Geometric Proof

Jim Ebert
Durham Academy
Durham, North Carolina

Peter Flusser
Kansas Wesleyan University
Salina, Kansas

Gail Kaplan
The Key School
Annapolis, Maryland

Rebecca Kessler
Roberts Wesleyan College
Lima, Ohio

Ed Sandifer
Western Connecticut State University
Danbury, Connecticut

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Overview

Why do we have to do proofs?

This is a question most, if not all, geometry teachers have been asked through the years by students. After all, proofs have been with us for over 2500 years. A proof is basically a convincing argument, using logic, that a proposed result is true. Many early civilizations used logical arguments, but it is the Greeks who are generally credited with developing the idea of producing proofs using logical arguments based on explicitly stated axioms, that is, truths assumed without proof. We do not know exactly why they developed such proofs. One hypothesis is that because the Greeks' basic political organization was the city-state, which controlled a population of only a few thousands, and because no matter what the type of government, it was ruled by law, the citizens were motivated to argue and to debate (Katz, 43). Out of this characteristic, "there developed the necessity for proof in mathematics, that is, for argument aimed at convincing others of a particular truth." (Katz, 43) Thus, the nature of Greek society provided a conducive environment for the birth of the mathematical notion of a proof.

There are many ways to give a "convincing argument." Some of these are through diagrams; others through verbal means. Since diagrams may on occasion be misleading, however, it is often a combination of these methods which works best. It is well to remember, however, that not only should a proof "convince," but it should also explain. That is, it should show the reader or listener why the proposed result is true. It follows that the purpose of a proof is to make mathematics easier, not harder. It is to help students, not scare them to death.

In this module we will look at various ways of dealing with proofs, some coming directly from historical sources and others more distantly related. We should keep in mind, however, that "although Western civilization owes a great debt to Greek society in literature, art, and architecture, it is to Greek mathematics that we owe the idea of mathematical proof, an idea at the basis of modern mathematics and, by extension, at the foundation of our modern technological civilization." (Katz, 43)

Rationale

This module is designed to help the teacher get across the idea that proofs are often necessary when we work with mathematical ideas. Yet there are various kinds of proofs, and it is important also for the student to understand when and where to use these kinds. Students using this module will also learn some of the historical background of mathematical proof.

Each of the lessons in this module are designed to actively involve the student. The lessons revolve around the development and dissemination of active approaches to learning in the geometry classroom. They are designed to introduce the student to the mathematical notion of proof. The following NCTM standards are incorporated throughout the module: Mathematics as Problem Solving, Mathematics as Communication, Mathematical Connections, Geometry From a Synthetic Perspective, Geometry From an Algebraic Perspective, and Mathematical

Structure. These lessons are not designed as history lessons with a mathematical flavor, but rather mathematics lessons from a historical perspective. The various lessons are created as independent choices; it is possible to choose a single part of the module and use it at an appropriate time in your course during the year.

Activity Summaries

The various activities in this module are designed to help students recognize the need for mathematical proofs. The unit is designed so that the students will have fun while learning significant mathematics.

I The Checkerboard Challenge

This engaging brain teaser is an excellent tool for illustrating the necessity of proof. It is simple enough to be understood by very young children, but the analysis and proof required to properly answer the query is significant. It highlights the difference between proof and observation.

II The Missing Square!

This project illustrates the necessity of proof. Our eyes may provide deceptive results, but logical proofs do not allow us to be fooled.

III An Excerpt from Plato's Dialogue *Meno*

This excerpt is a discussion between a teacher, Socrates, and the slave of one of his associates, Meno. Socrates wants to convince Meno that there is no such thing as learning, but only recollecting. Both men know that the slave boy has had no formal mathematical training. Through a series of questions Socrates leads the boy to construct a square with double the area of a given square. The conversation illustrates several mathematical ideas. In particular, if the lengths of two similar figures are in a given proportion k, the area of the figures are not in proportion k, but k^2.

IV Indirect Proofs

Two classical theorems are proved. Attention is paid to the structure of the proofs, especially their use of proof by contradiction.

V The Irrationality of $\sqrt{2}$

This activity takes the student through a Greek style proof that the side and diagonal of a square are incommensurable. The modern proof of the irrationality of $\sqrt{2}$ is virtually identical to this proof.

VI The Declaration of Independence and Proof

Jefferson's masterwork is analyzed as an example of a Euclidean-style proof by logical reasoning from explicit axioms.

VII Introduction to Congruence Through Folding Activities

This hands-on activity of the ancient art of folding provides a serious connection between the properties of the folded objects and the proofs that show the result does indeed satisfy the desired conditions.

VIII Euclid Through the Path of Tangrams

Part A: A Series of Tangrams:
This series of exercises shows the basic ideas behind several of Euclid's proofs.

Part B: The Connection:
This portion illustrates proofs that confirm the properties apparent from the tangram exercises.

IX Geometric Algebra and Quadratic Equations

This section shows how Euclidean geometry may be used to solve quadratic equations.

X Figurate Numbers

Students begin with triangular numbers and discover that consecutive triangular numbers add to square numbers. They then discover the formula for the nth triangular number and a few facts about oblong numbers.

XI Heron's Formula and the Euler Line

This section contains detailed derivations of some theorems in Euclidean geometry which were discovered long after Euclid.

XII The Pythagorean Project

Pythagoras and his followers were members of a secret society that lasted for hundreds of years. We suggest that the teacher dress up in a toga and as each student enters the classroom make them an honorary member of the Pythagoreans. This will set the stage for the activities that illustrate the Pythagorean Theorem and its various applications.

The Checkerboard Challenge
Teacher Notes

Level: This activity is suitable for high school students..

Materials: Paper and Pencil

Time Frame: 20 minutes

Objective: To convince students of the necessity of a proof.

When to Use: This is an excellent exercise during the first week of classes. Students find the question engaging and fun. It helps establish a positive atmosphere in the geometry classroom.

How to Use: Distribute the one page description to the students and have them work on the problem in small groups. You can help guide the students as necessary.. Although you may want to provide your students with a checkerboard grid on which to work, perhaps even with the corners cut out and the squares colored, we suggest that you *not* provide them, so that the students discover for themselves how important it is to draw a picture.

The Checkerboard Challenge
Student Page

A puzzle? A game? A proof?

The popular game of checkers may have begun more than 4000 years ago when the Egyptians played games on a board like today's checkerboard. Today's game of checkers seems to have its origins in France during the 11th century. Chess, a different game played on the same board, was invented in India, also about a thousand years ago.

The same checkerboard pattern has been used for thousands of years in almost every culture on textiles, tiled floors, ceramics and other decorated objects. Over the years, many brain teasers have used the checkerboard. Try the checkerboard challenge. Good Luck!

First, let's call the shape you get by putting two squares side by side a *domino*. You can think of a domino as a rectangle that is twice as long as it is wide.

Now, start with an ordinary checkerboard, eight squares on each side. It has 64 squares. Remove two squares, one from the top right corner and one from the lower left, leaving 62 squares. Can the squares which are left be filled with 31 dominos, each domino covering two adjacent squares?

Note: After you have determined whether the answer is yes or no, describe the pattern that works or prove that no pattern will work.

The Checkerboard Challenge
Solutions and Suggestions

Solution: It is *not* possible to cover the modified checkerboard with 31 dominoes.

Many students will initially be convinced that the board can be covered since 2 x 31 = 62. Students generally approach the problem by drawing a picture, a super way to start. After a few attempts students change their minds and believe that a pattern does not exist. To these students pose the question, "how do you know that the pattern truly does not exist or do you only know that you have been able to find it?" Suggest that student take another approach. Provide a helpful hint to consider the colors of the board.

To prove that it is impossible to cover the board with 31 dominoes, note first that the two corners that are removed from the corners of the checkerboard are both the same color, say black. This means there are 30 black squares and 32 red squares left on the board. Each domino covers two adjacent squares. Adjacent squares are different colors, so each domino covers one black and one red square. The first 30 dominos cover 30 red squares and 30 black squares. The two squares remaining are the same color. Therefore, they cannot be covered with the remaining domino.

The Point of this Exercise: This exercise is designed to show students how the things they know because they prove them are different from the things they know because they observe them. It is one thing to *believe* the problem has no solution because you tried a hundred times and you couldn't find a solution. It is quite another thing to *prove* it has no solution, for then you know for certain.

You may want to discuss with your students how they know things. Some things they know because people told them, or they read about them. Other things they know because they saw them with their own eyes. A few things, mostly mathematical truths, they know because they have proved them. Experience and observation are important ways to learn what *might* be true, and what to try to prove, but knowledge of mathematical truths comes from proofs.

