being in the top of a tower on the leads, or out of his prison window, hee might ... know the depth to the ground, to see if he were able with anie device to let himselfe downe without danger." (Although Blagrave used similar triangles and the staffe's scales rather than trigonometry, the point here is that the problems arose from the "real world".)

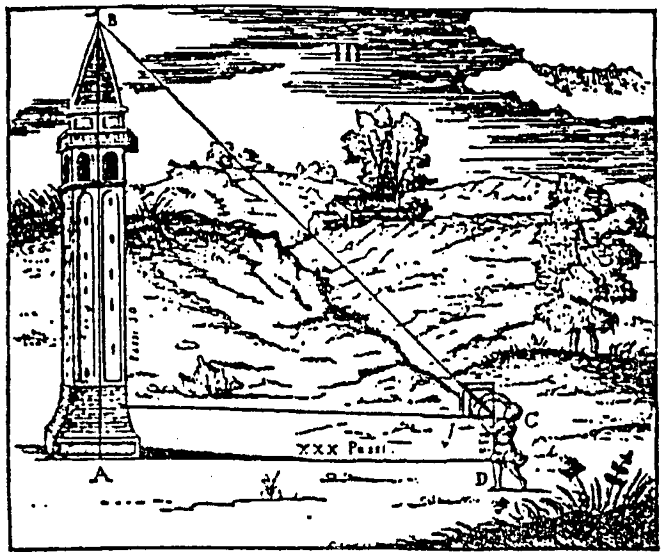


Source: *A Booke of the making and use of a Staffe, newly invented by the Author, called the Familiar Staffe*, by John Blagrave, publ. Hugh Jackson, London 1590 – facsimile publ. Da Capo Press, New York 1970.



A picture of 16th century trigonometry activities, from the textbook *De quadrante geometrico libellus* (Nuremberg, 1594) by Cornelius de Judaeis

Exercise: This is suggested by a problem published in 1752. Wishing to calculate the height *AB* of the building, this surveyor has stepped away 30 paces from the axis of the tower, on level ground. Standing at point *D*, he reads an angle of elevation 44° on the scale of his quadrant. A pace is an average stride in walking, from 2.5 to 3 feet in length. Suppose that the surveyor's eye is 5 feet above the ground and his pace is 2.5 feet. How tall is the building?

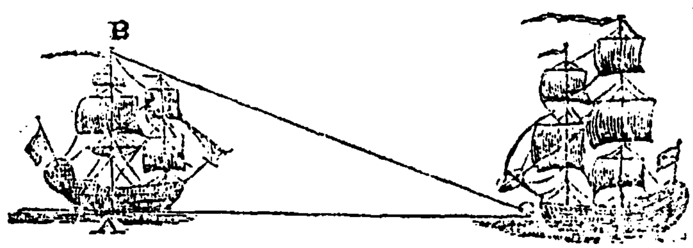


The use of a quadrant instrument, in *L'Uso della Squadra Mobile* by Ottavio Fabri, Trent 1752, shown in *History of Mathematics, Volume II*, by David E. Smith, publ. Dover Publications 1958, page 355.

A Textbook Problem from 1808

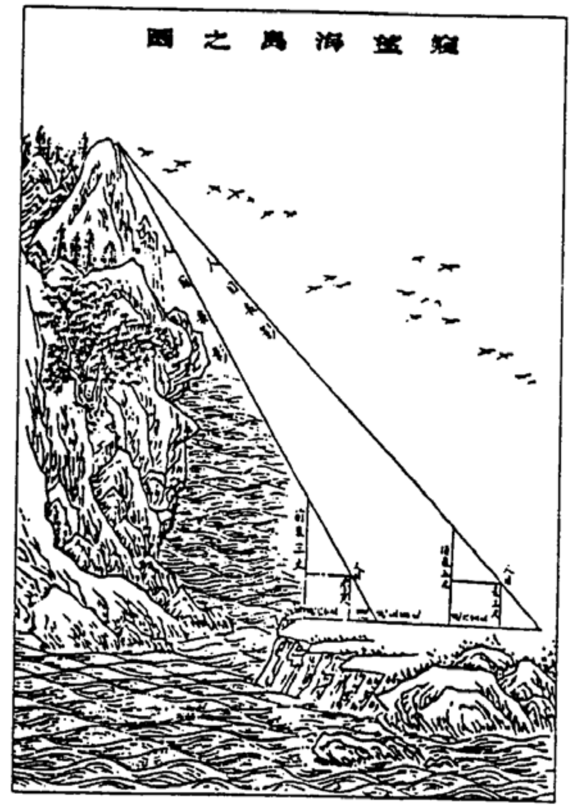
At the start of the 19th century, students learned a type of trigonometry called "*Heights and Distances,* or *Altimetry and Longimetry".* According to *Mathematics* by Samuel Webber (England 1808): "By the mensuration and protraction of lines and angles we determine the lengths, heights, depths, or distances, of bodies and objects." Students used logarithms to shortcut multiplying and dividing by numbers with many digits.

Exercise: From a known height to find the distance of an inaccessible object on a level. From the top of a ship's mast, which was 80 feet above the water, the angle of depression of another ship's hull, at a distance upon the water, is 20°; what is their distance? (Source: Webber, page 96, Problem III)



The Inaccessible Height and Distance: A Problem Crossing Many Centuries and Cultures

Liu Hui of China posed the question of finding the height of an island as the first problem in *The Sea Island Mathematical Manual* (year 263). The observer cannot reach the perpendicular point at ground level under the island's peak, since it is across water as well as inside the mountain. Without knowing a measured distance to that point, one cannot simply apply right triangle ratios.



The Chinese mathematician solved the problem as follows: He chose two observation points separated by a known distance, and from each point he sighted the island's peak over a pole of known length. However, since Liu Hui did not use angles, his solution was not trigonometric. He used side ratios of similar right triangles. Some math historians consider his work to be a precursor of trigonometry.

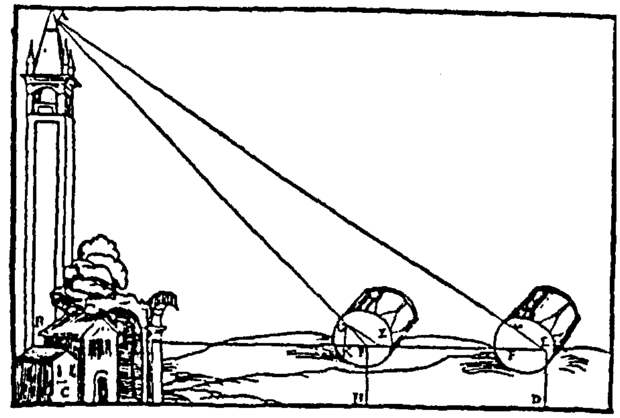
The astronomer Aryabhata of India (born 476) wrote of the same type of problem. His solution was basically the same as the Chinese one.

The Islamic mathematician Al–Qabisi (tenth century) measured the angles of elevation at the 2 poles and worked out a trigonometric form. He used only the sine ratio, and the expression is cumbersome. Al-Biruni (973–1055) used isosceles right triangles and similar triangles.

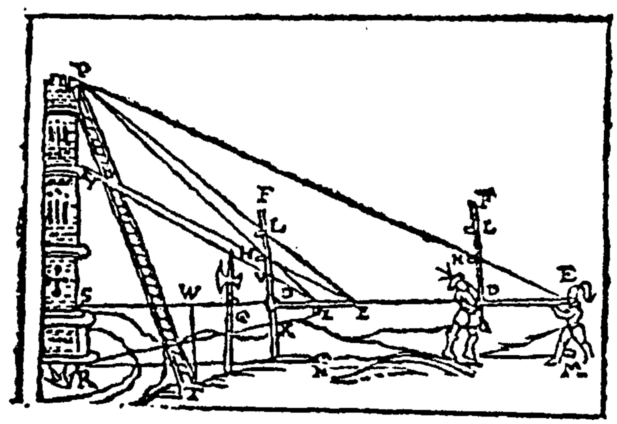
Hugo of St. Victor (Paris, 1096–1141) gave the problem in a surveying text but still used only geometry to solve it. As mentioned before, plane and spherical trigonometry was used for astronomy and heavenly triangles, not for surveying earthly ones, well into the fourteenth century.

In the real world of sixteenth and seventeenth century Europe, waging battles involved geometry and calculations as part of strategy. The problem of indirectly measuring an inaccessible height appeared widely.

Drumheads served as a convenient "paper" for recording angles and lines in the field (including the battlefield). Mathematics historian Smith gives an illustration, from *Libro del Misurar* (1569) by Belli of Venice, of "drumhead trigonometry, a common method of triangulating in the 16th century".

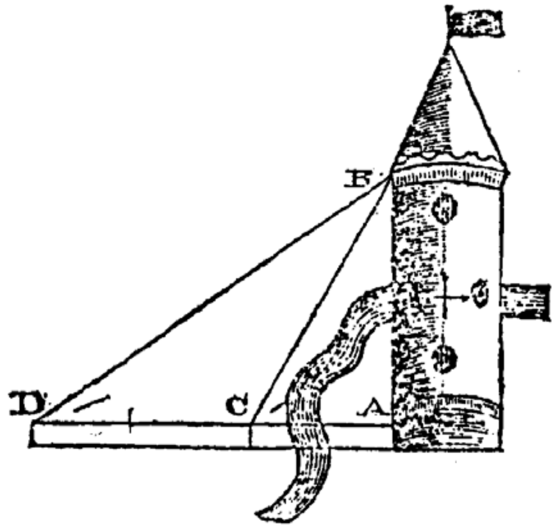


In a field manual (1590) for his invention, a "familiar staffe" instrument, Blagrave gave good reason why combatants measured from afar: "If a Wall or tower were to be scaled ... How ... to get the height thereof, thereby to make your scaling ladders accordingly, ...Where you dare not come neere the base of the tower for daunger of shot or let by reason of some deepe mote or ditch." Blagrave based his methods on his familiar staffe's scales as well as similar triangles.



Around 1700 Murai Masahiro in Japan published a problem similar to the Sea Island problem in *Riochi Shinan*. It's trigonometric solution was influenced by the West.

Exercise:



Source: *Mathematics, Vol. II*, by Samuel Webber, publ. Cambridge University Press 1808, page 99.

Wanting to know the height of an inaccessible object; at the least distance from it, upon the same horizontal plane, I took its angle of elevation equal to 58°, and going 100 yeards directly farther from it, found the angle there to be only 32°: required its height, and my distance from it at the first station, the instrument being 5 feet above the ground at each observation.

a) In triangle *BCA*, write the ratio for cot $\angle BCA$, that is cot 58°.

b) Solve for *CA*.

c) Use the answer in b to express the length of *DA* in terms of *AB*.

d) In triangle *BDA*, write the ratio for cot $\angle D$, that is cot 32°; then use the answer in c to express this solely in terms of *AB*.

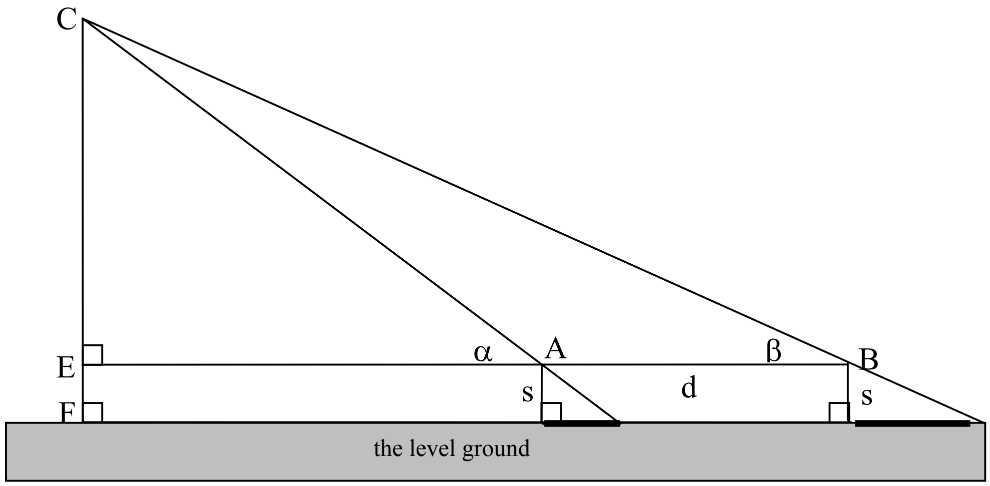
e) Solve the final equation of d for *AB*.

f) Add the height of the instrument to find the height of the tower in yards.

g) Replace your *AB* value into your *CA* expression in b, and find the distance to the tower.

Extensions
For Inquiry, Challenge, and Enjoyment

Exercise:



Complete this general solution for height, using right triangle trigonometry. Then compare the result with the geometric results of Liu Hui and Aryabhata.

Given: data by direct measurement: sticks or poles of equal length *s*
distance *d* between the poles
angles of elevation α, β sighting the object's top at *C*

To find: height *CF*

a) The plan is to find *EC* then add it to *s* for a total of *CF*.
For convenience, call $EC = y$ and call $EA = x$. In right triangle *AEC*, write an expression for cot α, and solve for *x* in terms of *y*.

b) In right triangle *BEC*, write an expression for cot β. Then substitute the value for *x* found in a to get an equation in *y*. Solve for *y*.

c) Now write an expression for the total height *CF*.

Liu Hui solved this problem as follows, where the Chinese word *fa* means the divisor, and *shi* means the dividend (*The Sea Island Mathematical Manual,* edited by Frank Swetz):

> Multiply the distance between poles by the height of the pole, giving the *shi*. Take the difference in distance from the points of observations as the *fa* to divide [the *shi*], and add what is thus obtained to the height of the pole. The result is the height of the island.

The first sentence gives, from our diagram, d times s, so the dividend will be ds. The second sentence tells us to take the bolded bases of the 2 small triangles and subtract the shorter (call it a) from the longer (call it b). This difference, $b - a$, will be the divisor. So the height of the island is $\frac{ds}{b-a} + s$.

d) Derive this value for the height by using the similarity ratios in two sets of similar triangles in the diagram. You should get two different expressions for the quantity sx. If you equate them and solve for y, you should be able to derive Liu Hui's expression.

Aryabhata noted in his solution that he was dividing by the difference of the poles' shadows. Also remember that these shadows along the ground, the *umbra recta*, represent our modern cotangent, $b = s \cot \beta$ and $a = s \cot \alpha$.

e) Show that your trigonometric solution from c and Liu Hui's solution from d are the same, by using the expressions for the shadows noted above.

As a follow–up, look into *The Sea Island Mathematical Manual* and learn how Liu Hui found the distance to the island.

Resources: *The Sea Island Mathematical Manual: Surveying and Mathematics in Ancient China*, by Frank J. Swetz, Pennsylvania State University Press 1992, a translation and commentary on the *Haidao Suanjing* written by Liu Hui in 263.

"Lecture 4 A Chinese surveying problem" at the web page http://www.maths.uwa.edu.au/ Staff /schultz/3M3/L4Chinese trig.html.

Triangle Applications
Right Triangle Problems
Teacher Notes, Answers, and Solutions

Sun's Angle by Shadow Reckoning

Exercise:

a) $c = \sqrt{1+b^2}$

b) csc α $= c / 1 = \sqrt{1+b^2}$; sin α.$= a / c = 1 / \sqrt{1+b^2}$; the cosecant and sine are reciprocals of one another. To find α, look up the sine of α just calculated in a table of sines and then find α. If you use a calculator, you can just press the inverse sine button to accomplish the same thing.

c) tan α = (length of gnomon) / (length of shadow). Nowadays we would press the [$\tan^{-1}$] key on a calculator and enter the above quotient, to evaluate α..

Navigation: Miles of Longitude and Pedro Nunes's Quadrant

Teacher's Note: Show students that on a globe the equator is at 0° latitude, and the poles 90° N and 90° S; also, that longitude 0° starts at a prime meridian, and circles of constant latitude are smaller for higher latitudes. Circles of constant longitude are called meridians, and circles of constant latitude are called parallels. There are approximately 69.4 miles 1° of a great circle. We get this by dividing the circumference of the Earth (25,000 miles) by 360. So this is the length in miles of 1° at the equator. The length of 1° is shorter at higher latitudes.

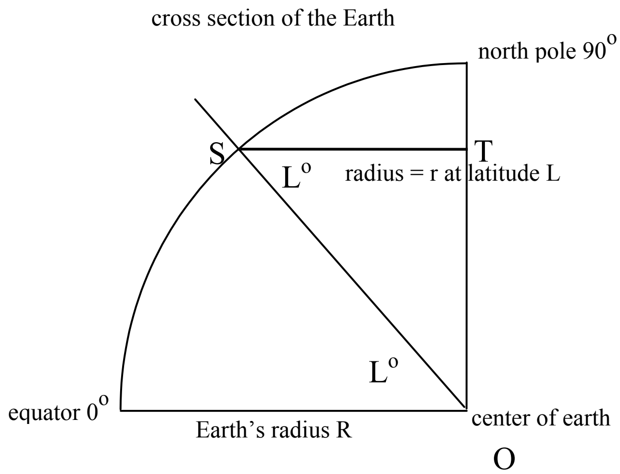
Exercise:

a) Turn the bead to the *AB* scale.

b) Students will refer to standard sources such as dictionary or almanac. Unfortunately, there were different "leagues" in use. In particular, the English land league was about 3 miles, or 4.83 km., the English nautical league was 5.56 km. or 18,240 ft, while the French league was 5.85 km. If we use Nunes' value of 17.5 leagues per degree at the equator, neither of these three leagues gives the correct value for the earth's circumference. But that value was probably not known accurately by Nunes either.

c) Solution: Let *D* = position of bead on the semicircle. Triangle *ADB* is a right triangle since it is inscribed in a semicircle. In that triangle, $\cos \angle DAB = DA / AB = DA / AC =$

DA / (100%). Thus the position (length DA) of the bead on scale AC is just cos $\angle DAB$. Now look at the diagram



(**Teacher's Note:**
This diagram is not included in the student handout.)

of a cross section of the Earth. The circle of higher latitude has radius $r = ST$, and cos $\angle OST = r / R$. Also $\angle OST = L°$, so $r = R \cos L°$. In terms of the circumference of a circle, on which the sailing route is, $2 \pi r = 2 \pi R \cos L°$. Therefore Nunez's quadrant works because the distance per 1° at the equator is scaled down by a factor of cos $L°$ at the sailor's latitude $L°$.

Teacher's Note: Students may explore navigation and mapping, which are profound sciences in themselves. A few suggestions are:

Edmund Gunter's trigonometry methods and instruments for mariners
Gunter's biography, http://es.rice.edu/ES/humsoc/Galileo/Catalog/Files/gunter.html. This web page was accessed on 28 June 2001.
The Navigation chapter in *Math and Civilization* by Resnikoff and Wells, publ. Holt 1973.

The Tower Problem of Pitiscus

Exercise:

a) $\angle P = 29° \ 40'$; $\tan 29° \ 40' = QR / 200$; $QR = 200 \tan 29° \ 40'$

b) $QR = 200 \tan 29° \ 40' = 113.9238$ feet, or approximately 114 feet.

c) $\tan 60° \ 20' = 200 / QR$; $QR = 200 / \tan 60° \ 20'$. In terms of tan $\angle P$, we multiply by 200. In terms of tan $\angle Q$, we divide into 200.

Teacher Note: The next exercise is stated in such a way that it can be done without a lesson on the Law of Sines. Triangle *PQR* is a right triangle, and the solution is simple enough to here wrap up Pitiscus' work. At this point in time, students will be satisfied with being given the statement of the Law of Sines without proof; they are resilient in accepting that the proof will come later and that further applications will be fleshed out at the proper chapter.

d) $QR \sin\angle\ 60° 20' = PR \sin\angle\ 29° 40'$, so $QR = 200 \sin\angle\ 29° 40' / \sin\angle\ 60° 20'$. Therefore, QR = 113.9238 feet, the same answer as before.

e) If we use Pitiscus' values, we get $QR = 9899000/86892 = 113\frac{80204}{86892} = 113.9230$.

Pitiscus' sine tables were not as accurate as ours.

Finding a Height if You Know It's Distance and the Angle of Elevation

Exercise:

Since 30 paces = 75 feet, the height of the building above eye level is 75 tan 44°. Thus the actual height of the building is 5 + 75 tan 44 = 77.43 feet.

A Textbook Problem from 1808

Exercise:

We have $\cot 20° = x / 80$, so $x = 80 \cot 20° = 80/\tan 20° =$ 219.798 feet

The Inaccessible Height and Distance: A Problem Crossing Many Centuries and Cultures

Exercise:

a) $\cot 58° = CA / AB$

b) $CA = AB \cot 58°$

c) $DA = 100 + AB \cot 58°$

d) $\cot 32° = DA / AB = (100 + AB \cot 58°) / AB$

e) $AB \cot 32° = 100 + AB \cot 58°$; so $AB = \frac{100}{\cot 32 - \cot 58}$ = 102.51 yards.

f) Since 5 feet = 1.67 yards, the height of the tower is 104.18 yards.

g) $CA = AB \cot 58° = 102.51(0.6249) = 64.06$ yards.

Extensions for Inquiry, Challenge, and Enjoyment

Exercise:

a) $x = y \cot \alpha = EA$

b) $\cot \beta = (x + d) / y$
$\cot \beta = (y \cot \alpha + d) / y$
$y \cot \beta = y \cot \alpha + d$
$y (\cot \beta - \cot \alpha) = d$
$y = d / (\cot \beta - \cot \alpha) = EC$

c) $CF = \dfrac{d}{\cot\beta - \cot\alpha} + s$

d) $\dfrac{s}{a} = \dfrac{y}{x}; \dfrac{s}{b} = \dfrac{y}{x+d}$
$sx = ay;\ \ sx + ds = by$, so $sx = by - ds$
$ay = by - ds$ or $by - ay = ds$ or $(b - a)y = ds$
$y = \dfrac{ds}{b-a}$
It follows that the height is $\dfrac{ds}{b-a} + s$.

e) $CF = \dfrac{d}{\cot\beta - \cot\alpha} + s = \dfrac{d}{\frac{b}{s} - \frac{a}{s}} + s = \dfrac{ds}{b-a} + s$.