Some students may be disturbed by the conclusion of this proof, that something is impossible. They cannot follow the rules of the game and cover the region with dominoes. They cannot do it by trying harder, or by getting a faster computer. It cannot be done. Young people, who are broadening their horizons and testing their limits, may find it hard to accept that some things have absolute limits and are actually impossible.

Additional Exercise: Ask the students to look at the formula $p = n^2 - n + 41$. They should substitute various small numbers for n and calculate p. For example, if you substitute 1 for n, you calculate that $p = 41$. If you substitute 2 for n, you get $p = 43$. If you substitute 7 for n, you get $p = 83$. If you substitute 10 for n, you get $p = 131$. The interesting point here is that each of these numbers is a prime number. The challenge for the students is to decide on whether this formula *always* produces a prime number. If the answer is yes, they need to produce a proof. If it is no, they need to give a value for n in which the resulting value for p is not prime.

The Missing Square!

Teacher Notes

Level: This activity can be used in middle school or high school

Materials: Paper and Pencil

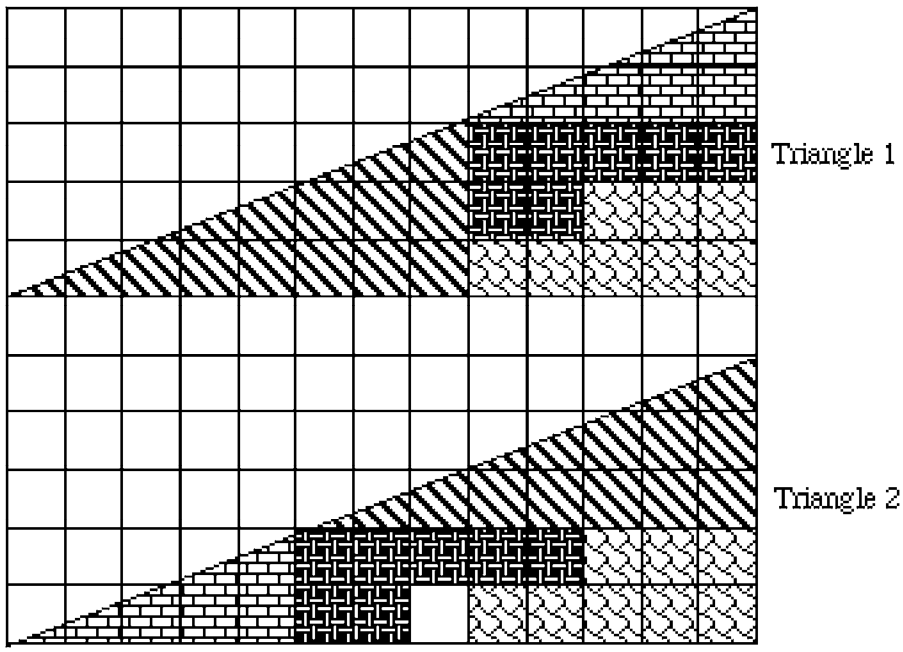
Time Frame: 20 minutes

Objective: This activity is designed to persuade students of the necessity of a proof.

When to Use: This is an excellent exercise anytime during the semester.

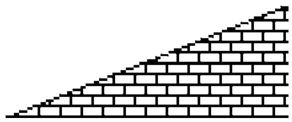
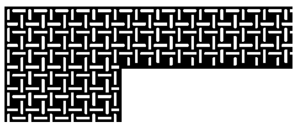
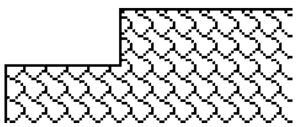
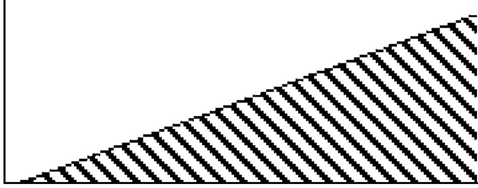
How to Use: We recommend having students discuss their ideas with a small group. The teacher can help to guide the students to a solution.

The Missing Square!
Student Pages



Carefully examine the above diagram. Triangle 1 is divided into several pieces. Those pieces are rearranged and made into triangle 2; however, there is a missing square in triangle 2. How can this happen? We will now explore the answer to this question. Complete the following charts.

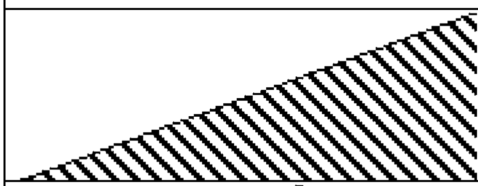
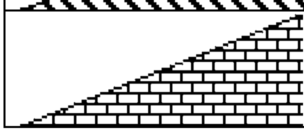
Triangle	Base	Height	Area of Triangle	Total Shaded Area
Triangle 1				
Triangle 2				

Piece	Area of Shaded Piece
	
	
	
	

The area of triangle 1 is equal to the sum of the area of its four shaded parts. What is the area of triangle 1 using this sum? Is this the same as the area you calculated in the first chart?

The area of triangle 2 is equal to the sum of the area of its four shaded parts plus the unshaded square unit. What is the area of triangle 2 using this sum? Is this the same as the area you calculated in the first chart?

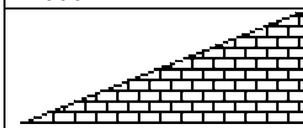
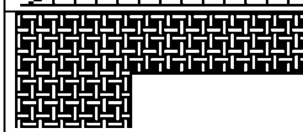
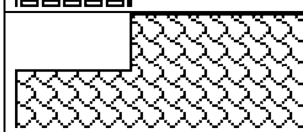
Let's try to reconcile these measurements. To calculate the slope of a line passing through the points (x_1, y_1) and (x_2, y_2) evaluate the quotient $\frac{y_2 - y_1}{x_2 - x_1}$. Recall that for any given line the slope of the line is always the same, regardless of the points chosen to calculate the slope.

Triangle	Coordinates of Endpoints of Hypotenuse	Slope of Hypotenuse
1		
2		
		
		

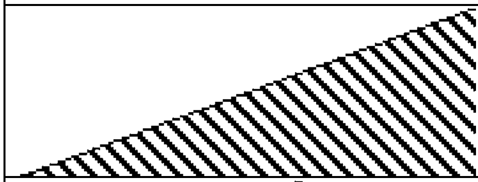
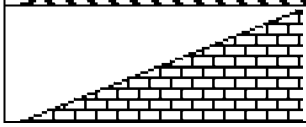
Do all of the slopes in the above chart have the same value? Describe what you have discovered. Where did the missing square come from?

The Missing Square!
Solutions

Triangle	Base	Height	Area of Triangle	Total Shaded Area
Triangle 1	13	5	$\frac{65}{2}u^2$	$\frac{65}{2}u^2$
Triangle 2	13	5	$\frac{65}{2}u^2$	$\left(\frac{65}{2}-1\right)u^2$ because one square is not shaded

Piece	Area of Shaded Piece
	$\frac{1}{2}\cdot 5\cdot 2=5u^2$
	$7u^2$ [Just count the squares!]
	$8u^2$ [Just count the squares!]
	$12u^2$ [triangle with base 8 and height 3]

Area of Triangle 1 = $5u^2+7u^2+8u^2+12u^2=32u^2$. This is not the same as the area previously computed.
Area of Triangle 2 = $5u^2+7u^2+8u^2+12u^2+1u^2=33u^2$. This is not the same as the area previously computed.

Triangle	Coordinates of Endpoints of Hypotenuse	Slope of Hypotenuse
1	(0, 0) and (13, 5)	5/13 ≈ 0.3846
2	(0, 0) and (13, 5)	5/13 ≈ 0.3846
	(0, 0) and (8, 3)	3/8 = 0.3750
	(0, 0) and (5, 2)	2/5 = 0.4000

The large triangles are not really triangles. Neither has a straight "hypotenuse".

An Excerpt from Plato's Dialogue *Meno*

Teacher Notes

Level: This activity is designed for use in high school.

Materials: Whiteboard, Blackboard, or Overhead; script of part of the *Meno*; historical background on Plato's Academy and on Plato's philosophy

Time Frame: One or two class periods.

Objectives: To understand the reason for proof; to see examples of Plato's philosophy; to get a feel for the nature of geometry in Greek civilization.

When to Use: This activity is appropriate for any time during the year. It might work quite well on the day before a holiday when a more unusual task is needed to capture the student's attention

How to Use: The historical background to Plato and the Academy can be given to students to read beforehand. The reproduction of Raphael's *School of Athens* could be tacked to the wall and referred to at various times throughout the geometry course. Student projects dealing with the individuals in the picture could well be assigned, perhaps in collaboration with the history department.

Before staging the play, we suggest a brief discussion on how questions can be used to teach. We suggest that you choose three students to play the roles of Socrates, Meno, and the slave and a fourth student to draw the diagrams as the skit progresses. You may want to change actors part way through the play to give more students a chance to participate. There is no necessity of having the students memorize the lines. They should just read them. But they should make sure that the audience understands the relevant diagrams as they proceed.

Your students may also want to discuss the institution of slavery in ancient Greece. It is a sad truth that, despite their brilliant contributions to mathematics, philosophy, science and politics, ancient Greek civilization is stained by its reliance on and tolerance of slavery.

After the students are finished the play, use the worksheet to ensure their understanding of the mathematics involved. There is an additional page of questions which might be discussed to help place this dialogue into its historical context and further illuminate the nature of proof.

Background: The play is part of one of Plato's dialogues, *Meno.* In the play, Plato describes Socrates' idea that people inherently know mathematics. To understand it, they merely need to be shown that they know it. Socrates claims to demonstrate his theory by asking a slave boy questions about triangles and squares and leading the boy to realize what he already knows about mathematics. A few more details about Plato's philosophy are dealt with on a separate page and may be discussed in advance with the class. It would be helpful if this could be done in conjunction with the students' study of ancient Greece in history class.

An Excerpt from Plato's Dialogue *Meno*
Student Pages

The Academy

The picture reproduced here is a fresco which is in the Vatican and which was created around 1510. It is one of four frescoes painted by Raphael called *Stanze della Senatura* depicting humanity's varied efforts towards understanding. Each of these frescoes stands for one of the paths by which mankind approaches Truth: the spiritual (Theology), the intellectual (Philosophy), the imaginative (Poetry), and the social (Law). The subject of this fresco is Philosophy. That subject included what we now call science and mathematics.

This fresco is also called *The School of Athens*. It shows us how Raphael imagined that Plato's Academy to look. At the center stand Plato and Aristotle, who represent the two chief schools of philosophy of the Renaissance period. Plato, venerable and bearded, holds in one hand the *Timaeus*, the dialogue in which he not only describes the creation of the universe but also explains why and how it occurred. With the other hand he points upwards, an allusion to the Platonic theory of Ideas, as interpreted by the philosophers of the day. Aristotle, holding his *Ethics*, gestures towards the earth, indicating his belief in the study of the real world. The two systems of thought are brought into aesthetic harmony by the calm attitudes of the figures and by the soaring Roman architecture in which they stand and which was inspired by Bramantes' designs of St. Peter's Cathedral, which was just being started.

Around the two central figures are grouped the philosophers of the ancient world. The man seen in profile to Plato's left, bearded and snub-nosed, is Socrates. Others include Euclid, bending down in the right foreground using his compasses to demonstrate a theorem; Ptolemy, to Euclid's right, crowned and holding a globe; and Pythagoras, the bald man in the left foreground, writing. In front of Pythagoras is a young boy, holding a tablet inscribed with a diagram of musical harmonies and the Pythagorean decad. The man sitting on the steps in the foreground is Diogenes, the Cynic, a philosopher who taught that the only distinction that matters is the distinction between virtue and vice.

The real Academy probably did not look at all how Raphael depicted it. In particular, the figures in this fresco are all from different time periods and most of them were not contemporaries of Plato. The school was founded in 387 B.C.E., probably at a site sacred to the Athenian hero Academus. The best idea we have of the subjects studied there comes from Plato's dialogues themselves and from Aristotle's testimony. When Plato died, the leadership of the Academy went to his nephew Speusippus. The Academy was destroyed once when Athens fell to the Romans in 88 B.C.E., but it was rebuilt. It was finally closed by the Emperor Justinian in 529 C.E.

The School of Athens

Plato and the Meno

The *Meno* is a dialogue written by Plato, the founder of the Academy. Plato was one of the greatest philosophers who ever lived. He taught that everything in this world is changing and transient. However, there exists an unchanging and eternal world of "ideals." All physical objects are mere reflections of ideal objects that exist in this ideal world. Before birth, human souls reside in this ideal world to which they will return after death only to be born anew.

The objects that mathematicians study, Plato taught, exist in the world of ideals. Diagrams and symbols are merely tools to help mathematicians think about these objects. Mathematics is thus of primary importance since it deals with ideal objects *par excellence*. For example, mathematics deals with an ideal square; after all, any figure that you actually draw will fail to be a true square because the sides will not be "exactly" equal to one another. So because mathematics deals with ideal objects, they can be understood only through our reasoning abilities and not through our senses. Thus mathematics helps develop our reasoning powers.

In the *Meno*, Plato's spokesman is a character called Socrates. There was a real person named Socrates, who lived from 470 to 399 B.C.E. The facts known about him are few. We know that he was a well-known philosopher and teacher who attracted a large following of students among whom was Plato. He did not write any books nor leave us any other personal record of his ideas. He was a volunteer in the Athenian army who fought bravely in the war against Sparta. He was physically not very handsome, but there exist at least two statues that purport to be copies of copies of actual portraits of Socrates. Although a character named Socrates is the main character in many of Plato's dialogues, we do not know whether the ideas expressed by this character are those of the real Socrates. We do know, however, that the real Socrates was executed in 399 B.C.E. by being made to drink a cup of hemlock, a poison. He had been convicted of misleading the youth of Athens through his teaching. And one of Plato's dialogues does deal with this historical event.

In the *Meno*, Socrates and his friend Meno consider the question whether virtue can be taught. Socrates claims that no one can teach anyone anything. Knowledge comes from recalling what the soul has seen before birth. All a teacher can do is help the soul to remember. To convince Meno of his ideas, Socrates proposes an experiment. By merely asking questions and not telling him anything, he will get an uneducated slave boy to understand that a square constructed on the diagonal of a given square has twice the area of that square. We leave it up to the reader to decide whether Socrates' experiment was successful.

An Excerpt from *Meno* by Plato

MENO . . . what do you mean by saying that we do not learn, and that what we call learning is only a process of recollection? Can you teach me how this is?

SOC I told you, Meno, just now that you were a rogue, and now you ask whether I can teach you, when I am saying that there is no teaching, but only recollection; and thus you imagine that you will involve me in a contradiction.

MENO Indeed, Socrates, I protest that I had no such intention. I only asked the question from habit; but if you can prove to me that what you say is true, I wish that you would.

SOC It will be no easy matter, but I will try to please you to the utmost of my power. Suppose that you call one of your numerous attendants, that I may demonstrate on him.

MENO Certainly. Come hither, boy.

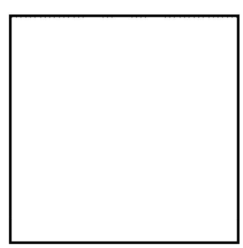
SOC He is Greek, and speaks Greek, does he not?

MENO Yes, indeed; he was born in the house.

SOC Attend now to the questions which I ask him, and observe whether he learns of me or only remembers.

MENO I will.

SOC Tell me, boy, do you know that a figure like this is a square?

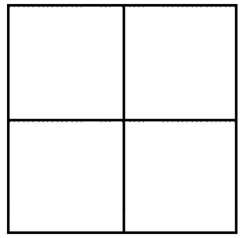


BOY I do.

SOC And you know that a square figure has these four lines equal?

BOY Certainly.

SOC And these lines which I have drawn through the middle of the square are also equal?

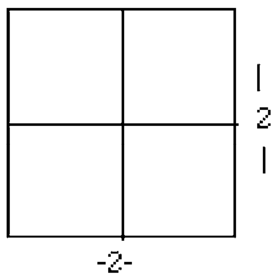


BOY Yes.

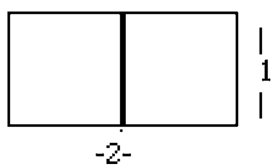
SOC A square may be of any size?

BOY Certainly.

SOC And if one side of the figure be of two feet, and the other side be of two feet, how much will the whole be?

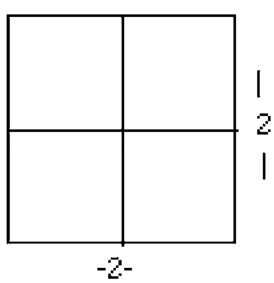


Let me explain: if in one direction the space was of two feet, and in other direction of one foot, the whole would be of two feet taken once?



BOY Yes.

SOC But since this side is also of two feet, there are twice two feet?



BOY There are.

SOC Then the square is of twice two feet?

BOY Yes.

SOC And how many are twice two feet? Count and tell me.

BOY Four, Socrates.