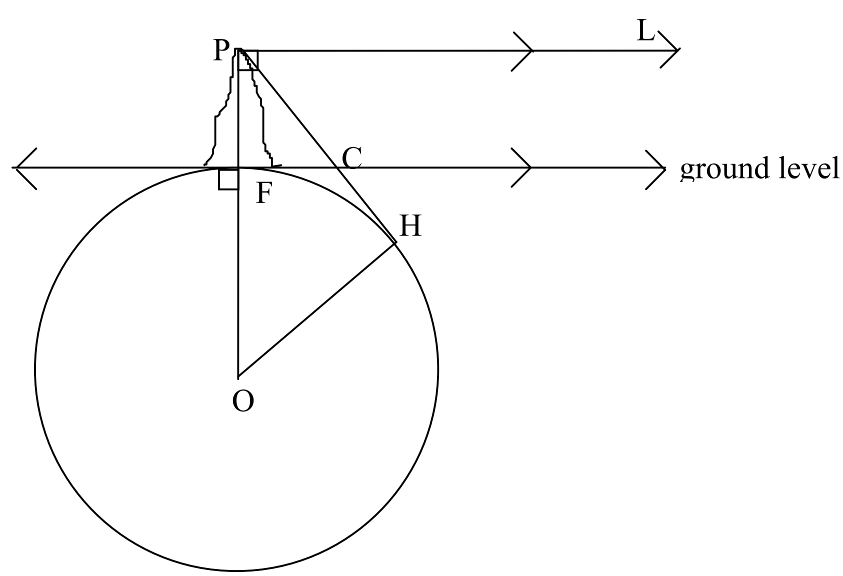
Teacher's Note: Students who like to do mathematics, and not just read about it, could write a precise explanation of the relationship between the similarity approach to finding height and the trigonometric approach. They could present this material in a paper, poster, or talk. The inaccessible height and distance problem connects mathematics within itself (geometry, algebra, trigonometry) as well as diverse peoples and times. Some students may even appreciate the mathematical beauty.

Triangle Applications
General Triangle Problems
Student Pages

Al-Biruni's Calculation of the Earth's Size

The Islamic mathematician Al-Biruni (973–1055) gave a "method for the determination of the circumference of the earth. It does not require walking in deserts." The standard method of determining the circumference did involve such walking, because it required measuring the distance on the earth between two points one degree of latitude apart. This had been done on the orders of a caliph. Al-Biruni had a new idea. He calculated the height of a mountain in Nandana, India, with a scaled square board and similar triangles. Then he went to the mountain top with his astrolabe and measured the angle of depression to the horizon. Since he had a table of sine values, Al-Biruni applied the Law of Sines to find the radius of the Earth.

Al-Biruni's historical solution:



The scale of the mountain has been very exaggerated to make a clear diagram. In this diagram, *P* represents the peak of the mountain; the length *PF* is the known height of the mountain; *O* is the center of the earth, and *H* is the horizon point as viewed from the mountain's peak. We want to calculate *OF* = *OH*, the radius of the earth. Note that *PH* is tangent to the earth at *H* and is therefore perpendicular to *OH*.

a) Al-Biruni's value of the mountain's height *PF* was 652.055 cubits. His dip angle *LPH* was 34 minutes (which is less than 1 degree). In theory taking this measurement is fine, but be aware that a small error will be greatly magnified. Al-Biruni applied the Law of Sines to right triangle *PFC*. In triangle *PFC*, label the values of the mountain's height and the two acute angles. (Note: Your textbook probably applies the Law of Sines only to oblique triangles. People used to apply it to right triangles too.) Write the Law of Sines ratio for the 2 legs in triangle PFC. Evaluate *FC*. Then use the Pythagorean Theorem to evaluate *PC*.

b) *FC* = *CH* because from an exterior point two tangents to a circle are equal. Calculate the distance *PH* to the horizon.

c) Now look at triangle *PHO*. Update your diagram by labeling in the values of the hypotenuse and $\angle O$. Write the Law of Sines ratio in terms of $\angle OPH$, $\angle O$, side *PH*, and unknown side *OH*. Solve for *OH*. Then use Al-Biruni's value $\pi = 3\ 1/7$ to compute the circumference of the Earth.

d) Al-Biruni's results were: radius 12,803,337.036 cubits, and circumference 80,478,118.511 cubits, although your results may differ slightly. Thus the length of a degree on a meridian was this circumference divided by 360. Find the length of one degree in cubits. Then, given that 4000 cubits are equal to one Arabian mile (in the measurements of his day), determine the length of a degree in miles. Al-Biruni judged this value "very close" to what a geodetic survey some years earlier had already shown.

e) There were many different cubits used in the ancient world. If we assume that the cubit of Al-Biruni was approximately 20", calculate how close Al-Biruni came to Earth's actual circumference.

Source: *Episodes in the Mathematics of Medieval Islam*, by J. L. Berggen, publ. Springer–Verlag 1986.

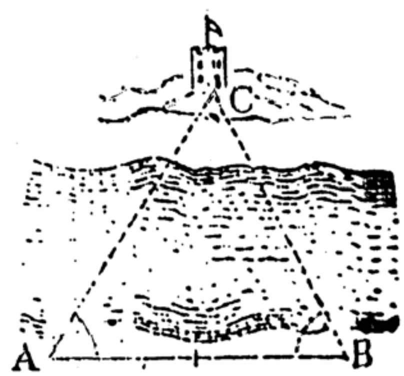
Modern Solution of Al-Biruni's Problem

a) In right triangle *PHO*, let r symbolize radius $OH = OF$. Since $OP = OF + FP$, we have $OP = r + 652.055$. Furthermore, we know that $\angle O = 34'$. (Why?) Choose one of the six modern trigonometric functions, to express a ratio involving radius r and $\angle O$.

b) Solve the equation from a for r algebraically, and find the value of r. Compare this to your previous calculation.

How Far to an Inaccessible Object

Someone at location A wishes to find the distance to some visible point C. However, he cannot obtain it directly by pacing or using measuring sticks because access is blocked by such things as a river, hills, a swamp, or enemy surveillance.



Exercise: Complete the general solution of the inaccessible distance problem. The person wants to know distance *AC*. He starts by measuring off a sideways distance *AB* and the angles *A* and *B*.

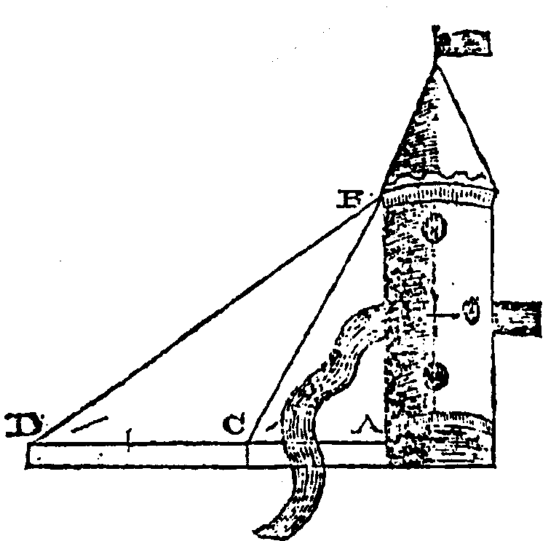
a) Apply the Law of Sines to relate *AC* and *AB*.

b) Express *AC* in terms of only the knowns *AB*, angle *A*, and angle *B*.

Height and Distance of an Inaccessible Building

By the 1800s students solved the inaccessible height and distance problem by applying the Law of Sines. (Source: *Mathematics,* by Samuel Webber, publ. Cambridge University Press 1808, page 99.) This problem was also stated in the Right Triangle unit.

"Wanting to know the height of an inaccessible object; at the least distance from it, upon the same horizontal plane, I took its angle of elevation equal to 58°, and going 100 yards directly farther from it, found the angle there to be only 32°; required its height, and my distance from it at the first station, the instrument being 5 feet above the ground at each observation."
Complete the steps:



a) Evaluate angle *DBC*.

b) Apply the Law of Sines to find side *BC*.

c) Find height *AB*, and the whole height of the building.

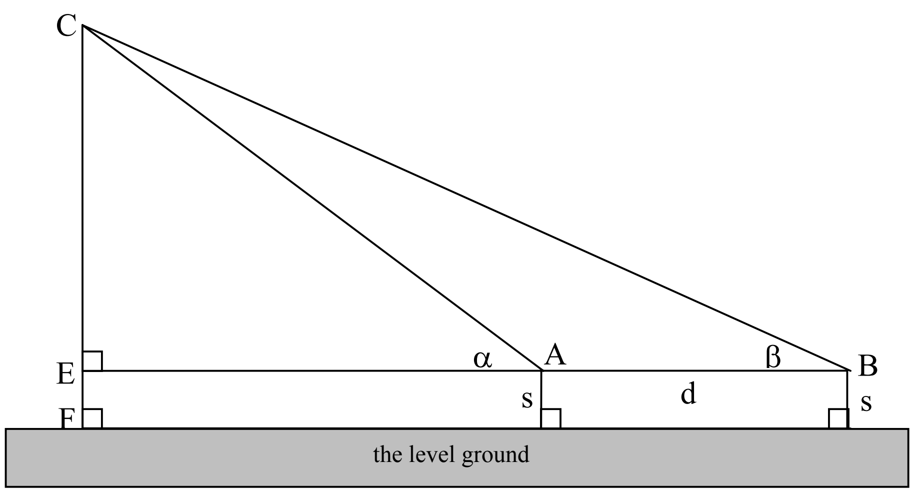
d) Find distance *CA*.

Inaccessible Height – General Case

Exercise: Complete the general solution. Use the Law of Sines as well as right triangle trigonometry.

By direct measurement, you have: sticks or poles of equal length s
distance d between the poles
angles of elevation α, β sighting the object's top *C*

You want to find height *CF*.



a) Express $\angle BCA$ in terms of α and β.

b) Look at triangle *BCA* and write an expression for *CA* by the Law of Sines.

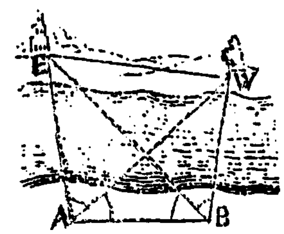
c) Look at right triangle *ECA* and write an expression for *CE*.

d) Write an expression for total height *CF*.

Distance between Edifice and Wood When I Can't Get to Either One

The following exercise is taken from a textbook printed in 1864.

"Wanting to know the horizontal distance between two inaccessible objects *E* and *W*, the following measurements were made: viz:



$AB = 536$ yards
$BAW = 40° 16'$
$WAE = 57° 40'$
$ABE = 42° 22'$
$EBW = 71° 07'$.
Required the distance *EW*."

The textbook's author gives the answer but not the process of solution. However, the following use of the Law of Sines is typical of the time. Complete the steps. The plan is to get to triangle *AEW*. From the given, we can find both *AE* in triangle *AEB*, and *AW* in triangle *AWB*. Then on triangle *AEW* we will apply the Law of Cosines to find *EW*.

a) In triangle *AEB*, calculate edifice angle $\angle AEB$. Then use the Law of Sines to calculate the distance *AE* between observation point *A* and the edifice *E*.

b) In triangle *AWB*, calculate angle $\angle AWB$ to where the woods begin. Use the Law of Sines to calculate the distance *AW* between *A* and the woods.

c) In triangle *AEW*, find the distance *EW* between the edifice and the woods.

Source: a textbook by Charles Davies (1798–1876), *Elements of Geometry and Trigonometry, from the works of A. M. Legendre*, adapted to the course of mathematics instruction in the United States, publ. Barnes & Burr, NY 1864. Problem 6 on page 49.

Sidelight: *Elements of Geometry* by A. M. Legendre (1752–1833) was translated into the Hawaiian language. It was published in 1843 by Lahainaluna High School Press under the title *Mole o ke anahonua.*

Triangle Applications
General Triangle Problems
Teacher Notes, Answers, and Solutions

Al-Biruni's Calculation of the Earth's Size

Al-Biruni's historical solution

Teacher Notes: This was a beautiful solution and amazing achievement. Al-Biruni had only sines and no calculator. To avoid numerical burn-out of your students, accept sufficiently close results. The TI-83+'s results are close to Al-Biruni's when rounded: radius 1.3×10^7 cubits, and circumference 8.0×10^7 cubits.

Images and explanations of astrolabes are available by linking from these websites:
http://www.astrolabes.org – Link to the circa 1200 astrolabe in the Getty Museum.
http://www.mhs.ox.ac.uk – Excellent on-line exhibits at Museum of History of Science at Oxford.

Applying the Law of Sines to a right triangle may seem a little strange to those of us who learned it in some textbook chapter entitled oblique triangles. Al-Biruni's solution here informs students that the Law of Sines, as indeed the Law of Cosines, is a property of all triangles – right as well as acute and obtuse. The Tower Problem of Pitiscus, in the Right Triangle unit, also applies Law of Sines to a right triangle. This was a creative and powerful technique for mathematicians of a millennium ago.

a) The Law of Sines ratio for the 2 legs in triangle *PFC* gives $FP / \sin\angle FCP = FC / \sin\angle FPC$. Since lines *PL* and *FC* are parallel, we have $\angle FCP = 34'$. The ratio therefore is
$652.055 / \sin 34' = FC / \sin 89° 26'$
$FC = 652.055 \sin 89° 26' / \sin 34'$, so $FC = 65{,}927.26124$
By the Pythagorean Theorem, $PC = \sqrt{FC^2 + 652.055^2} = 65{,}930.48574$.

b) $PH = PC + CH = PC + FC = 131{,}857.747$

c) $\angle O = 34'$. The ratio is $OH / \sin 89° 26' = PH / \sin 34'$. Therefore, $OH = 131{,}857.747 \sin 89° 26' / \sin 34' = 13{,}331{,}728.35$ cubits. The circumference is then equal to 83,799,435.36 cubits.

d) One degree is 83,799,435.36/360 = 232,776.2093 cubits. If we divide by 4000, we get 58.19 miles per degree.

e) If 1 cubit is 20" = 1 2/3 ft, then 83,799,435.36 cubits = 139,665,753 feet, or about 26,452 miles, a value slightly larger than the actual circumference.

Modern Solution of Al-Biruni's Problem

a) $\cos 34' = r / (PF + r)$

b) $r = (PF + r)(\cos 34')$

$r = PF \cos 34' + r \cos 34'$

$r - r \cos 34' = PF \cos 34'$

$r (1 - \cos 34') = PF \cos 34'$

$r = (652.055 \ \cos 34') \ / (1 - \cos 34') = 13{,}332{,}380.41$ cubits

Teacher Note: Notice here that since cos 34' is very close to 1, the denominator of the fraction is very close to 0. Therefore, any slight change in the denominator will change the answer by a large amount. Could Al-Biruni have measured the dip angle accurately to within one minute of angle?

How Far to an Inaccessible Object

a) $AC / \sin B = AB / \sin C$

b) $AC = AB \sin B / \sin C$

$AC = AB \sin B / \sin [180° - (A + B)]$

Height and Distance of an Inaccessible Building

Teacher Note: The author's concept is recreated below. However, he wrote his solution in logarithms. Of course today students are readily permitted and expected to employ calculators.

a) $\angle DBC = 26°$

b) $BC / \sin 32° = 100 / \sin 26°$, so $BC = 100 \sin 32° / \sin 26° = 120.884$

c) In right triangle ABC, $\sin 58° = AB / BC$, so $AB = BC \sin 58° = 102.515$.
The whole height = 5/3 (yards) + AB = 104.182 yards. The author's answer is 104.17 yards.
The discrepancy comes from rounding. Essentially the answer is 104.2 yards.

d) This needs only right triangles.
$\tan BCA = BA / CA$, so $CA = BA / \tan BCA = 102.515 / \tan 58 = 64.058$ yards. This should be rounded to 64.1 yards

Inaccessible Height – General Case

Teacher's Note: This is a revisit of the inaccessible height problem which was previously solved using right triangles. The advantage of using the Law of Sines is simplicity and brevity. The point here is not so much a formula as a process and reminding the students about the role of generalization in mathematics.

a) $\angle BCA = \alpha - \beta$.

b) In triangle BCA, we have, by the Law of Sines, $CA = (d \sin \beta) / (\sin \angle BCA)$

c) From right triangle ECA, we get $CE = CA \sin \alpha = (d \sin \beta \sin \alpha) / (\sin \angle BCA)$

d) $CF = s + (d \sin \beta \sin \alpha) / (\sin \angle BCA)$

Distance between Edifice and Wood When I Can't Get to Either One

Teacher's Note: This multi–triangle exercise applies the Law of Sines and the Law of Cosines. Instead of triangle AEW with sides AE and AW, we could choose triangle BEW and sides BE and BW just as well.

a) Edifice angle $\angle AEB = 39° 42'$. Then by the Law of Sines,
$AE / \sin \angle ABE = AB / \sin \angle AEB$, so $AE / \sin 42° 22' = 536 / \sin 39° 42'$ and
$AE = 536 \sin 42° 22' / \sin 39° 42' = 565.457$.

b) $\angle AWB = 26° 15'$. Then, by the Law of Sines, $AW / \sin \angle ABW = AB / \sin \angle AWB$, or
$AW / \sin 113° 29' = 536 / \sin 26° 15'$. So $AW = 536 \sin 113° 29' / \sin 26° 15' = 1111.506$

c) **Teacher's Note:** Here we depart from the method of solution of the nineteenth–early twentieth century. Before calculators, people used logarithms to shortcut multiplying and dividing by decimal numbers. Here they would use the Law of Tangents, which lent itself better to logarithms. Today we prefer the Law of Cosines, and that is what the rest of this solution will use. The Law of Tangents can be seen in the unit on Trigonometric Identities.

$EW^2 = AE^2 + AW^2 - 2(AE)(AW) \cos \angle EAW = 882{,}878.819$. Therefore, $EW = 939.616$ yards. The author's answer was 939.634 yards.

Teacher's Note: The above exercise is but one accessible example representing the vast application of both planar and spherical trigonometry to surveying, triangulation, and mapping. This area provides rich connections with the subject area of the history of a place and its peoples, perhaps as close and easily identified with as the students' own neighborhood.

References for Triangle Applications

General

Eagle, M. Ruth, *Exploring Mathematics through History.* Cambridge University Press 1995.

Grattan–Guinness, Ivor, *The Norton History of the Mathematical Sciences: The Rainbow of Mathematics*. W. W. Norton & Company 1998.

Katz, Victor J., *A History of Mathematics,* 2nd ed. Addison–Wesley 1998.

Maor, Eli, *Trigonometric Delights*. Princeton University Press 1997.

Smith, David E., *History of Mathematics, Volume II.* Dover Publications 1958.

http://www-groups.dcs.st-and.ac.uk/~history
The MacTutor History of Mathematics archive contains mathematicians, history topics, and famous curves. Accessed 26 June 2001.

Biographies

http://es.rice.edu/ES/humsoc/Galileo/galileo_resources.html
Lives of more than 600 scientists. Accessed 26 June 2001.

Distance & Angle in 16th & 17th century Battle

Blagrave, John, *A Booke of the making and use of a Staffe, newly invented by the Author, called the Familiar Staffe.* Hugh Jackson, London 1590. Facsimile published by Da Capo Press NY 1970.

Burke, James, *The Day the Universe Changed.* Little, Brown and Company (Back Bay Books) 1985. The chapter, "Infinitely Reasonable", describes how analyzing cannon ball motion in the sixteenth and seventeenth centuries led into Newton's gravitational model.

http://www.mhs.ox.ac.uk/geometry/summary.htm
Museum of History of Science at Oxford, "The Geometry of War" in the 16th century – Gunnery: "prediction of range in relation to the elevation of a gun," important practically and tough mathematically, and pictures of instruments. Range finding and Surveying: angles and distances, triangulation, instruments. Accessed 26 June 2001.

Also see http://www.mhs.ox.ac.uk/exhibits/index.htm for on–line exhibits such as "The Measurers," which contains some other uses of mathematics. Accessed 26 June 2001.

Islamic Mathematics

Berggen, J. L., *Episodes in the Mathematics of Medieval Islam.* Springer–Verlag 1986.

Navigation

Resnikoff, H. L., and Wells, R. O. *Math and Civilization.* Holt Rinehart Winston 1973.

Taylor, E.G.R., *The Haven–Finding Art*: *A History of Navigation from Odysseus to Captain Cook.* Abelard–Schuman Ltd NY 1957, and Hollis and Carter 1971.

Surveying and Mapping

Berton, Simon and Robinson, Andrew, *The Shape of the World: The Mapping and Discovery of the Earth.* Rand McNally 1991.

Calinger, Ronald, ed., *Vita Mathematica: Historical Research and Integration with Teaching*. The Mathematical Association of America 1996.

Textbook Problems

Davies, Charles, *Elements of Geometry and Trigonometry, from the works of A. M. Legendre,* adapted to the course of mathematics instruction in the United States. Barnes & Burr NY 1864.

Webber, Samuel, *Mathematics, Vol. II*, Compiled from the Best Authors. Cambridge University Press 1808.

Trigonometric Identities
Teacher Notes

Description of Unit: Three categories of identities are practiced here. "Applying Elementary Identities" is a worksheet with space for students to write. "Applying Sum and Difference Formulas, Double Angle and Half Angle Formulas" and "Applying the Law of Sines" are reproducible problem sets. To keep abreast with the text chapters, each category may be used when and where the teacher chooses. For time flexibility, some page breaks within categories allow the teacher to reproduce a portion consisting of a few short problems or one long problem, rather than assign an entire category. Solution keys give the process in steps. All the problems associate with mathematicians and therefore suggest historical reasons why the identities arose, while text exercises usually are disconnected and listed without rationale.

The product-to-sum formulas appear in the "Applying Elementary Identities" section because here they derive from right triangle ratio definitions of sine and cosine and some standard geometry. Problem #9 shows how the product-to-sum formulas unburdened people's calculations. Because those formulas have historical importance and are derived from geometry and elementary identities, every trigonometry course should include, not skip, them. If the teacher chooses, they can be placed conventionally after the sum and difference identities and can be derived from them.

The variety of problems serves students from geometry to precalculus levels. There is intrinsic value for trigonometry students as well as challenge toward facility in a future calculus course. Although calculators have reduced using identities to process numbers, the mathematical structure of trigonometry and the beauty of patterns justify continued interest.

Prerequisites: "Applying Elementary Identities" requires the right triangle ratio definitions of the six trigonometric functions, some geometry (Pythagorean property, and measures of central and inscribed angles), and the basic trigonometric identities. Problems requiring the right triangle definitions and geometry are #1, 4, and 6. Problem #5 requires sometimes laying aside the right triangle ratio definition of sine and using the Hindu sine, which was half the chord subtended by twice the central angle. In #5 step b)(3), students may need reminding that the circle has radius 3438 units and that the Hindus calculated sin 30^o that way.

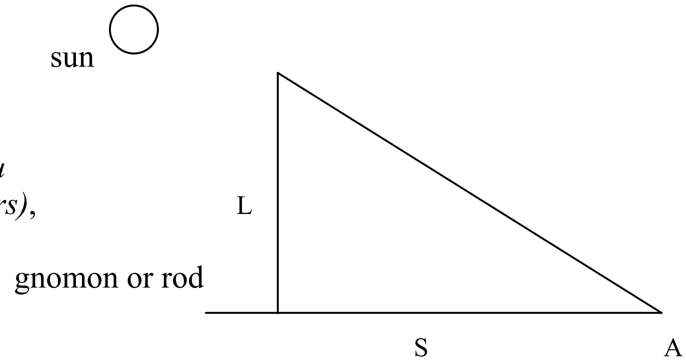
"Applying Sum and Difference Formulas, Double Angle and Half Angle Formulas" begins with a list of the identities required to solve the problems, namely sine and cosine of sum, difference, double, and half angles. One step in problem #8 provides the result of applying the Rational Root Theorem, which students might not have learned in Algebra 2. As an optional link, Ptolemy's derivation of these trigonometric identities can be found in the unit titled Development of Ptolemy's Table.

Finally, besides the Law of Sines, "Applying the Law of Sines" requires both preceding categories of identities.

Materials: The textbook provides the nature of trigonometric identities and the general method of verifying them. Reading it and understanding the examples should precede class use of this unit. Handouts of these historical problems may either supplement or replace the text exercises depending on the skill objectives set by the teacher. A calculator will ease the arithmetic in "Applying Elementary Identities" #5 c d.

Trigonometric Identities
Applying Elementary Identities
Student Pages

1. Al-Battani's Ratio of Sines. An Arab prince of Batan in Syria, Al-Battani from Harran (858–929) was also an astronomer. In *De Motu Stellarum (On the Motion of the Stars)*, he stated that



$$S = \frac{L \sin(90 - A)}{\sin A}$$

Al-Battani's ratio of sines enabled people to determine angle of elevation A of the sun. People measured the length S of the shadow made by a vertical gnomon (rod) whose length was L. The shadow studies allowed the Muslims to tell time accurately for prayer, as well as maintain a calendar.

From the time of the Greek astronomer Hipparchus (c. 190–120 B.C), who was the father of trigonometry, only the sine function had a name. Thus, Al-Battani used only sine. However, the quantity sin $(90^{\circ} - A)$ kept occurring in shadow studies.

a) Express sin $(90^{\circ} - A)$ simply by using another, modern trigonometric ratio.

b) Supposing you knew S and L and only had a table of sines. How would you determine A?

2. Abu'l-Wafa's Definitions. The Arab scholar Abu'l-Wafa (940–998) recorded the following relationships for the tangent, cotangent, secant, and cosecant functions:

$\tan A = \sin A / \cos A \qquad \sec A = \sqrt{1+\tan^2 A}$

$\cot A = \cos A / \sin A \qquad \csc A = \sqrt{1+\cot^2 A}$

Abu'l-Wafa created sine tables for every ¼ ^{o} of arc; his value for sin ½ ^{o} is accurate to 8 places. He made shadow tables of ratios of cos A / sin A for every 1^{o}. Mathematicians of his time used all 6 trigonometric ratios, the versine (radius – cosine), and half angle relations for sine and cosine.

Express Al-Battani's ratio of sines as a single function of Abu'l-Wafa's.

3. An Elementary Identity of Abu'l-Wafa's. Abu'l-Wafa gave this identity:

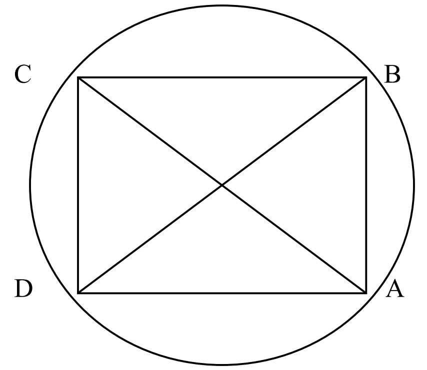
$$\tan A / \sec A = \sin A$$

Prove it using Abu'l-Wafa's definition of tangent and a modern reciprocal identity.

4. Ptolemy's Theorem, and the Fundamental Identity of Trigonometry.

Ptolemy (c. 85–165 AD) proves the following theorem in the *Almagest* in order to develop sum and difference formulas for chords:

The product of the diagonals of any quadrilateral inscribed in a circle equals the sum of the products of the opposite sides.



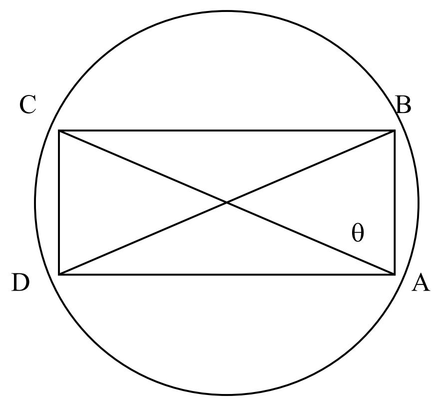
AC x BD= AB x CD + BC x AD

(See below for the proof of this result and some of its consequences.)

Here we derive from Ptolemy's Theorem the Fundamental Identity of Trigonometry, that $\sin^2 \theta + \cos^2 \theta = 1$.

a) Let quadrilateral *ABCD* be an inscribed rectangle.

Then side *CD* is equal to what other side?
Side *AD* is equal to what other side?
Diagonal *BD* equals what other side?



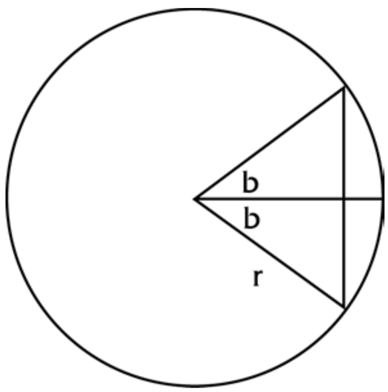
b) Substitute for *CD*, *AD*, and BD into Ptolemy's Theorem, to obtain the equation

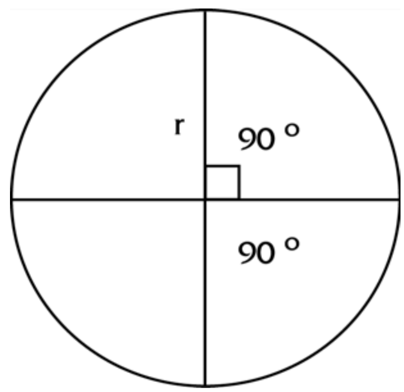
____.

The result is same as the property known as ________________________________ .

c) Let $AC = 1$. Let $\theta = \angle BAC$. Use the definitions that $\sin \theta$ = opp / hyp and $\cos \theta$ = adj / hyp, to develop $\sin^2 \theta + \cos^2 \theta = 1$.

5. Aryabhata's Table of Sines. The Indian astronomer Aryabhata (475–ca.550) was the first to formalize the concept of a sine as a half–chord in a circle. The Indian definition of the sine of an angle, *jya–ardha*, was half the chord subtended by twice the central angle. In this section, we will denote the Indian sine of an angle θ by Sine θ. For example,
Sine 90^o = ½ chord 180^o = ½ diameter = radius.

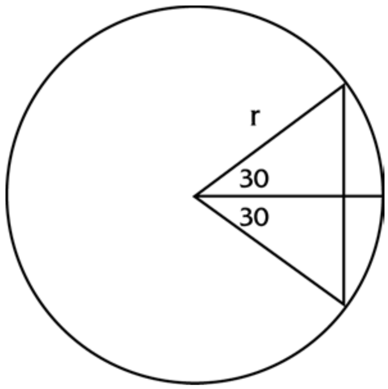




Hindu process for finding sine *b*:
1-Double the angle *b*, and draw the central angle whose size 2*b*.
2-Draw the chord intercepted by angle 2*b*
3-Find half the chord of angle 2*b*. The length of the half chord is sine *b*.

a) Indian radius of the circle: Aryabhata gave a table of Sines in his *Aryabhatiya* (about 510) for 0^o to 90^o in steps of 3 ¾ ^{o} (1/24 of a right angle). He used a circle of radius 3438, the same value that had been used by the early Greek astronomer Hipparchus (190–120 BCE). Why 3438? Because the angle or arc was to be measured in degrees and minutes, Hipparchus (and Aryabhata) decided to use the same measure for the radius of the circle. That is, they chose the circumference to be 21600 (which equals 6 × 60 × 60, the total number of minutes in a complete circle), and they knew the value of $\pi = 3.1416$. Show that the radius, to the nearest whole number, is 3438.

b) Calculating Sine 30°. To calculate Sine 30°, Aryhabhata first doubled 30° to obtain 60°. He did it because, by the Indian definition, the Sine is half the chord subtended by twice the central angle.

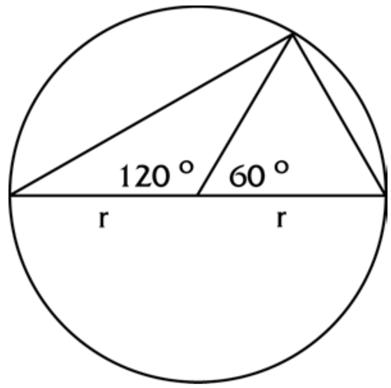


Complete the following steps and reasoning:

Step	Reason
(1) Sine 30° = ½ chord 60°	(1) __?__
(2) Sine 30° = ½ radius	(2) __?__
(3) Sine 30° = 1719	(3) __?__

c) Calculating Sine 60°.

Step	Reason
(1) Twice 60° is __?__.	(1) __?__
(2) $(\text{chord } 120^\circ)^2 + (\text{chord } 60^\circ)^2 = \text{diameter}^2$	(2) __?__ (See figure directly below. Hint: If an inscribed triangle has the diameter as a side, then it is a right triangle.)



(3) But Sine 60° = ½ chord 120° and Sine 30° = ½ chord 60°. Then 2 Sine 60° = chord 120° and 2 Sine 30° = chord 60°.

(3) __?__ and algebra.

(4) Then $(2 \text{ Sine } 60^\circ)^2 + (2 \text{ Sine } 30^\circ)^2 = (2r)^2$

(4) Substitution into step __?__.

(5) Simplifying,

__?__ then __?__ $= r^2$

(5) Algebra.

Now substitute the Indian values of Sine 30° and the radius into that equation.

Then solve for the value of Sine 60° to the nearest integer.

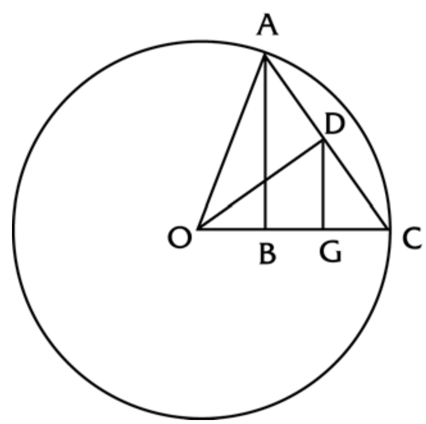
To compare with the modern value of sin 60°, divide Aryabhata's value by the circle's radius.

You should have obtained ≈ 0.86591... The modern value of sin 60° is √3 / 2 ≈ 0.86602....

d) Aryabhata's Cosines and the half-angle formula.

Aryabhata called the sine of the complementary angle Cosine or *kotijya*. He tabulated Cosine values by having the same column represent Sine A and Cosine$(90^\circ - A)$ for A from 0° to 45°. For example, Sine 30° = Cosine 60°, where we are using the notation Cosine θ to represent the Indian cosine. It follows from the Pythagorean Theorem that $\text{Sine}^2 A + \text{Cosine}^2 A = r^2 = 3438^2$.

Sines of 15°, $7\tfrac{1}{2}^\circ$, and $3\tfrac{3}{4}^\circ$ were computed by a half-angle formula. To determine the half-angle formula, the Indians probably proceeded somewhat like the following. Consider the central angle $\theta = AOC$ inscribed in the circle of radius $r = 3438$. Then if AB is drawn perpendicular to OC, we know that AB is equal to Sine θ. Now connect AC, and draw OD bisecting angle θ. Draw DG perpendicular to OC. Then DC is Sine(θ/2). (Why?)



Step		Reason	
(1)	Triangle DGC is similar to triangle ODC.	(1)	$\angle ODC$ = ? $\angle DCO$ = ? AA
(2)	$DC/GC = OC/DC$	(2)	?
(3)	$GC = \tfrac{1}{2} BC$	(3)	Triangle ABC is similar to triangle DGC (why?) and $DC = \tfrac{1}{2} AC$ (why?)
(4)	OB = Cosine θ	(4)	Why?
(5)	$BC = r - OB = r -$ Cosine θ	(5)	From the diagram and substitution.

(6) $\text{Sine}(\theta/2)/\ \tfrac{1}{2}(r - \text{Cosine}\ \theta) = r/\text{Sine}(\theta/2)$ (6) Substitution in (2).

(7) $\text{Sine}^2(\theta/2) = \tfrac{1}{2}\, r(r - \text{Cosine}\ \theta)$ (7) Algebra. Explain.

(8) $\text{Sine}(\theta/2) = \sqrt{1719(3438 - \text{Cosine}\theta)}$ (8) Substitution for r and taking square roots.

(9) By replacing r by 1 in (7) and replacing the Indian Sine and Cosine by the modern sin and cos, derive the formula for sin ½ θ.

(e) Calculating Sine 15˚.

(1) Calculate Sine 15° from Aryabhata's formula from (d) (8) and your knowledge of Cosine 30˚ = Sine 60˚. Express your answer to the nearest integer.

(2) Calculate sin 15° by using the identity $\sin \tfrac{1}{2}\theta = \sqrt{\dfrac{1-\cos\theta}{2}}$ and the value cos 30° = $\sqrt{3}$ / 2. Compare the values from (1) and (2) by dividing the Indian value by 3438.

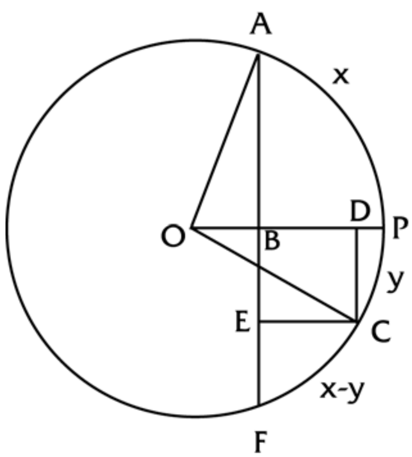
(3) Calculate the Indian values of Sine 7 ½ ˚ and Sine 3 ¾ ˚. You will first need to calculate Cosines, but these can be found from the Sine values by using the equation $\text{Sine}^2 A + \text{Cosine}^2 A = r^2 = 3438^2$. By use of the Sine values for 7 ½ ˚ and 3 ¾ ˚, the Indians obtained tables of Sines of angles from 3 ¾ ° to 90° in steps of 3 ¾ °. They attained greater precision for astronomical work by discovering interpolation methods and approximation formulas. In 1150 Bhaskara gave a system for finding the sine for arcs closer than 3 ¾ °.

6. Viète's Sum of Sines. Viète (1540–1603) showed that

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y).$$

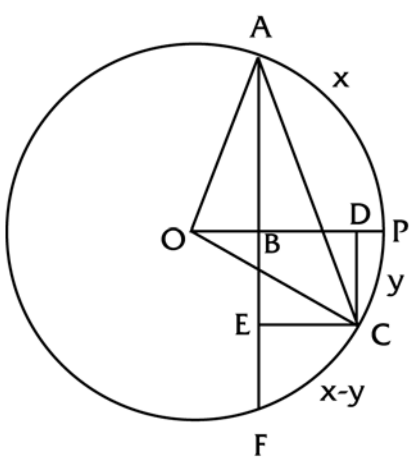
He did this using geometry, and the definitions sin A = opposite / hypotenuse and cos A = adjacent / hypotenuse. Complete the steps in Viète's development:

1) Choose the radius of the circle to be 1, with $\angle AOP = x$ = measure of arc AP and $\angle COP = y$ = measure of arc CP. Then $\sin x$ = length of segment __?__, and $\sin y$ = length of segment __?__.



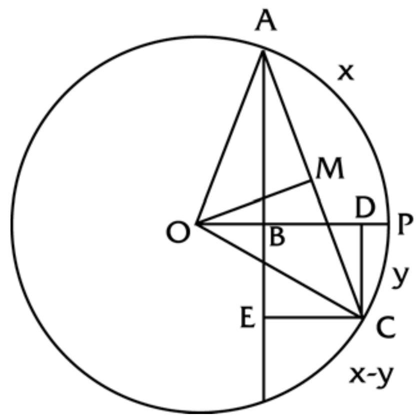
2) But length __?___ = CD. Then $\sin x + \sin y = AB + BE$ = length of segment __?__. Notice arc AP = arc $PF = x$, and that therefore arc $CF = x - y$.

3) Notice that arc CF is intercepted by the inscribed angle $\angle FAC$. Thus $\angle FAC$ = __?__.



4) In the right triangle AEC, cos $\angle FAC$ = __?__ / AC. Then AE = __?__.

5) We now show that $AC = 2 \sin \frac{1}{2}(x + y)$ as follows: Look at triangle AOC. It is isosceles, with equal sides __?__ and __?__.



Draw the altitude OM from O to AC. Then triangle AOM is a right triangle, with sin $\angle AOM$ = __?__.

$\angle AOC$ intercepts arc $AC = x + y$.

In isosceles triangle AOC, the altitude OM bisects $\angle AOC$ and also side AC. It follows that $\angle AOM = \frac{1}{2}\ \angle$ __?__ $= \frac{1}{2}(x + y)$, and sin $\angle AOM$ = sin __?__. It also follows that $AC = 2$ __?__ , and we conclude that $AC = 2 \sin \frac{1}{2}(x + y)$.

6) Finally, the facts are gathered:

From step (2), $\sin x + \sin y = AE$
From step (4), $AE = AC \cos \frac{1}{2}\ (x - y)$
From step (5), $AC = 2 \sin \frac{1}{2}(x + y)$

Therefore, $\sin x + \sin y$ = __?__.

We can derive a similar formula for $\cos x + \cos y$ by using the identity $\cos x = \sin(90 - x)$ and the formula just derived:

$\cos x + \cos y = \sin(90 - x) + \sin(90 - y) = 2 \sin \frac{1}{2}(180 - x - y)\cos \frac{1}{2}(y - x)$

$= 2 \sin(90 - \frac{1}{2}(x + y))\cos \frac{1}{2}(y - x) = 2 \cos \frac{1}{2}(y + x)\cos \frac{1}{2}(y - x)$.

7. Difference of Sines. An identity for $\sin x - \sin y$ can be developed from Viète's procedure for the sum of sines (see problem 6), by drawing x and y on the same side of the radius OP. However, in this exercise, use an "opposite angle" relationship instead, to show from Viète's formula for $\sin x + \sin y$, that

$$\sin x - \sin y = 2 \sin \tfrac{1}{2} (x - y) \cos \tfrac{1}{2} (x + y).$$

The opposite angle relationships we need here are $\sin(-A) = -\sin A$ and $\cos(-A) = \cos A$.

It follows that $\sin x - \sin y = \sin x + (-\sin y) = \sin x + \sin(-y) =$ __?__

Now, using the same idea as in the previous problem, derive the formula

$$\cos x - \cos y = 2 \sin \tfrac{1}{2}(y - x) \sin \tfrac{1}{2}(y + x)$$

8. Viète's Product-to-sum Identity for sin *A* cos *B*. Into his sum of sines identity, Viète put $\frac{1}{2}(x + y) = A$ and $\frac{1}{2}(x - y) = B$. Substitute just as he did, and simplify, to develop a "product-to-sum" formula for $\sin A \cos B$. You should get

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B\,.$$

Using similar techniques with the other three formulas developed in 6 and 7, derive the following three "product-to-sum" and "product-to-difference" formulas:

$$\cos(A-B) + \cos(A+B) = 2\cos A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2\sin B \cos A$$

$$\cos(A-B) - \cos(A+B) = 2\sin B \sin A$$

9. Using the Product-to-Sum and Product-to-Difference Identities. By the sixteenth century, trigonometry had to meet the demands of accuracy in surveying, navigation, and calendar making as well as astronomy. Tables by Georg Joachim Rheticus (1514–1576), who chose his circle radius to be 10,000,000, contained sine and cosine values of up to 9 digits. Rheticus started tables of tangent and secant based on a radius of 1,000,000,000,000,000 ($=10^{15}$), a feat completed after his death by his student Otho. But often, to solve problems involving triangles, mathematicians had to multiply sine values or cosine values together. Multiplication was done by hand without benefit of modern decimal notation, and it was therefore extremely tedious to multiply nine–digit numbers. Besides, such long multiplications were very prone to error. So mathematicians used the product-to-sum and product to difference formulas to replace multiplication by addition or subtraction, much simpler operations.