SOC And might there not be another square twice as large as this, and having the lines equal like this one?

BOY Yes.

SOC And of how many feet will that be?

BOY Of eight feet.

SOC And now try and tell me the length of the line which forms the side of that double square: this is two feet-what will that be?

BOY Clearly, Socrates, it will be double.

SOC Do you observe, Meno, that I am not teaching the boy anything, but only asking him questions; and now he fancies that he knows how long a line is necessary in order to produce a figure of eight square feet; does he not?

MENO Yes.

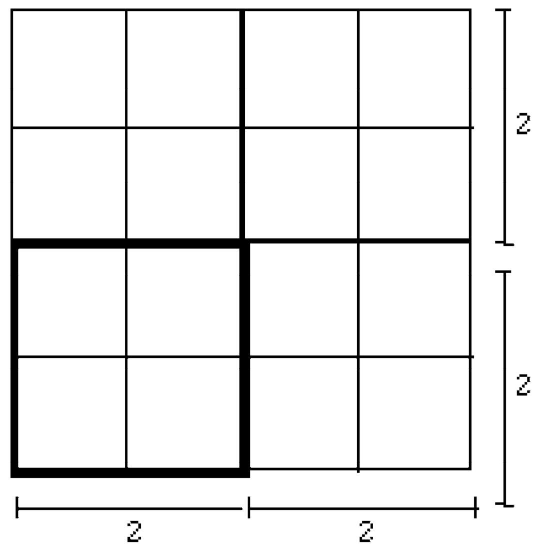
SOC And does he really know?

MENO Certainly not.

SOC He only guesses that because the square is double, the line is double.

MENO True.

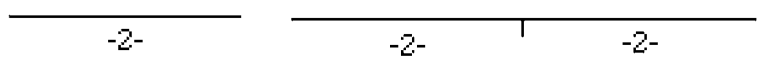
SOC Observe him while he recalls the steps in regular order. (*To the Boy*.) Tell me, boy, do you assert that a double space comes from a double line?



Remember that I am not speaking of an oblong, but of a figure equal every way, and twice the size of this-that is to say of eight feet; and I want to know whether you still say that a double square comes from double line?

BOY Yes.

SOC But does not this line become doubled if we add another such line here?

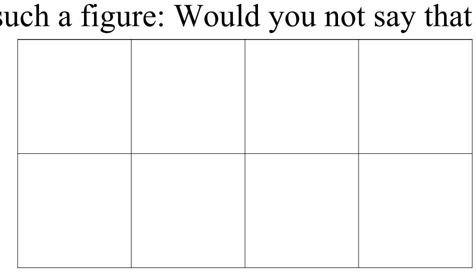


BOY Certainly.

SOC And four such lines will make a space containing eight feet?

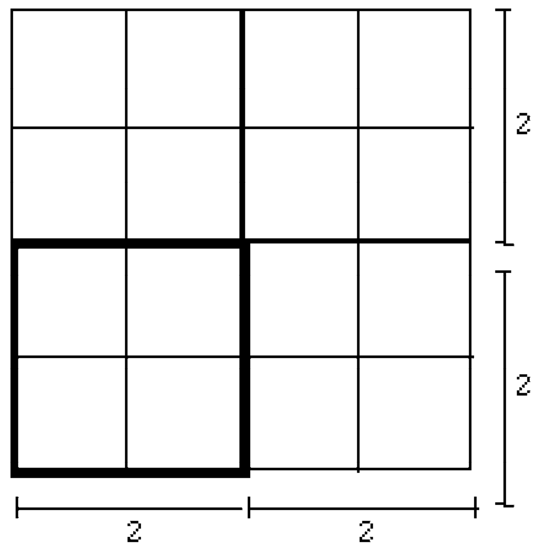
BOY Yes.

SOC Let us describe such a figure: Would you not say that this is the figure of eight feet?



BOY Yes.

SOC And are there not these four divisions in the figure, each of which is equal to the figure of four feet?

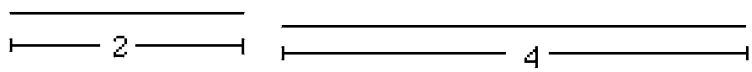


BOY	True.
SOC	And is not that four times four?
BOY	Certainly.
SOC	And four times is not double?
BOY	No, indeed.
SOC	But how much?
BOY	Four times as much.
SOC	Therefore the double line, boy, has given a space, not twice, but four times as much.
BOY	True.
SOC	Four times four are sixteen-are they not?
BOY	Yes.
SOC	What line would give you a space of eight feet, as this gives one of sixteen feet;-do you see?
BOY	Yes.
SOC	And the space of four feet is made from this half line?
BOY	Yes.

SOC Good; and is not a space of eight feet twice the size of this, and half the size of the other?

BOY Certainly.

SOC Such a space, then, will be made out of a line greater than this one, and less than that one?



BOY Yes; I think so.

SOC Very good; I like to hear you say what you think. And now tell me, is not this a line of two feet and that of four?

BOY Yes.

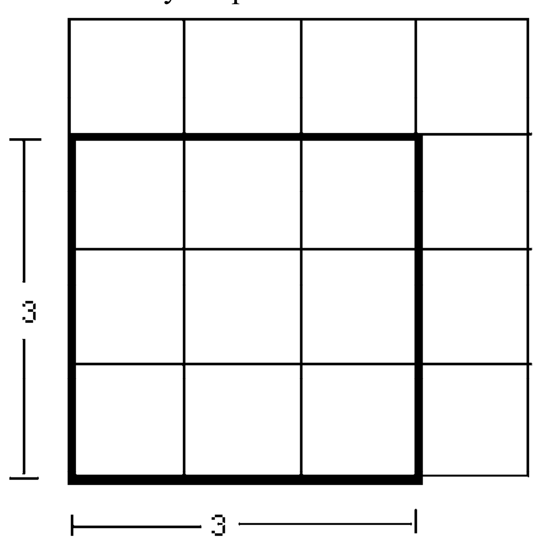
SOC Then the line which forms the side of eight feet ought to be more than this line of two feet, and less than the other of four feet?

BOY It ought.

SOC Try and see if you can tell me how much it will be.

BOY Three feet.

SOC Then if we add a half to this line of two, that will be the line of three. Here are two and there is one; and on the other side, here are two also and there is one: and that makes the figure of which you speak?



BOY Yes.

SOC But if there are three feet this way and three feet that way, the whole space will be three times three feet?

BOY That is evident.

SOC And how much are three times three feet?

BOY Nine.

SOC And how much is the double of four?

BOY Eight.

SOC Then the figure of eight is not made out of a side of three?

BOY No.

SOC But from what line?-tell me exactly; and if you would rather not reckon, try and show me the line.

BOY Indeed, Socrates, I do not know.

SOC Do you see, Meno, what advances he has made in his power of recollection? He did not know at first, and he does not know now, what is the side of a figure of eight feet: but then he thought that he knew, and answered confidently as if he knew, and had no difficulty; now he has a difficulty, and neither knows nor fancies that he knows.

MENO True.

SOC Is he not better off in knowing his ignorance?

MENO I think that he is.

SOC If we have made him doubt, and given him the "torpedo's shock," have we done him any harm?

MENO I think not.

SOC We have certainly, as would seem, assisted him in some degree to the discovery of the truth; and now he will wish to remedy his ignorance, but then he would have been ready to tell all the world again and again that the double space should have a double side.

MENO True.

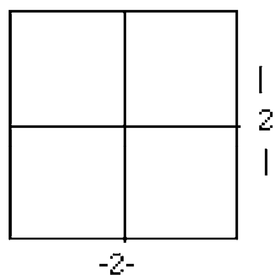
SOC But do you suppose that he would ever have inquired into or learned what he fancied that he knew, though he was really ignorant of it, until he had fallen into perplexity under the idea that he did not know, and had desired to know?

MENO I think not, Socrates.

SOC Then he was the better for the torpedo's touch?

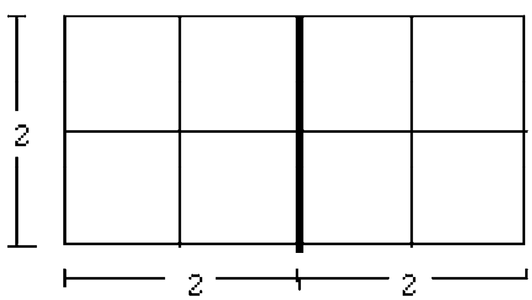
MENO I think so.

SOC Mark now the farther development. I shall only ask him, and not teach him, and he shall share the inquiry with me: and do you watch and see if you find me telling or explaining anything to him, instead of eliciting his opinion. Tell me, boy, is not this a square of four feet which I have drawn?



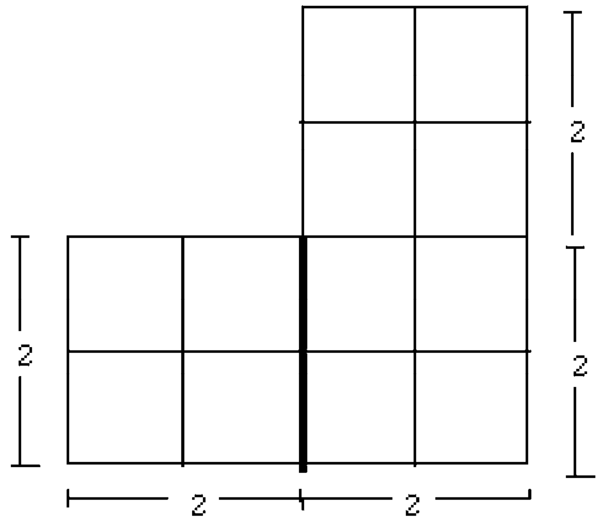
BOY Yes.

SOC And now I add another square equal to the former one?



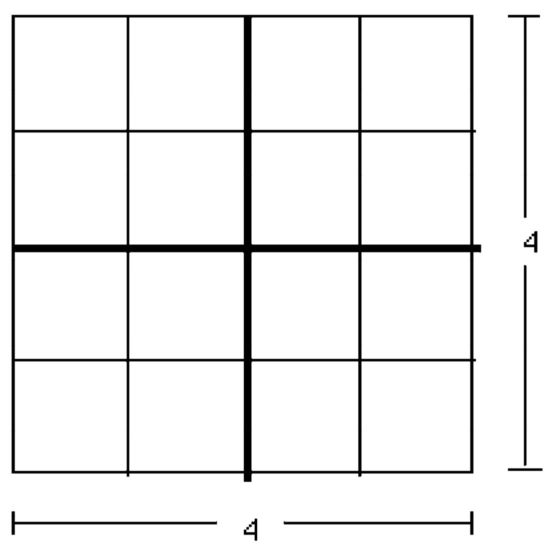
BOY Yes.

SOC And a third, which is equal to either of them?



BOY Yes.

SOC Suppose that we fill us the vacant corner?



BOY Very good.

SOC Here, then, there are four equal spaces?

BOY Yes.

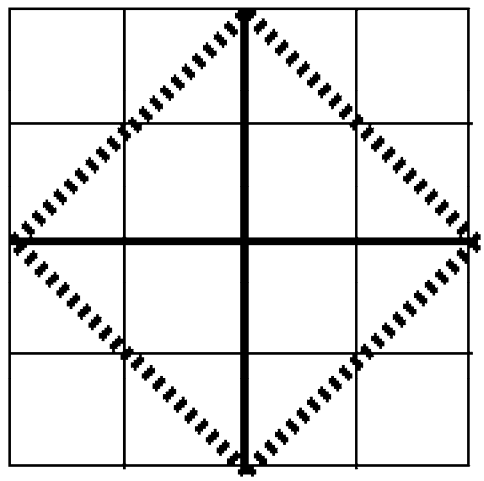
SOC And how many times larger is this space than this other?

BOY Four times.

SOC But it ought to have been twice only, as you will remember.

BOY True.

SOC And does not this line, reaching from corner to corner, bisect each of these spaces?



BOY Yes.

SOC And are there not here four equal lines which contain this space?

BOY There are.

SOC Look and see how much this space is.

BOY I do not understand.

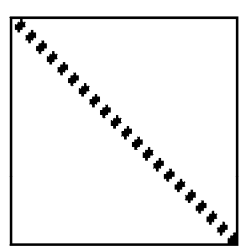
SOC Has not each interior line cut off half of the four spaces?

BOY Yes.

SOC And how many spaces are there in this section?

BOY Four.

SOC And how many in this?

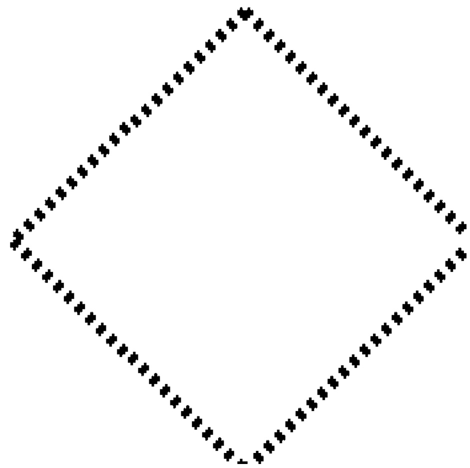


BOY Two.

SOC And four is how many times two?

BOY Twice.

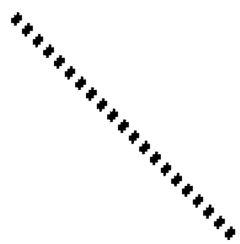
SOC And this space is of how many feet?



BOY Of eight feet.

SOC And from what line do you get this figure?

BOY From this.



SOC That is, from the line which extends from corner to corner of the figure of four feet?

BOY Yes.

SOC And that is the line which the learned call the diagonal. And if this is the proper name, then you, Meno's slave, are prepared to affirm that the double space is the square of the diagonal?

BOY Certainly, Socrates.

SOC What do you say of him, Meno? Were not all these answers given out of his own head?

MENO Yes, they were all his own.

SOC And yet, as we were just now saying, he did not know?

MENO True

SOC But still he had in him those notions of his – had he not?

MENO Yes.

SOC Then he who does not know may still have true notions of that which de does not know?

MENO He has.

SOC And at present these notions have just been stirred up in him, as in a dream; but if he were frequently asked the same questions, in different forms, he would know as well as any one at last?

MENO I dare say.

SOC Without any one teaching him he will recover his knowledge for himself, if he is only asked questions?

MENO Yes.

SOC And this spontaneous recovery of knowledge in him is recollection?

MENO True.

SOC And this knowledge which he now has must he not either have acquired or always possesses?

MENO Yes.

SOC But if he always possessed this knowledge he would always have known; or if he has acquired the knowledge he could not have acquired it in this life, unless he has been taught geometry; for he may be made to do the same with all geometry and every other branch of knowledge. Now, has any one ever taught him all this? You must know about him, if, as you say, he was born and bred in your house.

MENO And I am certain that no one ever did teach him.

SOC And yet he has the knowledge?

MENO The fact, Socrates, is undeniable.

SOC But if he did not acquire the knowledge in this life, then he must have had and learned it at some other time.

MENO Clearly he must.

SOC Which must have been the time when he was not a man?

MENO Yes.

SOC And if there have been always true thoughts in him, both at the time when he was and was not a man, which only need to be awakened into knowledge by putting questions to him, his soul must have always possessed this knowledge, for he always either was or was not a man?

MENO Obviously.

SOC And if the truth of all things always existed in the soul, then the soul is immortal. Wherefore be of good cheer, and try to recollect what you do not know, or rather what you do not remember.

MENO I feel, somehow, that I like what you are saying.

SOC And I, Meno, like what I am saying. Some things I have said of which I am not altogether confident. But that we shall be better and braver and less helpless if we think that we ought to enquire, than we should have been if we indulged in the idle fancy that there was no knowing and no use in seeking to know what we do not know; that is a them upon which I am ready to fight, in word and deed, to the utmost of my power.

An Excerpt from Plato's Dialogues

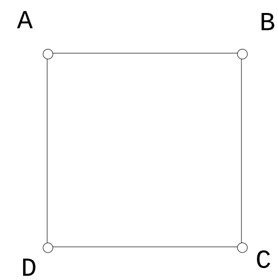
Worksheet

Half the Area???

This activity is to make sure you understand the areas involved in the dialogue. You will use the same squares that Socrates used and use folding to compare the areas involved.

Materials: Three squares of paper, at least two inches on each side.
One larger square of paper with sides twice the length of the three smaller ones.

Procedure: Name the corners of each square as shown in the figures at the right.



Step 1:

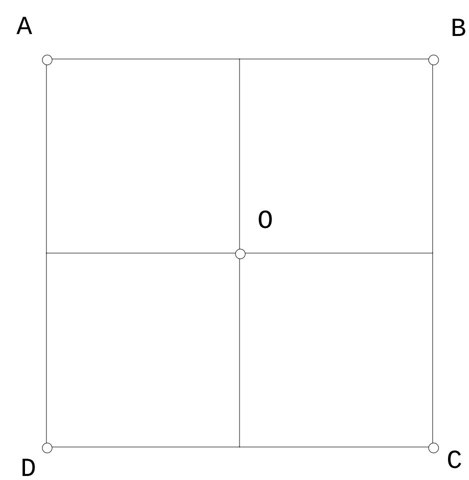
For the first small square make a vertical fold. Fold point A on top of point B and point D on top of point C. The new shape is a rectangle. How does the area of the rectangle compare to the area of the original square?