Do the following example *without* a calculator so that you can play the role of a sixteenth century mathematician.

Multiply: $\sin 28^{\circ} \times \sin 9^{\circ}$.

Method 1, straight multiplication.

In a table, you find that $\sin 28^{\circ} = 0.469472$ and $\sin 9^{\circ} = 0.156434$. (We are only using six places rather than nine or ten.) Now multiply the values for $\sin 28^{\circ}$ and $\sin 9^{\circ}$ by hand. You may not use your calculator.

Method 2, using a product-to-difference identity.

Use the identity $2 \sin B \sin A = \cos (A - B) - \cos (A + B)$ from problem 8 and this information:

$\sin 28^{\circ} = 0.469472$	$\cos 19^{\circ} = 0.945519$
$\sin 9^{\circ} = 0.156434$	$\cos 37^{\circ} = 0.798636$

Remember, you may not use your calculator even to add and subtract.

Which method is easier?

Solution Key
Applying Elementary Identities

1. Al-Battani's Ratio of Sines.

a) Cosine is the cofunction of sine; sin $(90^{\circ} - \text{A}) = \cos \text{A}$.

b) We know that $\dfrac{S}{L} = \dfrac{\sin(90 - A)}{\sin A}$. Let C be the known value S/L. Then

$$C^2 + 1 = \frac{\sin^2(90 - A)}{\sin^2 A} + \frac{\sin^2 A}{\sin^2 A} = \frac{1}{\sin^2 A}.$$

It follows that $\sin^2 A = \dfrac{1}{C^2 + 1}$ and therefore that $\sin A = \sqrt{\dfrac{1}{C^2 + 1}}$. Once we know sin A, we can find A by using the table of sines in reverse.

2. Abu'l-Wafa's Definitions.

$S = L \cos A / \sin A = L \cot A$

3. An Elementary Identity of Abu'l-Wafa's.

$\tan A / \sec A \quad = (\sin A / \cos A) / \sec A$ by Abu'l-Wafa's definition of tangent

$$= \left(\frac{\sin A}{\cos A}\right)\left(\frac{1}{\sec A}\right) = \frac{\sin A}{\cos A \sec A} = \sin A$$

since $\cos A \sec A = 1$ by a reciprocal identity.

4. Ptolemy's Theorem and $\sin^2 \theta + \cos^2 \theta = 1$.

a) AB, BC, AC

b) $AC^2 = AB^2 + BC^2$
Pythagorean Theorem

c) $\sin \theta = BC / AC$, then $BC = AC \sin \theta$
$\cos \theta = AB / AC$, then $AB = AC \cos \theta$
Substitute into $AC^2 = AB^2 + BC^2$.
Then $AC^2 = (AC \cos \theta)^2 + (AC \sin \theta)^2$
$AC^2 = AC^2 \cos^2 \theta + AC^2 \sin^2 \theta$
$1 = \cos^2 \theta + \sin^2 \theta$
Therefore $\sin^2 \theta + \cos^2 \theta = 1$.

5. Aryabhata's Table of Sines.

a) $C = 2\pi r$

$r = C/2\pi = 21600/2\pi$

$r = 3437.739 \approx 3438$

b) (1) The Indian Sine is half the chord subtended by twice the central angle.

(2) chord 60^o = radius, because the chord of 60° forms the base of an equilateral triangle, the two other sides being radii of the circle. Thus the chord is equal to a radius.

(3) 1719 is half of the radius 3438.

c) (1) 120^o. We need this because the Indian Sine is half the chord subtended by twice the central angle.

(2) An inscribed triangle with a diameter for one side, is a right triangle. (An inscribed angle intercepting a semicircle is a right angle.)

(3) Indian definition of Sine.

(4) Substitution into step #2.

(5) $4\ \text{Sine}^2\ 60^o + 4\ \text{Sine}^2\ 30^o = 4\ r^2$

Simplifying, then $\text{Sine}^2\ 60^o + \text{Sine}^2\ 30^o = r^2$.

Substituting the Indian values gives $\text{Sine}^2\ 60^o + 1719^2 = 3438^2$.

Then $\text{Sine}^2\ 60^o = 3438^2 - 1719^2 = 8864883$

$\text{Sine}\ 60^o = 2977.395338... \approx 2977$

In comparison, $2977/3438 = 0.865910413...$

d) In the diagram, DC is half the chord subtending the angle θ, which is twice the angle $\theta/2$. Since the Indian sine is half the chord subtended by double the angle, DC is Sine($\theta/2$). The reasons in the derivation of the half-angle formula are as follows:

(1) $\angle ODC = \angle DGC$, because both are right angles; $\angle DCO = \angle DCG$, because they are the same angle. Thus two angles in triangle DGC are equal to two angles in triangle ODC, and the triangles are similar by AA.

(2) Corresponding sides in similar triangles are proportional.

(3) Triangles ABC and DGC are similar, because they are both right triangles and they share a common angle. So they are similar by AA. Now $DC = \frac{1}{2} AC$, because the angle bisector OD bisects the base of the isosceles triangle AOC. Therefore, since corresponding sides in similar triangles are proportional, we have $GC/BC = DC/AC = \frac{1}{2}$; so $GC = \frac{1}{2} BC$.

(4) OB is the Sine of $\angle OAB$, so it is the Cosine of the complementary angle θ.

(5) This step follows from the diagram.

(6) This step follows by substituting the values for *DC*, *GC*, and *OC* into the proportion statement (2).

(7) Use cross multiplication to turn the proportion statement of (6) into this equation.

(8) Since $r = 3438$, we substitute this value into (7) and take the square root.

(9) If we replace r by 1 and the Indian functions by the modern ones in (7), we get the equation $\sin^2(\theta/2) = \frac{1}{2}(1-\cos\theta)$. By taking square roots of both sides, we get the modern half-angle formula:

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}}.$$

(e) (1) Sine 15° = Sin ½ (30°) = $\sqrt{1719(3438 - \text{Cosine}\,30)}$

$= \sqrt{1719(3438 - \text{Sin}\,60^\circ)}$

$= \sqrt{1719(3438 - 2977)} = \sqrt{1719(461)} \approx 890$

(2) We have sin 15° = sin ½ (30°) = $\sqrt{\frac{1-\cos 30^o}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \approx 0.25882$

To compare, $890 / 3438 \approx 0.25893$.

(3) We first calculate Cosine 15° = $\sqrt{3438^2 - \text{Sine}^2 15^\circ} =$
$\sqrt{3438^2 - 890^2} = \sqrt{11027744} = 3321$

Therefore, Sine 7 ½ ° = Sine ½(15°) =
$\sqrt{1719(3438 - \text{Cosine}\,15^\circ)} = \sqrt{1719(3438-3321)} = \sqrt{201123} = 448$.

Next, Cosine 7 ½ ° = $\sqrt{3438^2 - \text{Sine}^2 7\frac{1}{2}} = \sqrt{3438^2 - 448^2} = \sqrt{11619140} = 3409$.

So Sine 3 ¾° = Sine ½(7 ½°) =
$\sqrt{1719(3438 - \text{Cosine}\,7\frac{1}{2}^\circ)} = \sqrt{1719(3438-3409)} = \sqrt{49851} = 223$.

6. Viète's Sum of Sines.

(1) sin x = length of segment *AB*; sin y = length of segment *CD*.

(2) $BE = CD$; $AB + BE = AE$.

(3) $\frac{1}{2}(x - y)$

(4) $\cos\angle\ FAC = AE/AC$; $AE = AC\cos\frac{1}{2}(x - y)$

(5) The equal sides are OA, OC; $\sin\angle AOM = AM / OA = AM / 1 = AM$; $\angle AOM = \frac{1}{2}\ \angle AOC$; $\sin\angle AOM = \sin\frac{1}{2}(x + y)$; $AC = 2\,AM$

(6) $\sin x + \sin y = AE = AC\cos\frac{1}{2}(x - y) = 2\sin\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y)$

7. Difference of Sines.

$$\sin x - \sin y = \sin x + \sin(-y) = 2\sin\tfrac{1}{2}[x + (-y)]\cos\tfrac{1}{2}[x - (-y)]$$
$$= 2\sin\tfrac{1}{2}(x - y)\cos\tfrac{1}{2}(x + y)$$

$$\cos x - \cos y = \sin(90 - x) - \sin(90 - y) = 2\sin\tfrac{1}{2}(y - x)\cos\tfrac{1}{2}(180 - x - y)$$
$$= 2\sin\tfrac{1}{2}(y - x)\cos(90 - \tfrac{1}{2}(x + y) = 2\sin\tfrac{1}{2}(y - x)\sin\tfrac{1}{2}(y + x)$$

8. Viète's Product-to-sum Identity for sin *A* cos *B*

Let $A = \frac{1}{2}(x + y)$ and $B = \frac{1}{2}(x - y)$. Then $A + B = x$ and $A - B = y$. Substituting into the formula $\sin x + \sin y = 2\sin\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y)$ gives $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$.
If we let $A = \frac{1}{2}(x + y)$ and $B = \frac{1}{2}(y - x)$, the formula $\cos x + \cos y = 2\cos\frac{1}{2}(y + x)\cos\frac{1}{2}(y - x)$ becomes $\cos(A - B) + \cos(A + B) = 2\cos A\cos B$.
Similar substitutions produce the other two formulas.

9. How the Product-to-sum Identities Made People's Work Easier.

Method 1, straight multiplication:

```
        0.469472
        0.156434
         1877888
        1408416
       1877888
      2816832
     2347360
     469472
  0.073441382848
```

Method 2, using a product-to-difference identity:

$\cos(28 - 9)^{\circ} - \cos(28 + 9)^{\circ} = \cos 19^{\circ} - \cos 37^{\circ} = 0.945519 - 0.798636 = 0.146883.$

Therefore, $2\sin 28^{\circ}\times\sin 9^{\circ} = 0.146883$ and $\sin 28^{\circ}\times\sin 9^{\circ} = 0.0734415$

Trigonometric Identities
Applying Sum and Difference Formulas, Double Angle and Half Angle Formulas
Student Pages

This set of problems uses the following identities:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\sin \tfrac{1}{2} A = \pm\sqrt{\frac{1-\cos A}{2}}$$
$$\cos \tfrac{1}{2} A = \pm\sqrt{\frac{1+\cos A}{2}}$$

1. Ibn Yunus's Product-to-sum Formula. Ibn Yunus (950–1009) was an astronomer who lived in Cairo, Egypt. He compiled values of sine and tangent by 1° accurate to 8 places. The product-to-sum identities provided a way ("prosthaphaeresis", Greek for addition and subtraction) to rephrase a multiplication problem as addition or subtraction, which was easier as we have seen above. Ibn Yunus used this identity:

$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

Although we have already derived this formula one way, in the previous section, derive it anew by starting with the right side of the equation and using some of the identities above.

2. A Sine Formula from Viète's *Canon*. The *Canon mathematicus seu ad triangula* (1579) of Francois Viète (1540–1603) was the first European book to organize all six trigonometric functions in the solution of plane and spherical triangles. Verify this identity, which comes from Viète's *Canon*:

$$\sin \mathrm{A} = \sin(60^{\circ} + \mathrm{A}) - \sin(60^{\circ} - \mathrm{A}).$$

Begin by applying the sum and difference formulas for sine to the right side.

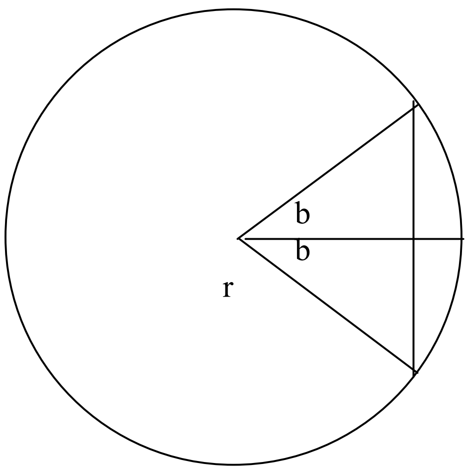
3. Abu'l-Wafa's Sine of a Double Angle. Abu'l-Wafa said that in a circle

$$\frac{\text{chord } A}{\text{chord } \tfrac{1}{2} A} = \frac{\text{chord } (180^\circ - \tfrac{1}{2} A)}{r}$$

Show that this is equivalent to the double angle sine identity, sin 2*B* = 2 sin *B* cos *B*, by completing the steps below:

a) The Indian definition of a sine, also used by Islamic mathematicians, was that in a circle, the sine of an arc is half the chord subtended by twice the central angle. That is,

$$\sin \theta = \tfrac{1}{2} \text{ chord } 2\theta.$$



Hindu process for finding sine *b*:

1- Double the angle *b*, and draw the central single whose size is 2*b*.
2- Draw the chord intercepted by angle 2*b*.
3- Find half the chord of angle 2*b*. The length of half chord is sine *b*.

In Abu'l-Wafa's formula, *A* is the "double angle" of ½ *A*. The left numerator should eventually contain sin 2*B*, and the denominator sin *B*. However, sin 2*B* = ½ chord 2(2*B*) = ½ chord 4*B*. This means just putting *A* equal to 2*B* won't work; we need to put *A* = 4*B*.

Let *A* = 4*B*. First, express chord *A* and chord ½ *A* in terms of *B*.

chord *A* =
and
chord ½ *A* =

b) Express sin 2*B* in terms of chord *A*, and express sin *B* in terms of chord ½ *A*.

sin 2*B* =
and
sin *B* =

c) Express chord (180° – ½ *A*) in terms of *B*.

chord (180° – ½ *A*) =

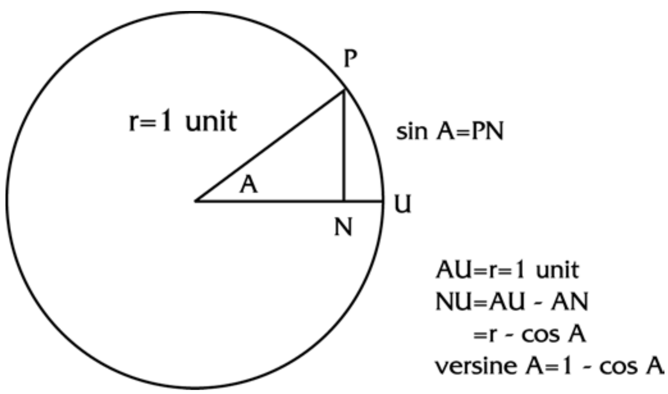
d) Abu'l-Wafa's formula said that

$$\frac{\text{chord } A}{\text{chord } \frac{1}{2} A} = \frac{\text{chord } (180^\circ - \frac{1}{2} A)}{r}$$

Substitute the expressions you found into the place of chord A, chord ½ A, and chord (180° – ½ A).

e) Use a cofunction identity to simplify that equation, and let radius $r = 1$ unit. Finally write the formula for sin $2B$.

4. The Indian Versine Function. Mathematicians of India used another function besides the six standard one, namely, the versine of an arc, defined, for circles of radius 1, as versin $A = 1 - \cos A$. They considered the versine to be the sine turned (versed) through 90° onto its side. In seventeenth century Europe, versine became the second most important function, after sine.



Prove this identity, used by Indian mathematicians: versin $2A = 2 \sin^2 A$

5. Viète's Tangent and Cotangent of a Half Angle. Prove these 2 identities from Viète's *Canon mathematicus seu ad triangula* (1579):

a) $\csc A - \cot A = \tan (A/2)$

b) $\csc A + \cot A = \cot (A/2)$

6. Ulugh Beg's Sine Cube Formula. Ulugh Beg (1393–1449), ruler of a domain in central Asia centered at Samarkand, computed sine tables for every 1' of arc. To do this, he used a method pioneered by Ghiyath al–Din Jamshid al–Kashi (d. 1429), an astronomer in his employ. Prove this identity used by al–Kashi:

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

Hint: Start with $\sin 3A = \sin (A + 2A)$.

Al-Kashi used this identity to calculate the sine of 1º to 18 decimal places. Let $x = \sin 1^\circ$. Then the identity becomes $3x - 4x^3 = \sin 3^\circ$. Since al-Kashi could calculate the sine of 3º to as many decimal places as he wanted, using the difference formula and the half-angle formula, this equation was a cubic equation whose solution was $x = \sin 1^\circ$. He then figured out an iterative method of solving this equation to as many decimal places as necessary.

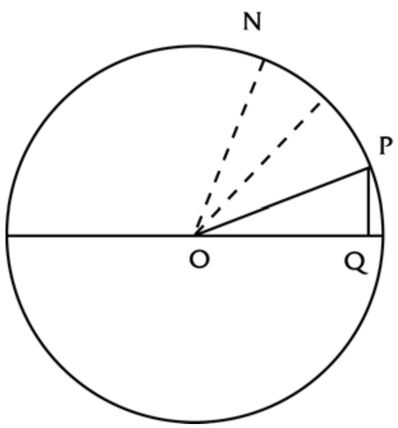
7. Viète's Formula for Cosine of a Multiple Angle. Viète (1540–1603) was the first mathematician to apply algebra to trigonometry in a systematic manner. He developed $\cos 2A$, $\cos 3A$, ..., $\cos 10A$ as functions of $\cos A$ by means of the recurrence formula

$$\cos nA = 2 \cos A \cos (n - 1) A - \cos (n - 2) A.$$

a) Illustrate the use of the formula by using it to calculate $\cos 2A$; that is, substitute $n = 2$ into the recurrence formula, and simplify. Afterwards, compare the result with your standard double angle formula for $\cos 2A$.

b) Use Viète's recurrence formula to develop an identity for $\cos 5A$ in terms of $\cos A$. (You will need to use the recurrence formula several times.)

8. Angle Trisection, a problem answered after 2000 years. The finding of a general method for trisecting an angle with only compass and straightedge was one of the three great unsolved problems of Greek antiquity. In 1837 Pierre Wantzel (1814–1848) used trigonometry to finally prove that the construction was impossible. Wantzel took 60^o as a counterexample and used an identity for cos 3*A*. Complete the following steps in order to learn his process:



a) In the unit circle (radius = 1) shown, $\angle NOQ$ is a central angle with measure 60^o. Suppose $\angle NOQ$ is trisected as shown with $\angle POQ = 20^o$. Draw $PQ \perp OQ$, making triangle *POQ* a right triangle. Express the ratio *OQ* / *OP* as a trigonometric function of 20^o. Then by simple algebra, find an expression for *OQ*.

b) Prove the triple angle identity: $\cos 3A = 4\cos^3 A - 3\cos A$. (Begin with $\cos 3A = \cos(A + 2A)$.)

c) Let $A = 20^o$. Then, substituting into the above equation, $\cos 60^o = \cos 3(20^o) =$ __?__

d) Evaluate $\cos 60^o$, which is one of the key values you should have memorized. Then multiply the equation by 2 to clear of fractions. Write the resulting equation.

e) Substitute $x = 2\cos 20^o$, and write the resulting equation in terms of *x*.

f) Wantzel reasoned that constructing a length, such as base *OQ* in triangle *POQ*, would be equivalent to finding an algebraic solution to an equation which was expressible in terms of rational numbers and square roots. In step e, you should have arrived at the equation $x^3 - 3x - 1 = 0$. Although this cubic equation is solvable, its solution involves cube roots and therefore cannot be expressed using square roots alone. It follows that *OQ* cannot be constructed using straightedge and compass and therefore that an arbitrary angle cannot be trisected using those tools.

Solution Key
Applying Sum and Difference Formulas, Double Angle and Half Angle Formulas

1. Ibn Yunus's Product-to-sum Formula.

$$\cos(x+y)+\cos(x-y) = (\cos x\cos y - \sin x\sin y) + (\cos x\cos y + \sin x\sin y)$$
$$= 2\cos x\cos y$$

2. A Sine Formula from Viète's *Canon*.

$$\sin(60^\circ + A) - \sin(60^\circ - A)$$
$$= (\sin 60^\circ\cos A + \cos 60^\circ\sin A) - (\sin 60^\circ\cos A - \cos 60^\circ\sin A)$$
$$= (\cos 60^\circ\sin A) - (-\cos 60^\circ\sin A)$$
$$= 2\cos 60^\circ\sin A$$
$$= 2(\tfrac{1}{2})\sin A$$
$$= \sin A$$

3. Abu'l-Wafa's Sine of a Double Angle.

Plan: The Indian sine was defined as $\sin\theta = \frac{1}{2}$ chord 2θ. Looking at Abu'l-Wafa's formula, A is the "double angle" of $\frac{1}{2}A$. We want the left numerator to represent $\sin 2B$, and the denominator to represent $\sin B$. However, $\sin 2B = \frac{1}{2}$ chord $2(2B) = \frac{1}{2}$ chord 4B. This means just putting A equal to $2B$ won't work; we need to put $A = 4B$.

a) Let $A = 4B$; then chord A = chord $4B$ and chord $\frac{1}{2}A$ = chord $2B$.

b) $\sin 2B = \frac{1}{2}$ chord $2(2B) = \frac{1}{2}$ chord $4B$, and $\sin B = \frac{1}{2}$ chord $2B$.

c) Since $A = 4B$, then $\frac{1}{2}A = 2B$, and chord $(180^\circ - \frac{1}{2}A)$ = chord $(180^\circ - 2B)$.

d) chord A = chord $4B$ by a), and chord $4B = 2\sin 2B$ by b), so chord $A = 2\sin 2B$.
Also, chord $\frac{1}{2}A$ = chord $2B$ by a), and chord $2B = 2\sin B$ by b), so chord $\frac{1}{2}A = 2\sin B$.
Likewise, chord $(180^\circ - \frac{1}{2}A)$ = chord $(180^\circ - 2B) = 2\sin\frac{1}{2}(180^\circ - 2B) = 2\sin(90^\circ - B)$.
Substituting, we get:

$$\frac{2\sin 2B}{2\sin B} = \frac{2\sin(90^\circ - B)}{r}$$

e) $$\frac{\sin 2B}{\sin B} = \frac{2\cos B}{r}$$

Then $\sin 2B = \sin B\,(2\cos B)$, or $\sin 2B = 2\sin B\cos B$.