Step 2:

For the second small square make a horizontal fold. Fold point A on top of point D and point B on top of point C. The new shape is a rectangle. How does the area of the rectangle compare to the area of the original square?

Step 3:

For the third small square make a diagonal fold. Fold point A on top of point C. The new shape is a triangle. How does the area of the triangle compare to the area of the original square?



Step 4:

Now consider the large square. Fold point A on top of point O. Fold point B on top of point O. Fold point C on top of point O. Fold point D on top of point O.

What is the shape of the figure you obtain?

How does the area of this new piece compare to the area of the original large square? How does it compare to the areas of the small squares?

An Excerpt from Plato's Dialogues

Answers

Half the Area ???

Step 1: The area of the rectangle is exactly half the area of the square.

Step 2: The area of the rectangle is exactly half the area of the square.

Step 3: The area of the rectangle is exactly half the area of the square.

Step 4: The shape of the new figure is a square. Its area is exactly half the area of the original large square and twice the areas of each of the smaller squares.

An Excerpt from Plato's Dialogues

Questions for Discussion

1. Did Socrates convince you that you don't learn – that your soul remembers? If not, what could Socrates have done to convince you of this?

2. What, if any, conclusions can you draw about the nature of Greek society in 380 B.C.E. from watching this play?

3. Why, do you think, did Socrates choose to have the slave boy "recall" mathematical rather than other facts?

4. What do you think of Socrates as a teacher? How would you like it if your teacher treated you or another student the way Socrates treated the slave?

5. Notice the structure of the argument Socrates draws out of the slave boy as to why a square on the doubled line is not equal to double the original square. He in fact has the slave boy discover that it has four times the original area. What kind of proof is one which makes an assertion and then shows that the assertion is incorrect?

6. What is the meaning of Socrates' statement to the boy about finding the line from which the correct square can be made: "Tell me exactly; and if you would rather not reckon, try and show me the line." Why could the boy not "tell him exactly?" Why would he have to "show" him?

7. Socrates says at one point that "this line, reaching from corner to corner, bisects" the square. How do we know that statement is correct? Can you prove it? The slave boy seems to accept this fact. What would Socrates have said if the boy had not accepted this result as "obvious."

8. Near the end of the excerpt, Socrates says of the slave, "he always either was or was not a man." Why is this statement true? Can you generalize this statement?

9. Did Socrates convince you that if you are given a square S with diagonal d, then the square constructed on d has twice the area of S?

10. Are you now convinced that if you wanted to construct a square with twice the area of a given square, you would construct it on the diagonal of the given square?

Indirect Proofs

Teacher Notes

Level: This activity is designed for an introductory high school geometry class.

Materials: Student activity should be duplicated and distributed.

Time Frame: One class period.

Objective: To introduce students to indirect proofs.

When to Use: This material serves as an introduction to indirect proofs, so it can go near the beginning of a geometry course. The prerequisites for the first theorem are the result that the base angles of an isosceles triangle are equal and the SAS triangle congruence theorem. The prerequisites for the second theorem are an understanding of prime numbers, including the fact that every positive integer greater than 1 is divisible by some prime number.

How to Use: You should conduct a preliminary discussion on the idea that for any statement P, either P or not P must be true. Give plenty of examples of this.

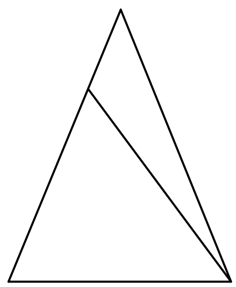
Students should work through the proofs of the two theorems in small groups. Each group should formulate answers to the questions, and then you can discuss the answers with the entire class. Note that the basic structure of the first proof is what is usually called "proof by contradiction." Namely, we assume that one of two alternatives is true and then derive its consequences to the point where we find a contradiction to something already known. It then follows that the other alternative must be true. This type of argument also occurs in the second proof. But the basic structure of the second proof is a bit different, in that one derives the existence of an additional prime number from either of two possible alternatives about the number N. Make sure that students understand the distinction here in the proofs.

Students may wonder why Euclid chose only to look at the case of three primes in the second theorem. One of the reasons is probably that he had no notation which would enable him to express an arbitrary number of primes, like our notation using subscripts and letters. We can, I think, call Euclid's proof here a proof by "generalizable example." You might want to discuss the difference between this kind of proof and "false proofs by example." Students may wonder why Euclid could just use 3 to "prove" his result, but they cannot use the example of 3 to conclude that the remainder of a prime number on division by 4 is always 3.

Indirect Proofs
Student Pages

We consider here two proofs from Euclid's *Elements* to try to understand a particular proof strategy.

Book I, Proposition 6: *If in a triangle, two angles are equal to one another, the sides opposite the equal angles will also be equal to one another.*



Let *ABC* be a triangle with angle *ABC* equal to angle *ACB*; I claim that side *AB* is also equal to side *AC*. For suppose *AB* is not equal to *AC*. Then one of them is greater. Assume that *AB* is greater. From *AB* the greater, let *DB* be cut off equal to *AC*, the lesser. Connect *DC*. Then, since *DB* is equal to *AC*, and *BC* is common, the two sides *DB*, *BC* are equal to the two sides *AC*, *CB* respectively; and the angle *DBC* is equal to the angle *ACB*. Therefore, by SAS, the triangle *DBC* is equal to the triangle *ACB*. But triangle *DBC* is less than triangle *ACB*, so it cannot be equal. Therefore, *AB* is not greater than *AC*. Similarly, *AB* cannot be smaller than *AC*. Therefore, *AB* must equal *AC*.

Questions to consider:

1. What is the statement in the proof in which Euclid says that something either is or is not?

2. Which alternative will Euclid take in order do disprove it?

3. What is the contradiction Euclid arrives at by looking at the consequences of the alternative he chose?

4. How does this contradiction prove the theorem?

Book IX, Proposition 20: *There are more prime numbers than any assigned quantity of prime numbers.*

Let p, q, r be the assigned prime numbers. I claim that there are more prime numbers than p, q, r. For consider the number $N = pqr + 1$. Then either N is prime or it is not. First, let it be prime. Then we have found a prime number different from p, q, r. Second, suppose that N is not prime. Then it is divisible by some prime number s. Now s cannot be the same as either of p, q, or r. For if it were, then one of those would divide N. Since that number also divides pqr, it would divide $N - pqr = 1$. But no prime number can divide 1. So s is different from p, q, r, and we have found a prime number different from those three. Therefore, in any case, there is a prime number different from the three assigned prime numbers.

Questions to consider:

1. What are the two statements in the proof in which Euclid says that something either is or is not?

2. How does he handle each of these two statements? Compare these to Euclid's way of handling the corresponding statement in Book I, Proposition 6.

3. Note that the proof only shows that if there are three prime numbers, there must be a fourth. How would you modify the proof to show that if there were four prime numbers, there must be a fifth? If there were five, there must be a sixth? If there are n prime numbers, there must be an $n + 1^{st}$?

4. Has Euclid proved the theorem he stated? In modern terms, the theorem states "There are infinitely many prime numbers." Why do you think Euclid stated the theorem in the way he did? How would a modern proof read?

The Irrationality of √2

Teacher Notes

Level: This activity is designed for an introductory high school geometry class.

Materials: Historical background to be handed out to students. Notebook paper.

Time Frame: One-two classes, with some time for homework

Objective: To introduce students to proof by contradiction.

When to Use: This activity helps provide an introduction to proof by contradiction. As such, it can be placed early in the course. But the ideas presented are rather sophisticated. It would be best if students have previously been exposed to the idea of irrational numbers.

How to Use: It would be best to go through the opening definitions with the class as a whole. The modern definitions of odd and even will probably be familiar, but students should be encouraged to translate between the algebraic (modern) definition and the more visual (Greek) definition. You should also go over the Greek proof of Theorem 1 with the students to get them familiar with manipulating dots. (Note that when we say "Greek proof," we mean a proof in the most ancient Pythagorean style. Euclidean proofs were, of course, closer in style to modern ones.) Discuss the idea that these proofs are ones in which you can just "see" that the result is true. Compare this with Socrates' statement in the *Meno* that the diagonal of a square divides it into two equal parts. Can you just "see" that that statement is true? Or does it need a formal proof?

The remainder of the activity would work well if students did this in small groups in class and then finished at home.

We present here modern proofs of the theorems to aid you in working these through with your students.

Theorem 1: $2n_1 + 2n_2 + . . . + 2n_k = 2(n_1 + n_2 + . . . + n_k)$; the latter expression is an even number.

Theorem 2: $(2n_1 + 1) + (2n_2 + 1) = 2(n_1 + n_2 + 1)$; the latter expression is an even number.

Theorem 3: Assume there are $2k$ odd numbers. Then split them into k pairs. The sum of each pair is even by Theorem 2. The sum of these k even numbers is even by Theorem 1.

Theorem 4: Assume there are $2k + 1$ odd numbers. The sum of the first $2k$ are even by Theorem 3. So the entire sum is the sum of an even and an odd number. This is odd, because $2n + (2m + 1) = 2(m + n) + 1$.

Theorem 5: The first part follows from Theorems 1 and 3 and the definition of product. The second part follows from Theorem 4 and the definition of product.

Theorem 6: These two parts are special cases of the two parts of Theorem 5.

Theorem 7: The side of an even square number cannot be odd, because the square of an odd number is odd. So it must be even. Similarly, the side of an odd square number cannot be even.

Theorem 8: $(2n)^2 = 4(n/2)^2$.

As a conclusion to this section, you may want to discuss the connection between the incommensurability of the side and diagonal of a square and the more modern statement that that square root of 2 is irrational. (This connection is alluded to in the historical background.) You could then lead the students through the modern proof of this fact, noting that it is virtually identical with the Greek proof.

The Irrationality of $\sqrt{2}$

Student Pages

Historical Background

Today, about 2000 C.E., we think of the length of a line segment as a number. We agree on a unit of length, like the meter, and then we may say that one line segment is 4 meters long, another may be 37/142 meters, and that a third, the length of the diagonal of a square whose side is 1 meter, is $\sqrt{2}$ meters long. These lengths, 4, 37/142, and $\sqrt{2}$, are all numbers. We think of the length of a line segment as a number that is attached to that segment to help us visualize how big it is.

It took humanity many years to arrive at this insight. In 380 B.C.E., the Greeks did not distinguish between a line segment and its length. It was the line segment itself, not its length, that was thought of as a quantity. The Greeks used the word "number" (*arithmos*) exclusively to denote what are today called natural numbers or positive integers. Thus a line segment could not have a number attached to it to denote its length, because what we call today the length of a line segment depends on the unit of measurement that is chosen. "Thus," an ancient Greek would say, "whether or not one can attach a number [positive integer] to a line segment depends on the unit of length that is selected. Since, from a theoretical point of view, one unit of length is as good as another, some segments will have length and others will not. Moreover, if you change your unit of length, segments that used to have lengths may no longer have them while segments that did not have lengths now do. Line segments, not their lengths, are quantities, and numbers and quantities are entirely different things."

As an example of what the ancient Greek mathematicians was saying, suppose you have a stick that is 3 meters long. We can certainly call the number "3" the length of this stick with the unit of measure 1 meter. But if we change the unit of measure to feet, then this stick will no longer have a number in the Greek sense associated to it. For us, the length would be approximately 9.84 feet – but 9.84 is not a "number" for the Greeks. Furthermore, this number is only an approximation. A better approximation would be 9.84252 feet. So rather than deal with a "number" associated to a line segment, the Greeks would simply deal with the line segments themselves as "quantities."

This brings us to the troublesome question regarding the nature of fractions. The Greeks, at least in their formal mathematical work, did not have fractions. They had ratios instead. A ratio was neither a number of a quantity, but instead a concept which enabled you to compare the size of one number with another number, or one quantity with another quantity. What we call today the fraction or rational number 3/4 did not exist for Greek mathematicians. They instead had a ratio 3:4 that helped them to conceptualize the relative sizes of the numbers 3 and 4. They would not divide 3 by 4, because the quotient of these two numbers was not a number. (They would, however, divide 8 by 2.) Today, we divide 3 by 4 and get as answer what we call a rational number, namely 3/4. So we and the ancient Greeks differ on this question about the nature of fractions.

Now two quantities, just as two numbers, can have a ratio to one another. The Greeks would only consider two quantities having a ratio when the two quantities were of the same type, such as two lengths or two areas or two times. When two quantities did have a ratio, there were two possibilities for the Greeks. Either the ratio was just like the ratio of two numbers or it was not. In the first case, what this meant was that you could find a unit of measure such that each of the two quantities was a numerical multiple of that unit. For example, the Greeks would say that two line segments L and M had ratio 3 to 4 if there were another line segment S such that $L = 3S$ and $M = 4S$. In this case, the two line segments L and M were said to be *commensurable* (have a common measure). But if the ratio of two quantities was not like the ratio of two numbers, then they were said to be *incommensurable* (have no common measure). In general, it was not easy to determine whether two quantities were commensurable or incommensurable. The Greeks developed an algorithm to do this, but it was often tricky to apply. Nevertheless, in certain special cases, they were able to show that two magnitudes were incommensurable. In what follows we will look at the argument in one of these cases, the case of the side and diagonal of a square.

Today, we no longer worry so much about the distinction between commensurable and incommensurable, because we no longer distinguish between "number" and "magnitude." Both of these concepts are subsumed under our more general concept of number. Nevertheless, we still preserve the distinction between rational and irrational numbers, those which can be expressed as fractions and those which cannot. So the proof below can be translated in modern terms into a proof that the length of the diagonal of a square of side 1, namely $\sqrt{2}$, is an irrational number.

The Irrationality of √2

This set of activities is designed to lead you through a proof that the side and diagonal of a square are incommensurable. We go through a number of preliminary definitions and theorems before reaching that result. These definitions and theorems deal with **numbers.** In what follows we will use the word **number** in the Greek sense, meaning positive integer. To remind you of this, we will always print that word in boldface.

One way the Greeks visualized **number** was as a set of pebbles arranged in a row or in some other geometric configuration. For example, the **number** seven would be represented as

* * * * * * *

This trend of thought led the Greeks to think of a single dot as a unit and to define a **number** as that which is obtained when units are added to each other. Numbers then come in different types, according to how they are represented with pebbles. We consider some basic definitions and ask you to prove some simple theorems using this pebble notation.

Definition 1: An **even number** is one that can be split into two equal parts. It will be represented like this:

* * * | * * *
The number six

Definition 2: An **odd number** is one that cannot be split into two equal parts. It will be represented like this:

* * * |*| * * *
The number seven

Modern definitions: An **even number** is one which can be written in the form 2n where n is a positive integer. An **odd number** is one which can be written in the from $2n + 1$, where n is a positive integer.

Activity 1: Convince yourself that the Greek definitions and the modern definitions define the same objects.

Definition 3: The **product of two numbers** is the **number** obtained when the first is added to itself as many times as there are units in the second.

Theorem 1: The sum of arbitrarily many even **numbers** is an even **number.**

Proof in Greek style: Arrange the **numbers** as follows:

```
  * * * | * * *
    * * | * *
* * * * | * * * *
       ...
    * * | * *
```

Now create a new **number** by arranging all the dots on the left of the vertical lines to the left of a new vertical line and all the dots on the right of the vertical lines to the right of the new vertical line. Because the new **number** has the same number of dots to the left of the vertical line as to the right, it is an even **number**.

Activity 2: Create a modern proof of this theorem, using modern definitions of even and odd.

Theorem 2: The sum of two odd **numbers** is even.

Proof in Greek style: Arrange the two numbers as follows:

```
  * * * |*| * * *
* * * * |*| * * * *
```

Now create a new **number** by drawing a vertical line. Put all the dots on the left of the bracketed dots to the left of that vertical line and all the dots to the right of the bracketed dots to the right of the vertical line. Finally, put the top bracketed dot to the left of the new vertical line and the bottom bracketed dot to the right of that line. Since there are now the same number of dots on each side of the vertical line, the **number** thus created is even.

Activity 3: Create a modern proof of Theorem 2.

Theorem 3: The sum of an even quantity of odd **numbers** is even.

Activity 4: Create a Greek style proof and a modern proof of this theorem.

Theorem 4: The sum of an odd quantity of odd **numbers** is odd.

Activity 5: Create a Greek style proof and a modern proof of this theorem.

Theorem 5: The product of an even **number** and any **number** is always an even number. The product of two odd **numbers** is an odd **number**.

Activity 6: Create a proof of this theorem using the results of some of the previous theorems.

Definition 4: A **square number** is the **number** obtained when some **number** is multiplied by itself. This latter number is called the **side** of the square number. A square **number** can be represented as follows, where the side is the top row:

```
* * * * *
* * * * *
* * * * *
* * * * *
* * * * *
```

The square **number** 25

Theorem 6: The square of an even **number** is even. The square of an odd **number** is odd.