4. The Hindu Versine Function.

versin $2A = 1 - \cos 2A = 1 - (1 - 2\sin^2 A) = 2\sin^2 A$

5. Viète's Tangent and Cotangent of a Half Angle.

a) $$\tan(A/2) = \frac{\sin(A/2)}{\cos(A/2)} = \frac{\sqrt{\dfrac{1-\cos A}{2}}}{\sqrt{\dfrac{1+\cos A}{2}}} = \sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$= \sqrt{\frac{(1-\cos A)(1-\cos A)}{(1+\cos A)(1-\cos A)}} = \sqrt{\frac{(1-\cos A)^2}{1-\cos^2 A}} = \sqrt{\frac{(1-\cos A)^2}{\sin^2 A}}$$

$$= \frac{1-\cos A}{\sin A} = \frac{1}{\sin A} - \frac{\cos A}{\sin A} = \csc A - \cot A\,.$$

Therefore, $\csc A - \cot A = \tan(A/2)$.

b) $$\cot(A/2) = 1/\tan(A/2) = \sqrt{\frac{1+\cos A}{1-\cos A}}$$

$$= \sqrt{\frac{(1+\cos A)(1+\cos A)}{(1-\cos A)(1+\cos A)}} = \sqrt{\frac{(1+\cos A)^2}{1-\cos^2 A}} = \sqrt{\frac{(1+\cos A)^2}{\sin^2 A}}$$

$$= \frac{1+\cos A}{\sin A} = \frac{1}{\sin A} + \frac{\cos A}{\sin A} = \csc A + \cot A\,.$$

Therefore, $\csc A + \cot A = \cot(A/2)$.

6. Ulugh Beg's Sine Cube.

$$\begin{aligned}
\sin 3A &= \sin A \cos 2A + \cos A \sin 2A \\
&= \sin A(\cos^2 A - \sin^2 A) + \cos A(2\sin A\cos A) \\
&= \sin A\cos^2 A - \sin^3 A + 2\sin A\cos^2 A \\
&= 3\sin A\cos^2 A - \sin^3 A \\
&= 3\sin A(1-\sin^2 A) - \sin^3 A \\
&= 3\sin A - 3\sin^3 A - \sin^3 A \\
\sin 3A &= 3\sin A - 4\sin^3 A
\end{aligned}$$

7. Viète's Formula for Cosine of a Multiple Angle.

a) Substituting $n = 2$, $\cos 2A = 2 \cos A = 2 \cos A \cos (2 - 1)A - \cos (2 - 2)A$

$$= 2 \cos A \cos A - \cos 0$$
$$= 2 \cos^2 A - 1$$

The result is the same as the standard double angle formula for $\cos 2A$.

b) $n = 5$, $\cos 5A = 2 \cos A \cos 4A - \cos 3A$

But $\cos 4A = \cos 2(2A)$ by double angle formula

$$= 2 \cos^2 2A - 1$$
$$= 2 (2 \cos^2 A - 1)^2 - 1$$
$$= 2 (4 \cos^4 A - 4 \cos^2 A + 1) - 1$$
$$= 8 \cos^4 A - 8 \cos^2 A + 1$$

Also $\cos 3A = 2 \cos A \cos 2A - \cos A$ by Viète's recurrence formula

$$= 2 \cos A (2 \cos^2 A - 1) - \cos A$$
$$= 4 \cos^3 A - 3 \cos A$$

Finally $\cos 5A = 2 \cos A \cos 4A - \cos 3A$

$$= 2 \cos A (8 \cos^4 A - 8 \cos^2 A + 1) - (4 \cos^3 A - 3 \cos A)$$
$$= 16 \cos^5 A - 16 \cos^3 A + 2 \cos A - 4 \cos^3 A + 3 \cos A$$
$$= 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$$

8. Angle Trisection, a problem answered after 2000 years.

a) $OQ / OP = \cos 20^o$, so $OQ = OP \cos 20^o$; since $OP = 1$, therefore $OQ = \cos 20^o$

b) One way is this:

$\cos 3A = \cos (A + 2A)$

$$= \cos A \cos 2A - \sin A \sin 2A$$
$$= \cos A (\cos^2 A - \sin^2 A) - \sin A (2 \sin A \cos A)$$
$$= \cos^3 A - \cos A \sin^2 A - 2 \sin^2 A \cos A$$
$$= \cos^3 A - 3 \sin^2 A \cos A$$
$$= \cos^3 A - 3 \cos A (1 - \cos^2 A)$$
$$= \cos^3 A - 3 \cos A + 3 \cos^3 A$$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

c) $\cos 60^o = \cos 3(20^o) = 4 \cos^3 20^o - 3 \cos 20^o$

d) Since $\cos 60° = ½$, we have, by substitution, $½ = 4 \cos^3 20^o - 3 \cos 20^o$.

Therefore, $1 = 8 \cos^3 20^o - 6 \cos 20^o$

e) $1 = (2 \cos 20^o)^3 - 3 (2 \cos 20^o)$

$1 = x^3 - 3x$ or $x^3 - 3x - 1 = 0$

Trigonometric Identities
Applying the Law of Sines
Student Pages

1. Mollweide's Formula for (a – b)/c, where a, b, and c are sides of a triangle.

Karl Mollweide (1774–1825) was an astronomer and teacher in Leipzig, Germany. In later life, Mollweide preferred mathematics over astronomy and became chair of mathematics at Leipzig. Mollweide designed a map projection of the world which preserves angles.

A form of Mollweide's identities is found in Newton (1707) and in a trigonometry text by Thomas Simpson (1748). These beautiful identities state that a certain ratio combining sides equals a ratio combining angles, with both numerators involving a difference, and with corresponding order! They are useful as a check after solving a triangle, since they contain all 3 sides a, b, c and all 3 angles A, B, C.

Complete the steps in the proof that $\dfrac{a-b}{c}=\dfrac{\sin((A-B)/2)}{\cos(C/2)}$

a) Let a, b, and c be the sides in a triangle, and the 3 corresponding angles opposite be A, B, and C, as standard.

$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$. Why?

b) By algebra, $\dfrac{a}{c}=\dfrac{\sin A}{?}$ and $\dfrac{b}{c}=\dfrac{?}{\sin C}$.

c) By subtracting rational expressions, we get $\dfrac{a-b}{c}=\dfrac{\sin A-\sin B}{?}$.

d) Tell why $\dfrac{a-b}{c}=\dfrac{2\cos((A+B)/2)\sin((A-B)/2)}{2\sin(C/2)\cos(C/2)}$.

Hints: See Applying Elementary Identities, problem 7, Difference of Sines.
Also see the sine double angle identity, and apply $\sin C = \sin 2((1/2)C)$

e) But $A + B = 180^{\circ} - C$, because __?__.
Then by algebra, $\frac{1}{2}(A + B) = 90^{\circ} - \frac{1}{2}C$.

f) Use the result of step e to write an expression for $\cos((A + B)/2)$, and simplify into a sine expression.

g) Step d said that $\frac{a-b}{c} = \frac{2\cos((A+B)/2)\sin((A-B)/2)}{2\sin(C/2)\cos(C/2)}$.

Substitute your result from step f for cos((A+B)/2). You should then have:

$$\frac{a-b}{c} = \frac{2\sin(C/2)\sin((A-B)/2)}{2\sin(C/2)\cos(C/2)}.$$

Simplify the ratio on the right side of the equation, and write the result as Mollweide's formula for (a – b) / c.

2. Mollweide's Formula for (*a* + *b*)/*c*, where *a*, *b*, and *c* are sides of a triangle.

Let triangle ABC have sides a, b, and c opposite its respective angles A, B, and C, as standard.

Derive the identity: $$\frac{a+b}{c} = \frac{\cos((A-B)/2)}{\sin(C/2)}$$

Hint: See problem 1, Mollweide's Formula for (a – b) / c.

3. The Law of Tangents (Viète, around 1580). This is another beautiful identity!

$$\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{a+b}{a-b}$$

Why is this true? Find out by performing the steps below:

a) Write the left side in terms of sines and cosines.

b) Simplify the compound fraction.

c) Apply Viète's identities for the sum of sines and the difference of sines. Hint: See Applying Elementary Identities, problems 6 and 7.

d) Use a Law of Sines relationship, and simplify. Awesome!

Solution Key
Trigonometric Identities
Applying the Law of Sines

1. Mollweide's Formula for ($a - b$)/c, where a, b, and c are sides of a triangle.

a) The equation is the Law of Sines.

b) $\sin C$, $\sin B$

c) $\sin C$

d) For the numerator on the right, see the identity in Applying Elementary Identities, problem 7, Difference of Sines.
For the denominator, let $\sin C = \sin 2(\tfrac{1}{2} C)$ and use the double angle sine identity.

e) The sum of the angles of a triangle is 180°. Thus angles $A + B + C = 180^\circ$. Then simply subtract C from both sides of the equation.

f) $\cos \tfrac{1}{2}(A + B) = \cos(90^\circ - \tfrac{1}{2} C)$
$= \sin \tfrac{1}{2} C$ because sine and cosine are cofunctions.

g) On the right side of the equation, divide numerator and denominator by their common factor $2 \sin \tfrac{1}{2} C$. Therefore

$$\frac{a-b}{c} = \frac{\sin \tfrac{1}{2}(A-B)}{\cos \tfrac{1}{2} C}$$

2. Mollweide's Formula for ($a + b$)/c, where a, b, and c are sides of a triangle.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} = \frac{\sin A}{\sin C} \text{ and } \frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin 2(\tfrac{1}{2} C)}$$

$$\frac{a+b}{c} = \frac{2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)}{2 \sin \tfrac{1}{2} C \cos \tfrac{1}{2} C}$$

by Viete's formula that $\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$

But $A + B = 180^\circ - C$, so $\tfrac{1}{2}(A + B) = 90^\circ - \tfrac{1}{2} C$.

Then $\sin \frac{1}{2}(A + B) = \sin(90^\circ - \frac{1}{2}C) = \cos \frac{1}{2}C$

Then $$\frac{a + b}{c} = \frac{2 \cos \frac{1}{2}C \cos \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C}$$

Therefore $$\frac{a + b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}$$

3. The Law of Tangents (Viète, around 1580).

a) $$\frac{\dfrac{\sin \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A + B)}}{\dfrac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A - B)}}$$

b) $$\frac{\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(\mathrm{A} + \mathrm{B}) \sin \frac{1}{2}(A - B)}$$

c) $$\frac{\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)} = \frac{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}$$

$$= \frac{\sin A + \sin B}{\sin A - \sin B}$$

d) According to the Law of Sines, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

Then $\sin A = (a / b) \sin B$, so

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{(a / b) \sin B + \sin B}{(a / b) \sin B - \sin B}$$

$$= \frac{(a / b) + 1}{(a / b) - 1} \times \frac{b}{b}$$

$$= \frac{a + b}{a - b}$$

Therefore, $\dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} = \dfrac{a + b}{a - b}$, and we are done.

Spherical Trigonometry
Teacher Notes

Description of Unit: Most high school texts do not include spherical trigonometry. Thus the commentary on the historical development of the subject is a relatively long description. This unit would be a good resource for students who wish to work independently on a mathematics topic not covered in a standard high school course. After the historical discussion, there is a list of the theorems used in spherical trigonometry. The first page covers the formulas most often needed to solve problems and the only ones necessary for use in the student exercises. The second page lists the half angle formulas. An advanced student might be interested in deriving these. They are included to show the connections between plane and spherical formulas.

Three spherical trigonometry proofs are given and student exercises included:

1. The proof of the right triangle theorems for spherical trigonometry provides a complete discussion of the relationships of plane and spherical triangles and shows how the sine and cosine of angles can be converted to sines or tangents of the sides. Napier's Rule is explained; it is suggested that students try to generate all ten right triangle formulas using the rule.
2. Student exercises on right triangles are short and are included to verify that the formulas studied can be applied to specific triangles.
3. Proof of the Law of Sines
4. Proof of the Law of Cosines
5. Exercises on Oblique Triangles include considering ambiguous cases. These can be extended to the development of rules for congruency of spherical triangles. The questions on distances between cities involve understanding how to connect longitude and latitude to solving spherical triangles and finding *qibla* directions (which way a person must turn to face Mecca).

A teacher wishing to introduce the idea of spherical triangles to a class without delving into proofs should give students the first page of formulas and then show them how to find the distance between two cities using longitude and latitude. If this is done, it is suggested that the proof of spherical right triangle relationships be given to students for reading at home. The table of *qibla* directions provides a wealth of questions for posing. Note: the answers are right there in the table.

The timing of the unit is dependent on the material selected and the background of the students involved. To cover the entire unit with an advanced class, a two week period is suggested.

Prerequisites: Historically, spherical trigonometry was developed along with plane trigonometry. In these units it is assumed that students are already familiar with plane trigonometry and know the right triangle relationships and the Laws of Sines and Cosines. If a student is not familiar with spherical geometry, it is recommended that some background work be completed before beginning this unit. If the class (or student) works well independently, this would be an excellent topic to research on the internet. The teacher could have a willing student prepare a short presentation on spherical geometry. The topics investigated should include: how to measure the distance between two points (the arc of the great circle between them), angle measure, and the sum of the angles of a spherical triangle.

Materials: It is highly recommended that students have visual aids to help them see the relationships among sides and angles of spherical triangles and how they compare to plane triangles. Inexpensive plastic spheres can be purchased at local craft stores and marked with overhead pens. The more elaborate Lenart Sphere is an excellent resource since it comes with a protractor for measuring spherical angles and lengths of great circles. Semispherical transparencies enable students to draw on the sphere and save their work. Transparencies can be stacked on one another to show further relationships. The Lenart Sphere and supplemental material, including an activities book, are available from Key Curriculum Press.

There are two pages of student exercises that can be duplicated: one on solving spherical right triangles and another on oblique triangles. A scientific calculator is required for computations. Answers are given at the end of each set of exercises.

Astronomer

depicted in *Das Stadebuch* (*The Book of Trades*)
A book of 114 woodcuts
by Jost Amman (1539-1591)
Frankfurt, Germany 1568
with descriptions written by Hans Sachs, "the shoemaker poet"

Source: http://cccw.adh.bton.ac.uk/schoolofdesign/MA.COURSE/01/LIAAmman.html

Historical Events in the Development of Spherical Trigonometry

The study of spherical trigonometry is closely related to the development of astronomy. Through trigonometry, one could determine the paths and positions of stars. As man sought to accurately measure time, make calendars, navigate the oceans, and develop a more precise geography, the relationships between angles and sides of spherical triangles became of great importance. Problems of astronomy involved finding arcs of great circles on a sphere. These arcs became sides of spherical triangles and, as theorems were developed, solutions were made possible by using the known parts of triangles that had been found by prior calculation or observation.

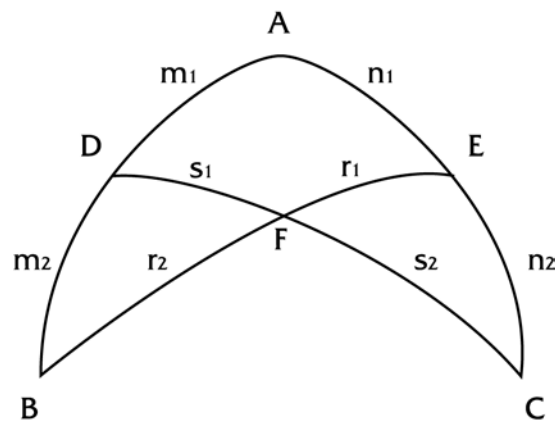
It is likely that spherical geometry was known by the school of **Pythagoras** (fifth century, BCE). The earliest extant texts on the subject, particularly dealing with the relationship of spherical geometry to astronomy, are *On the Moving Sphere*, by **Autolycus** (c. 300 BCE) and, the *Phaenomena*, by **Euclid** (c. 300 BCE). **Hipparchus** (~190 - 120 BCE) was probably the first person to study trigonometry extensively, both plane and spherical. He developed tables of chords and knew how to use these to solve triangles. Although Hipparchus and later mathematicians knew how to use plane trigonometry to solve triangles, they generally applied this knowledge to the heavens. They did not use plane trigonometry in indirect measuring and surveying on the earth. **Heron** (c. 100 CE) was interested in surveying and could have developed many theorems in plane trigonometry. However he was content to apply Euclidian Geometry to most situations. Surveyors themselves were uneducated and would not have been able to develop the necessary theorems.

Menelaus wrote *Sphaerica*, a major treatise on spherical geometry and trigonometry, around 100 CE and proved for spherical triangles many of the same theorems that Euclid had shown true for plane triangles. He was the first to define a spherical triangle. Menelaus then proved that, in a spherical triangle, the sum of the measures of two sides is greater than the measure of the third side, that the sum of the angles of a triangle is greater than two right angles, and that equal sides subtend equal angles. He proved the various triangle congruence theorems, including one that is not valid in plane geometry, that if three pairs of corresponding angles are congruent, then the triangles are congruent (AAA). He also proved numerous theorems in spherical trigonometry. Menelaus' most common proof method involved projecting a spherical diagram on the plane and then proving the result in the plane.

In dealing with spherical trigonometry, one must realize that both the angles of a spherical triangle and the sides are measured in degrees. Thus not only can one consider the sine of an angle, but one can consider the sine of a side. One can think of the sine of a side (or a spherical arc) as being equal to the sine of the corresponding central angle at the center of the sphere.

Perhaps the best known theorem in the *Sphaerica* is the one referred to as **Menelaus' Theorem**, from which many standard results of spherical trigonometry can be derived. Suppose two arcs AB, AC are cut by two other arcs BE, CD which intersect at F. With the arcs labeled as in the figure, and with $AB = m$, $AC = n$, $CD = s$, and $BE = r$, then Menelaus theorem, which we write using sines rather than the chords used by Menelaus himself, states:

$$\frac{\sin(n_2)}{\sin(n_1)} = \frac{\sin(s_2)}{\sin(s_1)} \cdot \frac{\sin(m_2)}{\sin(m)} \text{ and } \frac{\sin(n)}{\sin(n_1)} = \frac{\sin(s)}{\sin(s_1)} \cdot \frac{\sin(r_2)}{\sin(r)}.$$



The Egyptian astronomer Claudius **Ptolemy** (c. 100-178 CE) elaborated on what Hipparchus and Menelaus had written. He integrated trigonometry and astronomy and developed algorithms for solving spherical triangles. His famous work, the *Almagest*, had many examples of spherical trigonometry. In particular, he developed formulas equivalent to the following, which enabled him to solve spherical right triangles: Suppose a, b, c are sides of a spherical triangle, A, B, and C the angles opposite, with C a right angle. Then

$$\sin a = \sin c \sin A$$
$$\tan a = \sin b \tan A$$
$$\cos c = \cos a \cos b$$
$$\tan b = \tan c \cos A$$

(Remember that sides of spherical triangles are measured in degrees and are arcs of great circles.)

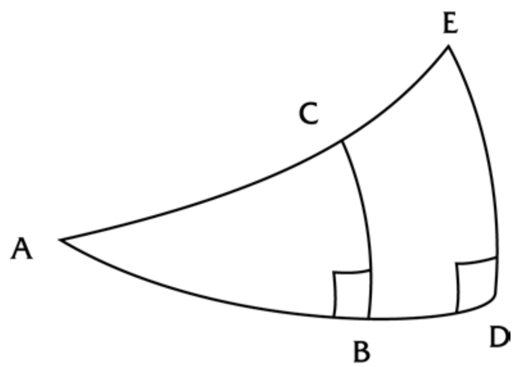
Ptolemy applied these results to solve such problems as the length of daylight at a given location at a given time, the position of the Sun when it rose, and the distance of the Sun from its zenith at noon. In addition, he developed mathematical models for the moon and the planets and used spherical trigonometry to help predict future positions of these bodies. He did not, however, develop general rules for solving non-right spherical triangles. Whenever these occurred, he broke them up into right triangles by drawing appropriate perpendiculars.

In another work, the *Planisphaerium*, Ptolemy developed the concept of stereographic projection, a mapping by which points on a sphere are represented on the plane of its equator by projection from the south pole . Under this mapping, angles between curves are preserved and circles are mapped onto circles (as long as the circle on the sphere does not go through the south pole).

Islamic mathematicians continued the work of Ptolemy in spherical trigonometry. They needed this subject not only for astronomical investigations, but also for performing religious duties.

The relationships above for spherical right triangles were used in Islam, but Islamic astronomers also developed methods for dealing directly with non-right triangles. For example, **Abū l-Wafā** (940-997), an astronomer of Baghdad, gave the "rule of Four Quantities" in his book *Zig al–Majisti, a* work based on Ptolemy's *Almagest.* This rule states that if *ABC* and *ADE* are two spherical triangles with right angles at *B* and *D* and a common acute angle *A*, then sin *BC*:sin *CA* = sin *DE*:sin *EA*. Using this rule, Abu'l-Wafa also proved the law of sines for a general spherical triangle ABC:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$



Other Islamic scholars also contributed to the development of spherical trigonometry**.** **Al–Bīrūnī** (973-1055) determined the qibla (the direction of Mecca relative to one's location) and **al–Bāttānī** (858-929) established the law of cosines for spherical triangles:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A}.$$

Nasir al–Din al–Tusi (1201–1274) wrote the *Treatise on Quadrilaterals*, the first work which treated plane and spherical trigonometry separately from astronomy. He developed an additional formula to solve spherical right triangles: $\cos c = \cot A \cot B$. The book also contained rules for solving general spherical triangles. In particular, al-Tusi showed how to solve a triangle when the three sides were known and how to solve one where the three angles were known. His book was the first to contain a solution of this last problem.

The Islamic scholars mentioned so far all resided in the Middle East. But it was a mathematician in Spain, **Jabir ibn Aflah** (also called **Geber**) (early twelfth century) whose works were central in the transmission of trigonometry to Europe. Jabir added another formula to solve spherical right triangles, one useful when a side and its adjacent angle were known: $\cos B = \cos b \sin A$. This statement is now referred to as **Geber's Theorem**.

Richard of Wallingford (1291–1336), an English monk, wrote one of the first trigonometry texts in western Europe, the *Quadripartium*, a treatise in four parts. He later revised and shortened this work in a treatise entitled *De sectore.* His goal was to explain how to solve problems in spherical trigonometry and therefore be able to apply them to astronomy. The work was in reality a very detailed explanation of the theorems of Menelaus plus many of the other theorems developed since *Spherica.* In the second version of the text, Richard included some of the spherical trigonometry of Jabir.

One of the most important developments in the study of trigonometry occurred in the mid-fifteenth century when **Regiomontanus** (1436–1476) wrote *De triangulis.* The work consisted of five books, two on plane trigonometry and three on spherical. Regiomontanus compiled all the known properties of plane trigonometry, spherical geometry, and spherical trigonometry in an organized form. All the mathematics was done independently of astronomy. Various methods of proofs were used. Interestingly, some of the proofs of results in spherical trigonometry were apparently taken directly from the work of Jabir.

Unfortunately, Regiomontanus' early death delayed the publication of his book until 1533. In the meantime, **Johann Werner** (1468-1528) improved on that book and published some of Regiomontanus' ideas in his own *De Triangulis Sphericis* of 1514. Further improvements were made by **George Joachim Rhaeticus** (1514-1576). In particular, Rhaeticus used all six trigonometric functions, which helped to simplify many formulas.

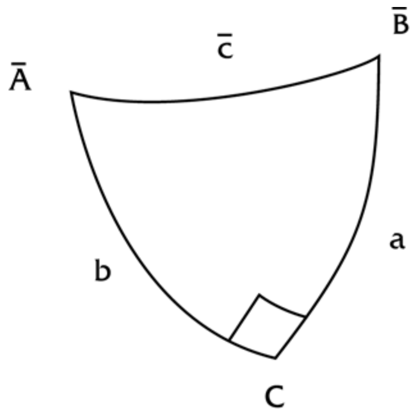
In 1579, trigonometry theorems were further systematized and extended by **François Viète** in his *Canon Mathematicus.* He gave a complete set of formulas to calculate any one part of a right spherical triangle in terms of two other known parts. Viete also wrote the rule for remembering this collection of formulas, the rule now called **Napier's rule** (a full description is given below).

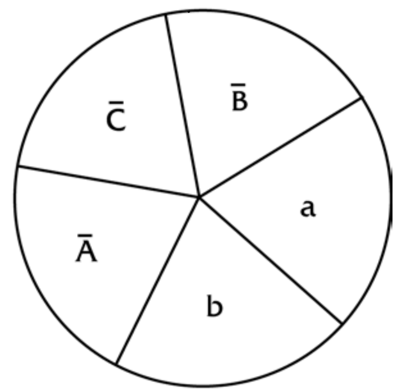
Though many others wrote of trigonometric relationships, a brief mention must be given to **Albert Girard** (1595–1632) who was the first to use the abbreviations sin, tan, sec. He also expressed the area of a spherical triangle in terms of its spherical excess, the difference between the angle sum and 180°.

John Napier (1550–1617) also wrote extensively on spherical trigonometry. His analogies ("analogy" referring to proportion) were useful in solving oblique spherical triangles. It is his rule of circular parts that is often used to remember the formulas used in solving right spherical triangles:

Arrange the sides and angles of a right triangle in order around a circle.

The bar above letters indicates the complement. Note that *C* is the right angle





Then the sine of any middle part is equal to the product of the cosines of the two opposite parts. Also, the sine of any middle part is equal to the product of the tangents of the two adjacent parts.

For example, sin a = sin A · sin c and cos A = sin B · cos a.

Teacher note: A list of theorems is in another section of this module.
Have students verify that all ten theorems can be generated by the two rules given above, and determine if there are any relationships resulting from the rules that are not in the list.

Theorems in Spherical Trigonometry
Teacher Notes

In spherical trigonometry, as in the plane, the three chief aims are brevity, clarity, and simplicity; a chapter on the Earth treated as a sphere being given to enliven an otherwise formal and lifeless subject.

Leonard M. Passano, Plane and Spherical Trigonometry
(a textbook from 1918)

Many formulas in spherical trigonometry are similar to those in plane trigonometry**. From the first set of exercises in this unit, it is important to remember that the sides of the triangles are arcs of great circles and thus measured in degrees**. The sum of the angles is not limited to 180º, but may total as much as 540º.

The next page contains a listing of some of the theorems of spherical trigonometry. Class discussion could involve stating either the Law of Sines or Cosines in words and asking the students to write the formulas. Following the list are several proofs written as exercises for the student (perhaps with teacher led discussion).

1. Basic Theorems of Right Triangles
   (including Napier's Rule and Student Exercises on Solving Spherical Right Triangles)
2. Law of Sines
3. Law of Cosines (including Student Exercises on Solving Oblique Triangles, Ambiguous Cases)

Hopefully the student will not find these "formal and lifeless." Any of the other theorems would be good challenges for the advanced student. It is important to stress that the triangles in these proofs must meet the following conditions:

1. Each angle is less than two right angles; each side is less than the semicircumference of a great circle (each element must be < 180º).
2. Any side of the spherical triangle is less than the sum of the other two sides.
3. The sum of the sides must be < 360º.
4. The sum of the angles must be between 180º and 540º.

Spherical Triangle Formulas

Right triangles (with right angle C):

$$\sin A = \frac{\sin a}{\sin c} \qquad \sin B = \frac{\sin b}{\sin c}$$

$$\sin A = \frac{\cos B}{\cos b} \qquad \sin B = \frac{\cos A}{\cos a}$$

$$\cos A = \frac{\tan b}{\tan c} \qquad \cos B = \frac{\tan a}{\tan c}$$

$$\tan A = \frac{\tan a}{\sin b} \qquad \tan B = \frac{\tan b}{\sin a}$$

$$\cos c = \cos a \cos b \qquad \cos c = \cot A \cot B$$

Oblique Triangles:

Law of Sines: In any spherical triangle, the sines of the sides are proportional to the sines of their opposite angles:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Law of Cosines: The cosine of any side is equal to the product of the cosines of the other two sides plus the product of their sines and the cosine of their included angle:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Similar results are true for the cosine of any angle:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$
$$\cos B = -\cos C \cos A + \sin C \sin A \cos b$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

Half Angle Formulas:

$$\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}} \qquad \sin\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}$$

$$\cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}} \qquad \cos\frac{a}{2}=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}$$

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} \qquad \tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}$$

In these formulas, s is half the sum of the sides of the triangle, while S is half the sum of the angles. Naturally, there are analogous formulas for the sine, cosine, and tangent of half of the other two angles or half of the other two sides.

Napier's Analogies:

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)} \qquad \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}$$

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)} \qquad \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)}$$

Delambre's Analogies:

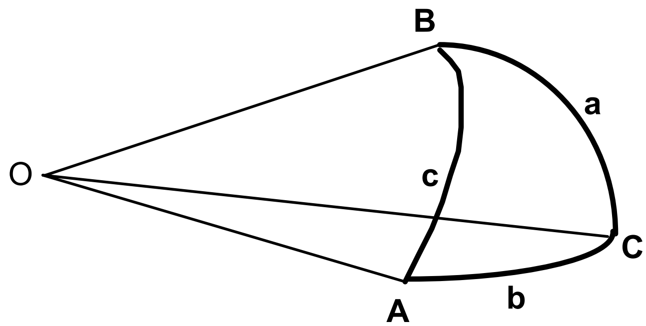
$$\sin\frac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\frac{C}{2} \qquad \cos\frac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\frac{C}{2}$$

$$\sin\frac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\cos\frac{C}{2} \qquad \cos\frac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\frac{C}{2}$$

Proof of Right Spherical Triangle Relationships

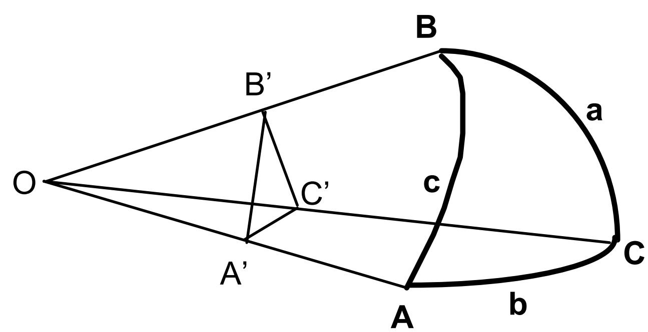
Teacher Note: The teacher during class discussion could present the following proof. The teacher may stop at any point in the discussion and ask students to make as many conclusions as they can. The ten conclusions listed may be written in equivalent forms; the ones listed are the most common found in texts. A page with only the two diagrams is given so that a transparency can be made for use in the classroom or for copying convenience.

Let C be the right angle of spherical triangle *ABC* and *O* be the center of the sphere:



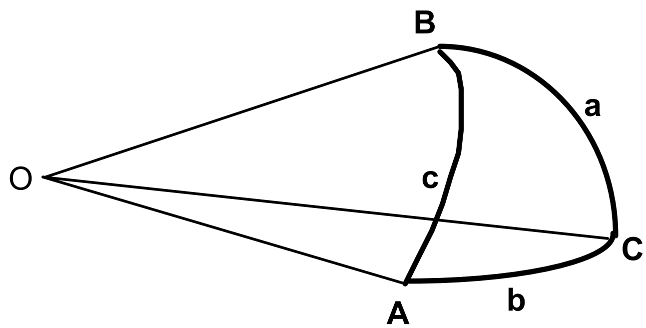
Draw radii *OA*, *OB*, and *OC*. At any point *A′* on *OA* , draw segments *A′B′* and *A′C′* in planes *OAB* and *OAC* respectively such that they are perpendicular to *OA*, meeting *OB* and *OC* at *B′* and *C′* respectively.

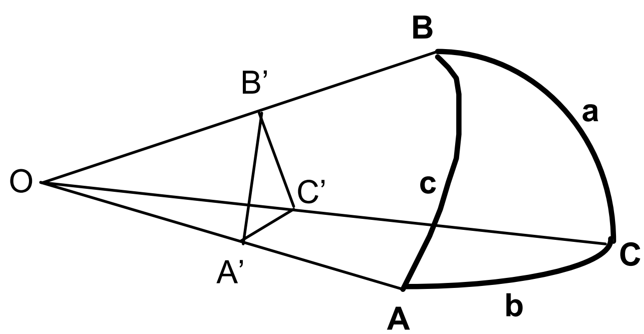
Draw *B′C′*.



Note: *OA* is perpendicular to plane *A′B′C′*, while planes *A′B′C′* and *OBC* are perpendicular to plane *OAC*. Therefore, *B′C′* is perpendicular to plane *OAC*, so B'C' is perpendicular to *A′C′* and *OC′*.

Diagrams for Spherical Right Triangle Proofs





Sides a, b, and c measure angles BOC, COA, and AOB respectively, and angle A of spherical triangle ABC is equal to angle $B'A'C'$.

In right triangle $OA'B'$:

$$\cos c = \cos A'OB' = \frac{OA'}{OB'} = \frac{OC'}{OB'} \cdot \frac{OA'}{OC'},$$

However, in right triangles $OB'C'$ and $OC'A'$

$$\frac{OC'}{OB'} = \cos a \quad \text{and} \quad \frac{OA'}{OC'} = \cos b.$$

It follows that $\cos c = \cos a \cos b$.

We also have $$\sin A = \sin B'A'C' = \frac{B'C'}{A'B'} = \frac{B'C'/OB'}{A'B'/OB'} = \frac{\sin a}{\sin c} \qquad \text{and}$$

$$\cos A = \cos B'A'C' = \frac{A'C'}{A'B'} = \frac{A'C'/OA'}{A'B'/OA'} = \frac{\tan b}{\tan c}.$$

Similarly, we get $$\sin B = \frac{\sin b}{\sin c} \quad \text{and} \quad \cos B = \frac{\tan a}{\tan c}.$$

We get the result for tan A by some manipulation, using the definitions and results already proved:

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin a}{\sin c} \cdot \frac{\tan c}{\tan b} = \frac{\sin a}{\cos c \tan b} = \frac{\sin a}{\cos a \cos b \tan b} = \frac{\tan a}{\sin b}.$$

Similarly, we get $$\tan B = \frac{\tan b}{\sin a}.$$

The second result for sin A also comes from algebraic and trigonometric manipulations:

$$\sin A = \frac{\sin a}{\sin c} = \frac{\cos a \tan a}{\cos c \tan c} = \frac{\tan a / \tan c}{\cos c / \cos a} = \frac{\cos B}{\cos b}.$$

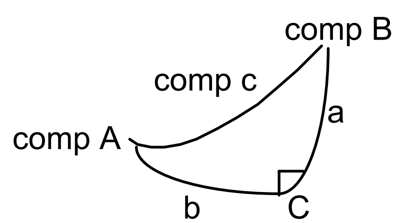
Similarly, $$\sin B = \frac{\cos A}{\cos a}.$$

The final formula in our list also comes from some manipulation:

$$\cos c = \cos a \cos b = \frac{\cos A}{\sin B} \cdot \frac{\cos B}{\sin A} = \cot A \cot B \,.$$

In our proof, we make the assumption that angles *A* and *B* are acute. However, a similar proof works if one or the other is obtuse.

Textbooks in the early 1900's usually had an entire chapter devoted to spherical trigonometry. Just as students today are expected to memorize the right triangle relationships for plane triangles, students at the turn of the century had to learn all the theorems related to spherical triangles. Looking at the ten theorems just proven, it would seem difficult to keep them straight. Many students use the letter combination sohcahtoa to remember the right triangle definitions. In the late 1500's, John Napier gave the following rule for the ten spherical theorems: the sine of the middle part equals the product of the tangents of the two adjacent parts or the sine of a middle part equals the product of the cosines of opposite parts, where the "parts" are as in the diagram. Do not consider the right angle *C*; the "parts" *a* and *b* are sides of the triangle, while the other three "parts" are the complements of angle *A*, side *c*, and angle *B*. As an example of using the rule, we calculate the sine of *a*. The adjacent parts to *a* are *b* and comp *B*, while the opposite parts are comp *A* and comp *c*. The rule then says that sin *a* = tan *b* tan (comp *B*) = tan *b* cot *B*. This is equivalent to the formula tan *B* = tan *b* / sin *a*. Similarly, we have sin *a* = cos(comp *A*) cos(comp *c*) = sin *A* sin *c*, and this formula is equivalent to sin *A* = sin *a* / sin *c*.



Can you generate all ten theorems using Napier's Rule?

Spherical Trigonometry
Student Pages

Exercises on Right Triangles

1. Given $B = 33°50'$, $a = 108°$. Find A, b, c.

   Note that, in general, it is preferable at each step to use the given values rather than any computed values. You can check your answer by using an identity not already used in the solution. You should also note that, as in plane triangle, the largest side is always opposite the largest angle and the smallest side is opposite the smallest angle.

2. Given $c = 70°30'$, $A = 100°$. Find a, b, and B.

3. Given $A = 105°59'$, $a = 128°33'$. Find b, B, and c. Note that in this case there are two possible solutions.

Answers:

1. $b = 32°31'$, $A = 99°54'$, $c = 105°6'$
   You can check this result with the identity $\cos A = \tan b / \tan c$.

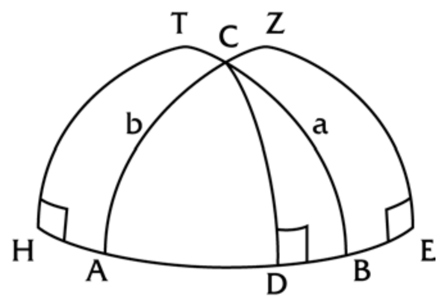
2. $a = 111°50'$, $b = 153°53'$, $B = 152°9'$
   Do not forget to check with an identity involving a, b, and B.

3. The two solutions are $b = 21°4'$, $c = 125°34'$, $B = 26°13'$ and
   $b = 158°56'$, $c = 54°26'$, $B = 153°47'$.
   Note that since $a > 90°$, the equation $\cos c = \cos a \cos b$ shows that when $b > 90°$, we have $c < 90°$ and conversely. Similarly, since $A > 90°$, the equation $\cos c = \cot A \cot B$ shows that when $B > 90°$, we have $c < 90°$ and conversely.

Proof of Law of Sines for Spherical Triangles

Around 1000 CE, Abū l-Wafā, an Islamic mathematician, demonstrated the Law of Sines for spherical triangles:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$



Given spherical triangle *ABC*, draw arc *CD* perpendicular to arc *AB*. Extend *BA* and *BC* to *BH* and *BT* respectively, where each of the latter arcs are quadrants (equal to 90º). Similarly, extend *AB* and *AC* to *AE* and *AZ* respectively, where both of these latter arcs are quadrants. Therefore *A* is a pole for great circle *EZ* and *B* is a pole for great circle *TH*. (This means that *A* is 90º away from the great circle *EZ* and similarly for *B* with respect to *TH*. That implies that *EZ* is perpendicular to both *AE* and *AZ*, while *TH* is perpendicular to both *BH* and *BT*. Look at a globe to confirm this fact.)

Consider right triangles *ADC* and *AEZ*. In the first, we know that sin *A* = sin *DC*/ sin *b*. In the second, we know that sin *A* = sin *ZE* / sin *ZA*. It follows that

$$\frac{\sin DC}{\sin b} = \frac{\sin ZE}{\sin ZA}.$$

Similarly, using right triangles *BDC* and *BHT*, we get:

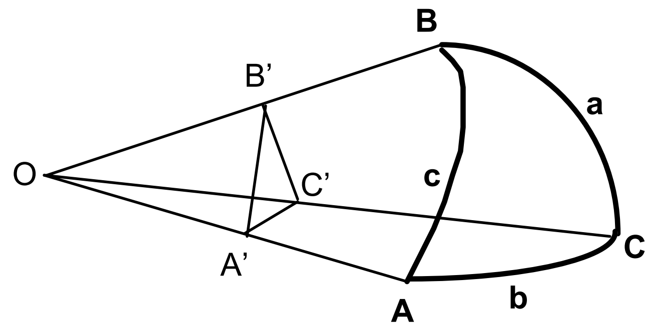
$$\frac{\sin DC}{\sin a} = \frac{\sin TH}{\sin TB}$$

But arc $ZE = \angle A$ and arc $TH = \angle B$, because of the nature of poles. Also, since *ZA* and *TB* are equal to 90º, their sines are equal to 1. It follows that

$$\sin DC = \sin b \sin A \quad \text{and} \quad \sin DC = \sin a \sin B .$$

Therefore, $\sin A \sin b = \sin B \sin a$ and we have one part of the law of sines. The rest follows by dropping a perpendicular from *A* onto *BC* and repeating the argument.

Proof of Law of Cosines for Spherical Triangles



ABC is a spherical triangle with center of sphere *O*, sides *b* and *c* < 90º. Through any point *A′* on *OA* pass a plane perpendicular to *OA* cutting planes *OAC*, *OAB*, *OBC* in *A'C'*, *A'B'*, and *B'C'*, respectively. Then $\angle B'A'C'$ is the measure of spherical angle *A* and $\angle OA'B'$ and $\angle OA'C'$ are right angles.

We now use the law of cosines from plane trigonometry:

In triangle *A'B'C'*: $B'C'^2 = B'A'^2 + C'A'^2 - 2B'A' \cdot C'A' \cos A$.
In triangle B'OC': $B'C'^2 = B'O^2 + C'O^2 - 2B'O \cdot C'O \cos a$
Thus $B'O^2 + C'O^2 - 2B'O \cdot C'O \cos a = B'A'^2 + C'A'^2 - 2B'A' \cdot C'A' \cos A$
Therefore, $2B'O \cdot C'O \cos a = B'O^2 - B'A'^2 + C'O^2 - C'A'^2 + 2B'A' \cdot C'A' \cos A$.

Since *B'OA′* and *C'OA′* are right triangles, $B'O^2 - B'A'^2 = OA'^2$ and $C'O^2 - C'A'^2 = OA'^2$.

Therefore, by substituting and dividing by 2, we get

$$B'O \cdot C'O \cos a = OA'^2 + B'A' \cdot C'A' \cos A$$

We now divide both sides by $B'O \cdot C'O$.
Since $OA'/C'O = \cos b$, $OA'/B'O = \cos c$, $B'A'/B'O = \sin c$, and $C'A'/C'O = \sin b$, we get

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

By interchanging the various arcs and angles, we get two other forms of the law of cosines:

$$\cos b = \cos c \cdot \cos a + \sin c \cdot \sin a \cdot \cos B$$
$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$

These relationships may be verbalized: the cosine of any side is equal to the product of the cosines of the other two sides plus the product of their sines and the cosine of their included angle.

Exercises on Oblique Triangles

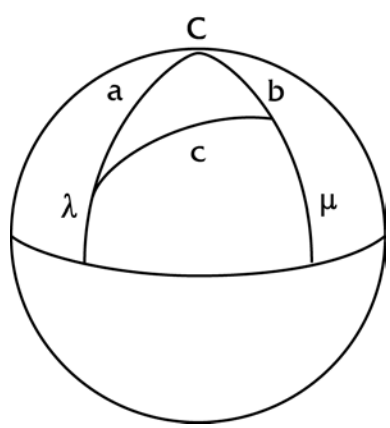
1. Justify the following statements concerning the existence of a spherical triangle with the given information:

   a. Given a side and the two adjacent angles, a triangle is always possible.
   b. Given two sides and the included angle, a triangle is always possible.
   c. Given three sides, a triangle is possible provided no side is greater than the sum of the other two and the sum of the sides < 360°.
   d. Given three angles, their sum must be between 180° and 540° and $B + C - A$, $C + A - B$, and $A + B - C$ must be between –180° and 180°.
   e. Given two sides and an angle opposite one of them, there could be 0, 1, or 2 solutions.
   f. Given two angles and a side opposite one of them there could be 0, 1, or 2 solutions.

2. Given $a = 58°$, $b = 116°$, $B = 94°50'$. Find A, C, and c. How many possible solutions are there? Hint: You will need to use one of Napier's analogies, as well as the Law of Sines.

3. Given $a = 62º$, $b = 126º$, $c = 70º$. Find A, B, and C.

4. We can use spherical trigonometry to find the distances between two cities on the earth's surface. Suppose that the latitude and longitude of the first city are λ and θ, respectively, and that the latitude and longitude of the second city are μ and φ, respectively. Then the spherical triangle whose vertices are the north pole and the two cities will have sides $a = 90° - \lambda$ and $b = 90° - \mu$. The included angle C will be equal to $\varphi - \theta$. (To verify this, it is handy to look at a globe and draw in the spherical triangle. Note that it may be convenient to represent west longitude with a plus and east longitude with a minus.) The distance between the two cities (in degrees) is then the length of the side c connecting them. You can find this value by using the spherical law of cosines. To determine the distance in miles, we need to convert degrees to miles using the fact that the circumference of the earth, which is equal to 360°, is also equal to 25,000 miles. (Note that depending on the relative positions of the two cities, the instructions here may need to be modified slightly.)

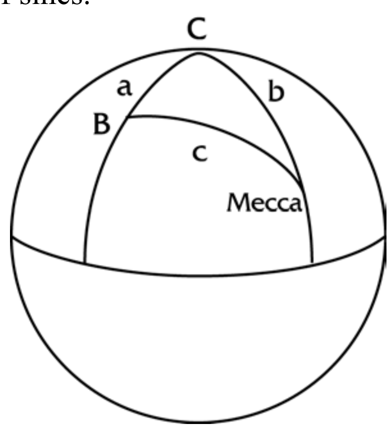


a. New York has latitude 41˚ N, longitude 74˚ W. London has latitude 52˚ N, longitude 0˚. Find the great circle distance between them in degrees and then convert to miles.

b. Find the great circle distance between London (whose coordinates are given in a) and Tokyo, whose latitude is 36˚ N and whose longitude is 140˚ E.

c. Select two cities on different continents and find the distance between them. Compare your answer with the mileage given in an atlas.

5. In the practice of their religion, Muslims need to know the direction to their holy city, Mecca, from their current location. As mentioned in the historical notes, al–Bīrūnī first determined a method for finding this direction, which is called the *qibla.* He felt his pursuit of a correct procedure would reward him in this world as well as the hereafter. To determine the direction to Mecca from a particular city, we need to solve a spherical triangle, the three vertices of which are the particular city, the North Pole, and Mecca. If the side from the city to the north pole is designated by *a*, the side from the north pole to Mecca by *b*, then the desired angle is angle *B*. Note that since we assume known the latitude and longitude of the particular city as well as that of Mecca (latitude 21˚ N, longitude 40˚ E) in this triangle we know *a, b,* and *C.* To find *B*, we need first to find *c* from the law of cosines and then use the law of sines.



a. Show that to face Mecca, a person in San Francisco (latitude 38˚ N, longitude 122˚W) needs to turn approximately 19° east of true north.

b. The table on the next page shows additional *qibla* values. Confirm the value for New York and for one other city of your choice.

:

QIBLA DIRECTIONS FROM AROUND THE WORLD

1 Muharram 1415
11 June 1994

Dr.Mohibullah N.Durrani
Copyright (c) 1994

[Permission for free distribution is given to all.]

D I R E C T I O N O F Q-I-B-L-A, EAST-of-NORTH (AZ)

CITY - Country	Lat (N +) (S -)	Long (E +) (W -)	From TRUE NORTH	From MAGNT NORTH	1992 Var Deg	Annual Drift Min	1995 Total Var-Min
MAKKAH-S.Arab	21.4	39.8	-	-	-	-	-
DAR SALAM-Tnz	-6.8	39.2	1.2	3.1	2.0W	1.2E	1.9W
MOSCOW - USSR	55.8	37.6	176.4	168.0	8.3E	1.3E	8.4E
ISTANBUL-Turk	41.0	28.9	151.5	148.1	3.4E	0.8E	3.4E
CAIRO - Egypt	30.1	31.3	136.7	34.3	2.3E	1.2E	2.4E
KHARTOUM-Sudn	15.5	32.6	48.1	46.7	1.3E	1.7E	1.4E
CAPETOWN-S.Af	-33.8	18.6	23.2	46.2	22.8W	3.0W	23.0W
LAGOS-Nigeria	6.5	3.4	63.3	68.2	5.2W	6.8E	4.9W
ALGIERS-Algra	36.8	3.0	105.5	107.1	1.9W	6.2E	1.6W
GREENWICH-Engl	51.5	0.0	119.2	123.7	4.8W	6.8E	4.5W
DAKAR-Senegal	14.7	-17.5	74.0	84.7	11.0W	7.1E	10.7W
RIO DeJENEIRO	-22.9	-43.2	67.6	88.3	20.4W	6.2W	20.7W
PARAMARIBO-Suri	5.9	-55.2	68.2	83.9	15.4W	6.3W	15.7W
BUNOS ARESarg	-34.7	-58.4	76.3	81.6	4.9W	7.8W	5.3W
LIMA-Peru S.A	-12.4	-77.0	72.2	70.7	2.0E	10.6W	1.5E
HALIFAX -CNDA	44.6	-63.6	65.6	85.9	20.4W	1.3E	20.3W
NEW YORK -USA	40.8	-74.0	58.5	73.3	14.6W	3.7W	14.8W
WASH DC -USA	38.9	-77.0	56.6	66.9	10.0W	5.2W	10.3W
MIAMI - USA	25.8	-80.2	56.6	60.7	3.7W	8.2W	4.1W
CHICAGO - USA	41.2	-87.6	48.8	50.7	1.6W	5.4W	1.9W
DALLAS - USA	32.8	-96.8	43.5	37.9	5.8E	4.0W	5.6E
DENVER - USA	39.7	-105.0	35.0	23.9	11.3E	3.3W	11.1E
SAN DIEGO-USA	32.7	-117.1	25.4	12.2	13.3E	1.7W	13.2E
SAN FRANCISCO	37.7	-122.4	18.9	3.3	15.8E	3.3W	15.6E
VANCOUVER-CND	49.3	-123.1	16.7	356.2	20.8E	6.7W	20.5E
ANCHORAGE-Als	61.1	-150.0	350.8	326.9	24.2E	5.5W	23.9E
HONOLULU -Hwi	21.3	-157.5	337.3	326.6	10.8E	2.8W	10.7E
SYDNEY-Austra	-33.9	151.2	277.5	265.1	12.4E	0.9E	12.5E
TOKYO - Japan	35.7	139.7	293.0	299.8	6.7W	1.2W	6.8W
BEIJING-China	39.9	116.4	278.9	284.8	5.9W	0.2W	5.9W
JAKARTA-Indon	-6.3	106.9	295.1	295.1	0.0W	0.7W	0.0W
DHAKA -B.Desh	23.8	90.3	277.5	278.2	0.7W	1.0E	0.7W
AGRA - India	27.2	77.9	269.0	268.9	0.0E	1.1E	0.1E
PESHAWAR -Pak	33.6	71.4	254.5	252.5	2.0E	0.7E	2.0E
BUKHARA -Uzbk	39.6	64.6	236.6	232.6	4.0E	0.3E	4.0E
TEHRAN - Iran	35.7	51.4	218.4	214.8	3.6E	0.0E	3.6E

Chart by Dr. Mohibullah N. Durrani, National Coordinator for Astronomical Information, Islamic Society of North America (ISNA)

Answers to Student Exercises on Oblique Triangles

2. Using the Law of Sines, $A = 70°5'$ or $109°55'$. But since $a < b$, we must also have $A < B$. Therefore, only one value of A is correct, namely $A = 70°5'$. To find c, we use one of Napier's analogies. To do that, we first calculate $\frac{1}{2}(A + B)$, $\frac{1}{2}(B - A)$, and $\frac{1}{2}(b - a)$. We then find that $c = 137°25'$. It is easiest to find C by another application of the law of sines. Again, this gives us two values, but we must pick the larger one, $C = 131°24'$. In this case, there is one solution to the triangle.

3. Use the law of cosines to determine $\cos A$. We get that $A = 28°7'$. The remaining angles can be found by using either the law of sines or the law of cosines. The solutions are $B = 154°25'$ and $C = 30°6'$. Whichever law you use to calculate the answer, it is wise to check using the other one.

4.a. The spherical triangle in this case has side $a = 49°$, side $b = 38°$, and angle $C = 74°$. We use the law of cosines to calculate $c = 49.83°$. The distance in miles is determined by multiplying this value by 25,000 and then dividing by 360. The answer is approximately 3460 miles.

b. In this case, $c = 85.32°$, and the actual distance is about 5925 miles.

5.a. The spherical triangle in this case has $a = 52°$, $b = 69°$, and $C = 162°$. From the law of cosines, we find that $c = 118.62°$. An application of the law of sines then gives $B = 19.2°$. (This differs slightly from the value in the table, because we have rounded off the latitude and longitude of our cities to whole degrees.)

Biographies

Since many of the units in this module reference the same mathematicians, biographies have been consolidated in a separate section. Most of these biographies contain quotations that can enrich classroom discussions and lead to further investigations.

An excellent web site for information on the lives of mathematicians (both the famous and the obscure) is The Mactutor History of Mathematics Archive. The address for this site is: http://www-history.mcs.st-and.ac.uk/history/. Accessed June 27, 2001

Biographies contained in this unit include:

Muhammad **Abū'l'Wafā** al-Būzjānī

Abu 'Abdallāh Muhammad ibn Jābir **al-Bāttānī**

Abu l-Rāyham Muhammad ibn Ahmaad **al-Bīrūnī**

Brahmagupta

Abraham **de Moivre**

Hipparchus of Rhodes (also of Bithynia and Nicaea)

Claudius **Ptolemy**

Johann Muller **Regiomontanus**

Thales of Miletus

Muhammad Abū'l-Wafā al-Būzjānī

Born: June 10, 940 in Buzjan, Khorasan region (now in Iran)
Died: July 15, 998 in Baghdad (now in Iraq)

Abū'l-Wafā is often referred to as the greatest Arab mathematician of the tenth century. His primary occupation was as an astronomer at the observatory at Sharaf al Daula. He also translated and wrote commentaries on the works of Euclid, Diophantus, and al-Khwarizmi. Unfortunately these writings have been lost. His interest in astronomy led him to prove theorems in both plane and spherical trigonometry. Like al-Biruni, Abū'l-Wafā introduced the secant and cosecant and studied the relationships among the six trigonometric functions and how they were associated with arcs on the unit circle. It should be noted that he focused on arcs of the unit circle rather than angles.

Abū'l-Wafā also devised a new method of calculating sine tables using the tangent function. His tables, with entries given at 15' intervals, were accurate to eight decimal places while Ptolemy's were only accurate to five. In doing this, he developed the double and half angle formulas. Abū'l-Wafā was one of the first mathematicians to prove the Law of Sines for spherical triangles.

Abu ‘Abdallāh Muhammad ibn Jābir al-Bāttānī (also known as Albategnius)

Born: about 860 in Haran (near Urfa), Syria
Died: 929 in Samarra, Iraq

As with many mathematicians of his time, al-Bāttānī primarily focused on the study of the stars and cataloged 533 of them. He was also able to give a more precise value of the length of a year (365 days, 5 hours, 48 minutes, 24 seconds). One of Al-Bāttānī’s goals was to improve on the *Almagest*. Many of his conclusions were similar or even more precise than those of Ptolemy. The main difference was that al-Bāttānī relied on trigonometric calculations rather than the geometrical models of Ptolemy. For some of his computations he used the sine of the complement of 90° (our current cosine). However this presented difficulties when the angle was between 90° and 180° since he did not take the sine of a negative number. Thus al-Bāttānī resorted to a versine function such that versine $\alpha = R + R\text{sine}(\alpha - 90°)$.

Abu l-Rāyhan Muhammad ibn Ahmad al-Bīrūnī

Born: September 15, 973 in Kath, Khwarazm (now Kara-Kalpakskaya, Uzbekistan)
Died: December 13, 1048 in Ghazna (now Ghazni, Afghanistan)

Quotations:

Once a sage asked why scholars always flock to the doors of the rich, whilst the rich are not inclined to call at the doors of scholars. "The scholars" he answered, "are well aware of the use of money, but the rich are ignorant of the nobility of science.''

Quoted in A. L. Mackay, *Dictionary of Scientific Quotations* (London 1994)

You well know...for which reason I began searching for a number of demonstrations proving a statement due to the ancient Greeks...and which passion I felt for the subject...so that you reproached me my preoccupation with these chapters of geometry, not knowing the true essence of these subjects, which consists precisely in going in each matter beyond what is necessary...Whatever way he [the geometer] may go, through exercise will he be lifted from the physical to the divine teachings, which are little accessible because of the difficulty to understand their meaning...and because the circumstance that not everybody is able to have a conception of them, especially not the one who turns away from the art of demonstration.

Book on Finding the Chords in the Circle

Al-Bīrūnī was a noted writer on a wide variety of topics including Indian life, language, religion and culture, as well as astronomy, mathematics, physics, and medicine.

His work on shadows and chords of circles were published in his *Exhaustive Treatise on Shadows* in which he described the relationships among all six trigonometry functions and showed the relation of them to those in the unit circle. Al-Bīrūnī also developed the Pythagorean identities such as $\cot^2\alpha + 1 = \csc^2\alpha$. He also worked on a table of sines with step $15'$ accurate to four sexagesimal places. The tables were only used for astronomical computations; similar triangles were applied to more earthly endeavors.

Especially important were his scientific investigations. Al-Bīrūnī was convinced that the Earth rotated on its axis. He made precise calculations of latitude and longitude and also wrote accurate observations of both a solar eclipse (April 8, 1019) and a lunar eclipse (September 17, 1019). He also wrote about calendars and collected observations of equinoxes. In addition, Al-Bīrūnī studied the specific weight of precious stones and metals and the properties of springs.

Brahmagupta

Born: 598 in Ujjain, India (place of birth not confirmed)
Died: 670 in India

Quotation: "As the sun eclipses the stars by its brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposed algebraic problems, and still more if he solves them."

Quoted in F. Cajori, *A History of Mathematics*

Like many of his time, much of Brahmagupta's writings focused on astronomy and dealt with solar and lunar eclipses, positions of the planets, and the length of the year. He wrote his major work *Brahmasphutasiddhanta* (*Correct Astronomical System of Brahma*) when he was only thirty years old. Interestingly, the work, which dealt with the solution of astronomical problems, was written in verse. He was also head of the astronomical observatory at Ujjain. However, our focus is on Brahmagupta's mathematical contributions.

In developing his mathematical formulas, he used new algebraic notations that eased computations and clarified his work. Many high school students are familiar with his formulas dealing with cyclic quadrilaterals:

Area = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where s is the semiperimeter and a, b, c, d are the four sides of the quadrilateral.

The product of the diagonals = sum of the product of opposite sides

Indian sine tables were computed for arcs 3¾° apart. Brahmagupta developed interpolation techniques to find other sine values. One used second order differences and another an algebraic formula (perhaps originated by his contemporary Bhaskara). His conclusions were not stated a formal proofs, but as algorithms for solving the problem in question.

Brahmagupta also studied arithmetic progressions, quadratic equations, and systems of linear congruences, and discovered theorems about right triangles and the surfaces and volumes of certain shapes. He was the first Hindu mathematician to state the rules for adding, subtracting, multiplying, and dividing positive and negative numbers. However he did make mistakes, believing, as many students do, that $0 \div 0 = 0$. [He did correctly state that $0 \cdot 0 = 0$.]

Abraham de Moivre

Born: May 26, 1667 in Vitry (near Paris), France
Died: November 27, 1754 in London, England

Abraham De Moivre was born in Vitry, in the French champagne country east of Paris. He was educated in the classics and mathematics, but in 1685 his life was drastically changed. The Edict of Nantes had been a legal bulwark of religious tolerance, but, when it was revoked in 1685, De Moivre, a Protestant, was sent to prison for a little over two years. In 1688, when he was freed, he departed for England where he settled. De Moivre continued to study mathematics and went on to a career of tutoring, solving mathematics problems for others, and doing his own research.

De Moivre's largest and most important work was the *Doctrine of Chances*, which first appeared in 1718 and went through several later editions. He also published the *Miscellanea analytica* (1730) which contained additional work on probability as well as the solution of polynomial equations. Though widely recognized for his mathematical achievements, even to the point of being elected a member of the Royal Society in 1697 and later to learned academies in Paris and Berlin, De Moivre was never able to obtain a university position. This may have been due to his being neither English nor Anglican.

Most high school students know of De Moivre when they study the theorem dealing with powers and roots of complex numbers in trigonometry. Though he never stated the relationship explicitly, there are indications he was familiar with "his" theorem as early as 1707. He was better known among his contemporaries for his work on probability theory and the development of analytic geometry.

De Moivre lived a long and productive life, but he died lonely and poor. Once he noticed that he was sleeping fifteen minutes longer each night and concluded that an arithmetic sequence existed and that he would die on the day he slept around the clock. De Moivre actually did die when he first slept round the clock, thus accurately predicting his death. Only a few close friends gathered to mourn the passing of this great mathematician.

Hipparchus of Rhodes
(also called Hipparchus of Bithynia or Nicaea)

Born: c. 190 B.C.E. in Nicaea, Bithynia (now Iznik, Turkey)
Died: c. 120 B.C.E. probably in Rhodes, Greece

Little is known of the life of Hipparchus and only one of his works has survived, a minor book entitled *Commentary on Aratus and Eudoxus*. Our knowledge of his mathematical investigations come from references made by later scientists and mathematicians referring to Hipparchus. Much of his investigations were celestial in nature. It is believed that he cataloged over 800 stars using various ways to determine their positions.

Hipparchus introduced into Greece the division of a circle into 360° and was one of the first to make use of the sexagesimal division of degrees into minutes and seconds. Hipparchus was also believed to be the first to form a table of chords that would enable right triangles to be easily solved. Thus many refer to him as the Father of Trigonometry. It was known that in a given circle the ratio of arc to chord decreased as the angle decreased from 180° to 0° (approaching a limit of 1). It appears that he organized these ratios in table form. Though many believe that Ptolemy relied heavily on the work of Hipparchus in writing his own tables, there is no clear cut proof.

Claudius Ptolemy

Born: about 85 in Egypt
Died: about 165 in Alexandria, Egypt

Quotation: "When I trace at my pleasure the windings to and fro of the heavenly bodies, I no longer touch the earth with my feet: I stand in the presence of Zeus himself and take my fill of ambrosia, food of the gods."

Quoted in C. B. Boyer, *A History of Mathematics* (New York 1968)

We know very little of Ptolemy's life, except that he lived in or near Alexandria, in Egypt. His title for his most important work is *The Mathematical Compilation*, but today it is generally known by the name given to it by Islamic astronomers: the *Almagest*, which means the *Greatest.* A treatise in thirteen books, it gives in detail the mathematical theory of the motions of the Sun, Moon, and planets. In the book, Ptolemy writes of the Earth being the center of the universe, a theory believed until the sixteenth century when Copernicus determined that the Earth and planets revolved about the Sun. Though Ptolemy was proven wrong, it is important to note that his work did treat astronomy as a science and was the most influential astronomical work for almost fifteen hundred years.

Mathematically what is important is that Ptolemy used geometric models to predict the movements of celestial bodies and then provided the theorems necessary to prove his conclusions. The formulas he devised for his chord function are equivalent to our angle addition formulas for sine and cosine. His table of the chord function at intervals of 1/2 a degree is equivalent to our sine table. In calculating his table, he found an approximation of $\sqrt{3}$ accurate to five decimal places (1.73205). He also has a very good approximation to π as 377/120 (3.141667).

Ptolemy's work was so influential, that even Copernicus, in replacing Ptolemy's earth-centered universe by a sun-centered one, followed the model of *Almagest* very closely.

Johann Muller Regiomontanus

Born: June 6, 1436 in Kőnigsberg, Archbishopric of Mainz (now Germany)
Died: July 8, 1476 in Rome, Italy

Quotation: You, who wish to study great and wonderful things, who wonder about the movement of the stars, must read these theorems about triangles. Knowing these ideas will open the door to all of astronomy and to certain geometric problems.

De Triangulis Omnimodis

Born Johann Műller of Kőnigsberg, Regiomontanus preferred the Latin version of his name (Kőnigsberg meaning King's mountain). He is considered by many to be the most influential mathematician of the fifteenth century.

After travels to Italy, where he became proficient in Greek, Regiomontanus returned to Germany and sat up a printing press and observatory. He hoped to print translations of the work of Greek mathematicians and scientists such as Archimedes, Heron, and Ptolemy. He was invited to Rome by Pope Sixtus IV to help with calendar revision and died before the could translate and publish the books. Some think he died of the plague; other believe he was poisoned by enemies.

As did many other mathematicians of his time, Regiomontanus focused much of his work on astronomy. He observed eclipses of the moon, traced the path of Halley's comet, and used lunar distances to determine longitude at sea. Though his methods were good, instruments at the time could not give lunar positions with enough degree of accuracy for use at sea.

Of most importance to our study of trigonometry is his work *De Tringulis Omnimodis* (*On triangles of Every Kind*). Published in 1464, it was a systematic account of methods for solving triangles. Each theorem was proven and most had diagrams and examples, including, for example, the Law of Sines for both plane and spherical triangles.. Previously trigonometry was used mainly to solve astronomical problems. Thanks to Regiomontanus, trigonometry became a separate discipline.

Thales of Miletus

Born: about 624 B.C.E. in Miletus, Asia Minor (now Turkey)
Died: about 547 B.C.E. in Miletus, Asia Minor (now Turkey)

Quotation: "I will be sufficiently rewarded if when telling it to others you will not claim the discovery as your own, but will say it was mine."

Quoted in H. Eves *In Mathematical Circles* (Boston 1969)

Though his occupation was that of engineer, Thales is often considered the first Greek philosopher and is known as one of the "Seven Wise Men" of Greece. Unfortunately, none of his work has survived so it is difficult to determine precisely the extent of his discoveries or his involvement in mathematical developments. Proclus who lived around 485 C.E. did write that Thales introduced the study of geometry into Greece. However this does not imply that he used deductive proofs as Euclid did in *The Elements*. Thales is often credited with five theorems of elementary geometry:

1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.
3. The angles between two intersecting straight lines are equal
4. Two triangles are congruent if they have two angles and one side equal.
5. An angle in a semicircle is a right angle.

Note that these statements are written using contemporary terminology.

Proclus also wrote that "his method of attacking problems had greater generality in some cases and was more in the nature of simple inspection and observation in other cases." The latter may apply to his observations of shadows and the heights of pyramids and similar objects. He found that unknown heights could be found by measuring the shadow of the object at the time when some other item and its shadow were equal in length.

Thales believed that all things come to be from water and explained many phenomena such as earthquakes as a result of the Earth floating on an infinite ocean. He is also credited with defining the constellation Ursa Minor.

Plato tells a story of how Thales was walking outside one starry night and fell in a ditch because he was so engrossed in heavenly observation. A servant girl helped him out of the ditch admonishing him "how do you expect to understand what is going on up in the sky if you do not even see what is at your feet." Thus, as Brumbaugh says, perhaps this is the first absent-minded professor joke.

History of Trigonometric Terms

This reference section describes the origins of the main vocabulary of trigonometry. It is organized by order of these topics:

the word trigonometry itself,
angle measures,
the 6 trigonometric functions,
and the inverse trigonometric functions.

Trigonometry

The term "trigonometry" first appeared in the title of a book in 1595 in Frankfort, Germany. Bartholomaus Pitiscus (1561-1613) published *Trigonometriae sive de dimensione triangulorum libri quinque*, the first text on trigonometry. The title is Latin for "On trigonometry, or concerning the measurement of triangles, in five books." The Greek words *trigonon* and *metrein* mean "three-angle" and "to measure." Pitiscus was a clergyman and also a mathematics professor at Heidelberg. Earlier, books on what we consider "trigonometry" often had the word for "triangle" in their titles.

Angle Measures

Degree

The Greeks described the 360 divisions at the center of a circle as *moira*, meaning steps. The Arab word was *daraja*, which the Europeans translated into the Latin *de gradus*. The symbol for degree(s) is °.

Minute

The Latin phrase *pars minuta prima* means "first small part": small in the sense of mini or miniature. Each of these "small parts" is one-sixtieth of a degree. The word "minute" then became the name of this "small part". The symbol for minute(s) is ′.

Second

The Latin phrase *pars minuta secunda* means "second small part": second in the sense of next after the first. Thus, a minute was divided into sixty "second small parts", and the word "second" became the name of this "part." The symbol for second(s) is ″.

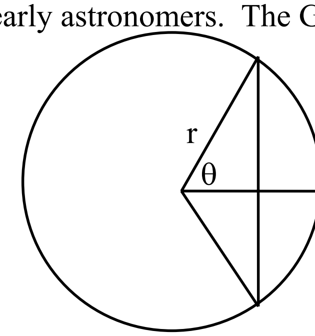
Radian

James Thomson, brother of the physicist Lord William Kelvin, used the word "radian" in his private papers in or before the year 1871. He first used radians publicly in a final exam that he gave at Queen's College in Belfast, Ireland, in 1873. Radian is an abbreviation for <u>ra</u>dius-<u>an</u>gle.

The 6 Functions: Sine and Cosine, Tangent and Cotangent, Secant and Cosecant

Sine and Cosine

The derivation of the name "sine" is a mini-history reflecting the multicultural origins of trigonometry. The chord and sine were most important for early astronomers. The Greek astronomer Ptolemy (c. 150 CE) tabulated lengths of chords corresponding to central angles in a circle. Aryabhata of India (c. 500 CE) streamlined astronomical calculations by creating tables of half chords. A half chord, or *jya-ardha*, is actually a side of a right triangle opposite $\angle\theta$, which is half a central angle. *Jya-arhda* is our modern concept of $r \cdot \sin\theta$ or simply $\sin\theta$ when radius $r = 1$ unit. It became shortened to *jya*. When the Arabs learned trigonometry from the Indians, they took this word to be *jiba*. In Arabic, however, vowels are generally omitted in writing. Thus, all that was written were the two consonants *jb*. When Latin translators translated the Arabic works, they took this word to be a different Arabic word, namely *jaib*, which means fold, bay, or inlet. Thus, they translated it by the comparable Latin word *sinus*, from which comes our word "sine." Fibonacci of Italy used and promoted the term *sinus rectus arcus* (1220).



The word "cosine" is simply short for "sine of the complement of an angle", or, simply r "sine complement."

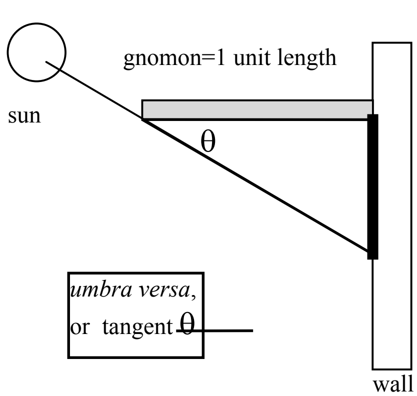
The abbreviations sin and cos first arose in 1626 in a trigonometry treatise by Girard (1595-1632). Other terms used for sine and cosine were:

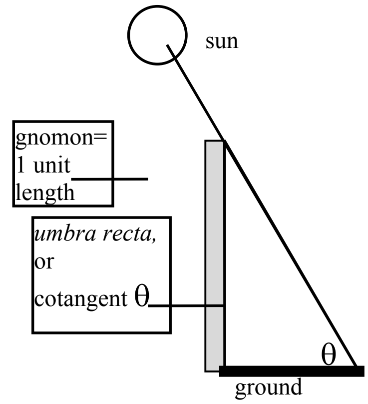
sinus residuae (meaning cosine) - by Viete, around 1580
co-sinus - Gunter, 1620
sin (in a drawing) - Gunter, 1624
sin (in a book) - Herigone, 1634
Si, Si.2 - Cavalieri, 1643
cosinus - John Newton, 1658
sin., cos. - Euler, 1748

Tangent and Cotangent

The concepts of tangent and cotangent originated in shadow reckoning performed by the Babylonians and the Greeks to mark the hours of the day. Cotangent was known as *umbra recta*, that is the "direct shadow" cast horizontally when a gnomon (standard rod of unit length) was stood vertically on the ground. Tangent was *umbra versa,* or "turned shadow"; it was cast vertically when a gnomon was affixed horizontally to a wall.

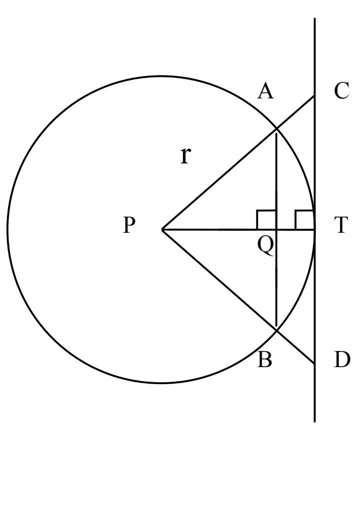
A **gnomon** was a straight pole or rod of standard length used in shadow reckoning. Also, L-shaped gnomons were used in sundials. *Gnomon* is a Greek word meaning “that enabling to be known.”





Viète (c. 1580) used the terms amsinus and prosinus for tangent and cotangent. The term tangent was first used by Fincke in 1583 and became popularized by Pitiscus in 1595. Other terms used were:

cotangens - by Gunter, 1620
Ta, Ta.2 - Cavalieri, 1643
Cot. - Jonas Moore, 1674
tang., cot. - Euler, 1748



The trigonometric tangent function is related to the geometric tangent line *CD*. The Latin word *tangere* means “to touch.” Defining tan $\angle APQ$ = opposite leg AQ / adjacent leg PQ, then by similar triangles:

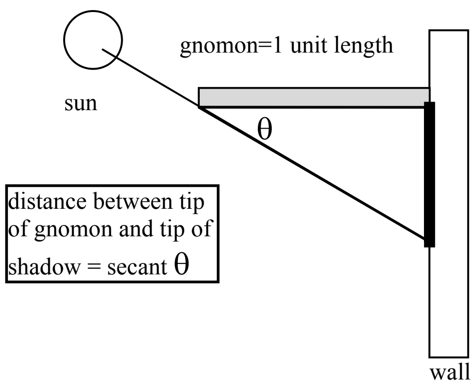
$$AQ / PQ = CT / PT$$

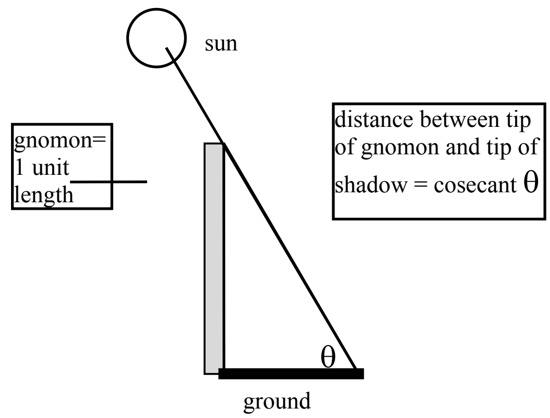
$$\tan \angle APQ = CT / \text{radius } r$$

If r is taken as a unit length 1, then tan $\angle APQ = CT$. which is half of the tangent segment intercepted by the central angle $\angle APB$. This development is like that of sine, sin $\angle APQ = AQ$, which is half of the *chord* intercepted by the central angle $\angle APB$.

Secant and Cosecant

Fincke first used the term “secant”, in 1583 in his book *Geometria Rotundi*. Rheticus was the first to name “cosecant” in his *Opus Palatinum*, published posthumously in 1596. In shadow reckoning, secant and cosecant were the distance between the tip of the gnomon and the tip of the shadow: secant for a horizontal gnomon, and cosecant for a vertical gnomon. Abu’l-Wafa (c. 980), was the first to formally treat these functions, but did not give them names.





Navigation in the 15th century sparked interest in tabulating acute angles and the hypotenuse.

The word Secante was used by Edward Wright 1599 in explaining the mathematics behind Mercator’s 1569 map. As this map preserved the correct direction between two locations, it served the needs of navigators and mariners. Here is a brief chronology of important names for secant and cosecant:

hypotenusa, meaning secant, used by Copernicus in 1542
secant - Fincke 1583
cosecant - Rheticus 1596 posthumous
Secante - Edward Wright 1599
Se and Se.2 - Cavalieri in 1643
sec. and cosec. - Euler in 1748

The Inverse Trigonometric Functions

Today, both the arcsine and $\sin^{-1}$ symbols (for example) are in general usage. Formal recognition of the inverse trigonometric functions came in the early eighteenth century, along with the following notation:

A S.	meaning the arc of the angle whose sine is x, was used by Daniel Bernoulli in 1729.
A t	arctangent, by Euler in 1736 indicating, on a unit circle, the arc whose tangent is t
A	arcsine, Euler
arcus sinui	Lambert 1758
arc. tang.	Sherferr 1758
arc(sin. = x)	Condorcet 1769
arc. sin. x	Lagrange 1772
arc. sin.	Lambert 1776
arcsin	J. Houel

An alternate notation, $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$, was used in 1813 by astronomer John F. W. Herschel of England (1792-1871). The inverse cosine of x, generally written arc(cos. = x), began appearing as $\cos^{-1} x$. The small raised numerals were carried over from calculus to indicate operations on functions, for example in calculus d^2x represented the derivative of a derivative. Herschel was motivated by exponents; where he meant $f(f(x))$, he wrote $f^2(x)$. He further adapted the symbolism of the Product of Powers Rule: $b^n b^{-n} = b^0$ (real number $b \neq 0$) and the Zero Exponent: $b^0 = 1$ ($b \neq 0$). He said that if f is the inverse function for $\cos(x)$, then $f \cos(x) = \cos(x) f = x$ implies that f can be written $\cos^{-1}(x)$. Herschel considered notation to have an important role in developing ideas.

A List of Important Developments in the History of Trigonometry

Suggested Student Usages for the Timeline:

1. Locate the geographic areas that are mentioned. (Keep a world map displayed in the classroom.) Look up the cities where the mathematicians lived, and locate the cities on the map.

2. Research details of the listed and other math events of the time period .

3. Draw a time scale horizontally. Then below it, for the appropriate years, list the math developments and the places where they occurred.

4. Do a project that explores an important development in the history of trigonometry.

5. Find out the life and times of mathematicians. Find out the cultural influences upon the mathematics of the time, and vice versa.

6. Add noteworthy events of the times, such as the Crusades, Shakespeare's writing Hamlet, or Gutenberg's printing press.

7. Answer questions such as: (Each answer about "when" would best include who, where, and why.)
   When did trigonometry get its name?
   When did the six trigonometric ratios get their names?
   For how long was trigonometry identified with astronomy?
   When did trigonometry first apply to right triangle ratios?
   When were trigonometric functions (circular functions) created?

The Timeline:

Century	Development
	Trigonometry begins with Greek astronomy, the belief being that heavenly bodies orbit the earth in circles. People analyze circular arcs and the related chords in order to predict positions and to tell time. Spherical trigonometry develops along with plane trigonometry.
6th century BCE	Thales (625-547 BCE) uses similar triangles for indirect measurement.
3rd century BCE	About 300 BCE, Euclid derives a geometry theorem equivalent to the Law of Cosines in Book Two, Propositions 12, 13 of the Elements. Archimedes (287-212 BCE) develops a geometry formula equivalent to that for sin(A $\pm$ B) in the Theorem of the Broken Chord.
2nd century BCE	Hipparchus (180-125 BCE), astronomer and father of trigonometry, creates tables of chords.
2nd century CE	Menelaus of Alexandria writes the earliest treatise on spherical trigonometry around 100 CE. Ptolemy writes *Almagest* around 150 CE. For central angles in a circle of radius 60, his table gives chords equal to 120 times the sine of 1/2 the angle
	By the 5th century, India becomes the center of mathematical advances.
5th century	A table of half-chords appears in an astronomical work, *Surya Siddhanta*, around 400 CE, probably based on Hipparchus' table.
6th century	Aryabhata associates sines with angles, today's sine function concept, around 500 CE. His sine table contains cosines, or sines of the complementary angles. Islamic contact with India occurs between 700 and 1000 CE.
7th century	In about 600, Bhaskara I gives an algebraic rule to approximate sine values without tables.
8th century	The Chinese Buddhist monk I-Hsing (683-727), the greatest astronomer of his time, makes the first table of tangents in 724.
	With the ascendancy of the Islamic world, Islamic mathematicians blend Greek, Hindu, and their own discoveries into a true trigonometry.
9th century	Al-Hasib develops tables of shadows, today's tangent and cotangent, around 860.
10th century	Al-Battani (c. 850-929) finds a trigonometric formula for the sun's height using shadows. Abu'l-Wafa (940-998), the greatest Islamic mathematician of the 10th century, puts trigonometry onto a unit circle. He originates the secant and cosecant concepts.
11th century	An Islamic method of prostaphaeresis exists which converts a product to a sum: $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$. Al-Biruni (973-1055) develops sin, cos, tan, cot, sec, csc for shadows and writes about astronomy and surveying.

	By the 12th century, Europeans learn trigonometry because Latin scholars translate Greek, Hindu, and Islamic mathematics.
13th century	Fibonacci's *Practica Geometriae* (1220) gathers trigonometry from Islamic works. Nasir al-Din al-Tusi (1201-1274) separates trigonometry from astronomy and writes a trigonometry text. Islamic mathematicians use all 6 trigonometric quantities from the 13th century onward.
15th century	Regiomontanus (1436-1476) is the first westerner truly able to explain Ptolemy's astronomy to Europeans.
16th century	Publication (1533) of *On Triangles*, by Regiomontanus, systematizes spherical and plane trigonometry. Rheticus (1514-1574) makes highly precise tables for radius 10^{15} and uses right triangle ratios rather than arcs. In 1579 Viete (1540-1603) blends trigonometry, algebra, and functions; e. g. he uses trigonometry to solve cubic equations. Pitiscus (1561-1613) publishes book (1595) with trigonometry as we know it today and coins the word "trigonometry". Age of Copernicus, Galileo, Kepler. Scientists of the 16th and 17th centuries study periodic phenomena.
17th century	Napier (1550-1617) invents logarithms of sines (to multiply sine values, it is much easier to add log sines). In 1635, Roberval publishes the first sketch of a sine curve. Trigonometry is changing from computations to functions.
18th century	In 1748, Euler (1707-1783) founds analytic trigonometry and periodic circular functions. He says that $e^{ix} = \cos x + i \sin x$. In 1796, Gauss uses trigonometry to prove that a 17-sided polygon can be constructed with compass and straightedge.
19th century	Fourier (1768-1830) in *Analytical Theory of Heat* (1822) shows that any function whatsoever can be written as a sum of sines and cosines.

Published Timelines:

Boyer, *A History of Mathematics*

Campbell and Higgins editors, *Mathematics - People, Problems, Results*, volume 1

Karl Smith, *The Nature of Mathematics*, inside the covers

Bibliography

Many wonderful books were used to create the module. Some of them stand out as all-purpose mathematical history reference books. These we encourage you to have in your library.

Boyer, Carl: *A History of Mathematics*, 1968, Rev. ed. New York: John Wiley, 1989

Katz, Victor: *A History of Mathematics: An Introduction*. Reading, Ma: Addison-Wesley, 2nd Ed. 1998

Eves, Howard: *An Introduction to the History of Mathematics*, 6th Ed. Harcourt Brace Jovanovich Publisher, 1990

Kline, Morris: *Mathematical Thought From Ancient to Modern Times*, Oxford University Press, New York, 1972

Smith, David Eugene: *History of Mathematics*, Volumes I and II, Dover Publications, 1958

These other books also had delightful sections relating to trigonometry and its history.

Aaboe, Asger: *Episodes from the Early History of Mathematics*, New York: Random House, 1964

Beckman, Petr: *A History of Pi*, Boulder, Colo.: Golem Press, 1977

Bekken, Fauvel, Johansson, Katz, and Swetz: *Learn from the Masters*, John Wiley and Sons, The Mathematical Association of America, 1995

Burton, David: *The History of Mathematics: An Introduction*. Boston: Allyn and Bacon, 1985

Dunham, William*: Journey Through Genius: The Great Theorems of Mathematics*. New York: John Wiley, 1990

Dunham, William: *A Mathematical Universe*. New York. John Wiley, 1994

Heath, Thomas: *Aristarchus of Samos*, Dover Publications, New York, 1981

Maor, Eli: *Trigonometric Delights*. Princeton University Press, 1998

NCTM 1995 Yearbook, *Connecting Math Across the Curriculum*, p. 104-114, "Connecting Mathematics with Its History: A Powerful, Practical Linkage:. Includes a list of resources.

Needham, Joseph: *Science and Civilization in China*, Vol. 3, Cambridge University Press, 1959

Peterson, Ivars: *Newton's Clock, Chaos in the Solar System*, W.H. Freeman & Co., New York, 1993

Polya, George: *Mathematical Methods in Science*. New Mathematical Library. The Mathematical Association of America, 1977

Resnikoff and Wells: *Mathematics in Civilization*, Holt Rinehart and Winston Inc. 1973

Websites

In addition we found these websites to be useful and enjoyable, and hope they are still around for your enjoyment.

http://www-history.mcs.st-and.ac.uk/history/
The MacTutor History of Mathematics archive, containing mathematicians, history topics, and famous curves. A must-see website. Accessed June 27, 2001

http://www.ies.co.jp/math/java/trig/index.html
Interactive investigation of the trigonometric functions, Laws of Sines and Cosines, Ferris wheel, and crane. Accessed June 27, 2001

http://www.skypub.com/tips/skywise/13-jul98.html
A cute cartoon explaining how our elliptical orbit with the sun at a focus causes us to have 94 days of summer and only 91 days of winter. Accessed June 27, 2001

http://nssdc.gsfc.nasa.gov/cgi-bin/database/www-nmc?69-059C-04
http://nssdc.gsfc.nasa.gov/cgi-bin/database/www-nmc?71-008C-09
http://nssdc.gsfc.nasa.gov/cgi-bin/database/www-nmc?71-063C-08
http://nssdc.gsfc.nasa.gov/cgi-bin/database/www-nmc?71-063C-08
Descriptions of the reflectors left on the Moon by the Apollo 11,14 and 15 missions, and some other cool stuff. Accessed June 27, 2001

http://www2.jpl.nasa.gov/files/universe/un940729.txt
http://www.ridgenet.net/~do_while/sage/v2i2f.htm
Information about how the reflectors were used to measure the distance to the Moon. Accessed June 27, 2001

http://spaceboy.nasda.go.jp/note/shikumi/e/Shi08_e.html
A nice website on Hipparchus's work finding the distances to the moon and to the sun. There's also a good method for finding the distance from the moon to the earth that uses only right triangles. Accessed June 27, 2001

http://cannon.sfsu.edu/~lea/courses/nexa/cwwplan.html
Shows retrograde motion and why Polaris is only our current North star. Accessed June 27, 2001