Activity 7: Create a proof of this theorem using some of the previous results.

Theorem 7: If a square **number** is even, its side is even. If a square **number** is odd, its side is odd.

Activity 8: Give a proof of this theorem, arguing from Theorem 6.

Theorem 8: An even square **number** is divisible into four equal parts.

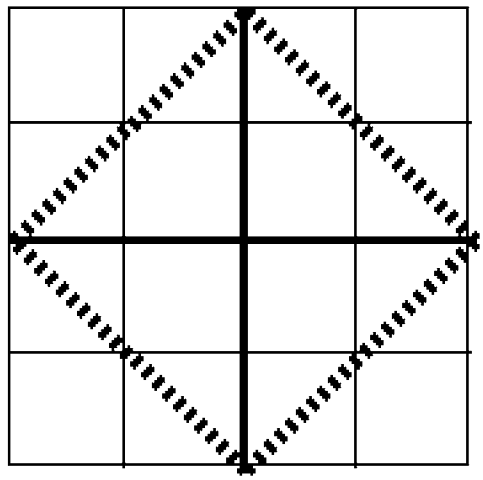
We begin a proof in Greek style by looking at a diagram of an even square number:

```
* * * | * * *
* * * | * * *
* * * | * * *
------
* * *   * * *
* * *   * * *
* * *   * * *
```

Activity 8: Finish the Greek style proof and give a modern proof of this theorem.

We are now ready to provide a proof of our main theorem, that the side and diagonal of a square are incommensurable lengths, that is, that they have no common measure. Recall that this statement means that there is no length, however small, such that both the side and diagonal are multiples of that length. In order to prove this result, we will assume that the two lengths are commensurable, that is, that they do have a common measure. We will then derive a contradiction which will show that this assumption must be false.

We begin with a diagram very similar to one which Socrates showed the slave boy in the *Meno*.



We will take *DH* to be the diagonal of the square *DBHI*, of which *DB* is the side. Thus, since our goal is to show that *DB* and *DH* are incommensurable, we will assume that they are commensurable. This means that there is some length, say *DX*, of which both *DH* and *DB* are multiples.

Basic Assumption: The side and diagonal of a square are commensurable.

Let us suppose that $DH = pDX$ and $DB = qDX$, where *p* and *q* are **numbers**. We therefore say that *DH* is represented by the **number** *p* and that *DK* is represented by the **number** *q*. It follows that the ratio $DH{:}DB = p{:}q$, that is, the ratio of the diagonal to the side is just like the ratio of two **numbers.** We may as well assume that *DX* is the longest possible segment of which both *DH* and *DB* are multiples. This means that *p* and *q* are the smallest possible **numbers** by which we can express the ratio of the diagonal to the side. In particular, this means that *p* and *q* cannot both be even. (Why?)

First Result: *p* and *q* are not both even.

The square *DBHI* is a square on the side *DB*. Since now *DB* represents a **number**, in fact *q*, the square itself represents a square **number.** Similarly, since the square *AGFE* is a square on the side *AG*, and since $DH = AG$, and since $DH = AG$ represents the **number** *p*, the square *AGFE* also represents a square **number**.

Now *AGFE* is the double of *DBHI*, as is clear from the diagram. Therefore *AGFE* represents a square which is double the square *DBHI* and is therefore an even **number.** By theorem 7, the side of this square must also be even. Therefore, $AG = DH$ represents an even **number**. In other words, the **number** *p* is even.

Second Result: *p* is even.

Since *AGFE* represents an even square **number**, it can be divided into four equal parts (by Theorem 8). One of those parts is the square *ABCD*. So that square also represents a **number**. Again from the diagram we see that *DBHI* is double the square *ABCD*. So *DBHI* represents the double of a **number** and must itself be an even **number**. Since it also represents a square **number**, *DBHI* represents an even square **number**. It follows that its side *DB* must also represent an even **number**. Because DB is represented by q, we know that the **number** q must be even.

Third Result: *q* is even.

Now compare the second and third results with the first result. They contradict each other! It follows that our basic assumption, that the side and diagonal of a square are commensurable must be false. We are thus led to the inevitable conclusion:

The side and diagonal of a square are incommensurable.

You should now translate this proof into the proof of the statement that the square root of 2 is irrational. Namely, begin with the assumption that $\sqrt{2} = p/q$, where p and q are relatively prime integers. Then derive a contradiction.

The Declaration of Independence and Proof

Teacher Notes

Level: This activity is suitable for a high school geometry class.

Materials: The Declaration of Independence.

Time Frame: One class for discussion. The students should read and analyze the Declaration of Independence before the discussion.

Objective: To show that an understanding of Euclidean proof was part of a liberal education in the eighteenth century.

When to Use: This activity can be done at any time after you have discussed the nature of a proof by logical argument from explicitly stated axioms.

How to Use: The Declaration of Independence is set up in the form of a mathematical proof, beginning with axioms. You should ask the students to read it carefully and analyze it in that form, using the questions as a guide. In particular, you can discuss the nature of axioms. How "self-evident" are Euclid's axioms – or the axioms which are stated in the geometry book you use? How self-evident are Jefferson's axioms?

For more information on this subject, consult chapter 2 of I. Bernard Cohen, *Science and the Founding Fathers* (New York: W.W. Norton & Company, 1995).

The Declaration of Independence and Proof

Student Pages

The Declaration of Independence

IN CONGRESS, JULY 4, 1776.
THE UNANIMOUS DECLARATION OF THE THIRTEEN UNITED STATES OF AMERICA,

WHEN in the Course of human events, it becomes necessary for one people to dissolve the political bands which have connected them with another, and to assume among the powers of the earth, the separate and equal station to which the Laws of Nature and of Nature's God entitle them, a decent respect to the opinions of mankind requires that they should declare the causes which impel them to the separation.

We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness. That to secure these rights, Governments are instituted among Men, deriving their just powers from the consent of the governed, That whenever any Form of Government becomes destructive of these ends, it is the Right of the People to alter or to abolish it, and to institute new Government, laying its foundation on such principles and organizing its powers in such form, as to them shall seem most likely to effect their Safety and Happiness. Prudence, indeed, will dictate that Governments long established should not be changed for light and transient causes; and accordingly all experience hath shewn, that mankind are more disposed to suffer, while evils are sufferable, than to right themselves by abolishing the forms to which they are accustomed. But when a long train of abuses and usurpations, pursuing invariably the same Object evinces a design to reduce them under absolute Despotism, it is their right, it is their duty, to throw off such Government, and to provide new Guards for their future security. Such has been the patient sufferance of these Colonies; and such is now the necessity which constrains them to alter their former Systems of Government. The history of the present King of Great Britain is a history of repeated injuries and usurpations, all having in direct object the establishment of an absolute Tyranny over these States.

To prove this, let Facts be submitted to a candid world. He has refused his Assent to Laws, the most wholesome and necessary for the public good. He has forbidden his Governors to pass Laws of immediate and pressing importance, unless suspended in their operation till his Assent should be obtained; and when so suspended, he has utterly neglected to attend to them. He has refused to pass other Laws for the accommodation of large districts of people, unless these people would relinquish the right of Representation in the Legislature, a right inestimable to them and formidable to tyrants only. He has called together legislative bodies at places unusual, uncomfortable, and distant from the depository of their public Records, for the sole purpose of fatiguing them into compliance with his measures. He has dissolved Representative

Houses repeatedly, for opposing with manly firmness his invasions on the rights of the people. He has refused for a long time, after such dissolutions, to cause others to be elected; whereby the Legislative powers, incapable of Annihilation, have returned to the People at large for their exercise; the State remaining in the mean time exposed to all the dangers of invasion from without, and convulsions within. He has endeavoured to prevent the population of these States; for that purpose obstructing the Laws for Naturalization of Foreigners; refusing to pass others to encourage their migrations hither, and raising the conditions of new Appropriations of Lands. He has obstructed the Administration of Justice, by refusing his Assent to Laws for establishing Judiciary powers. He has made Judges dependent on his Will alone, for the tenure of their offices, and the amount and payment of their salaries. He has erected a multitude of New Offices, and sent hither swarms of Officers to harass our people, and eat out their substance. He has kept among us, in times of peace, Standing Armies without the Consent of our legislatures. He has affected to render the Military independent of and superior to the Civil power. He has combined with others to subject us to a jurisdiction foreign to our constitution, and unacknowledged by our laws; giving his Assent to their Acts of pretended Legislation: For quartering large bodies of armed troops among us: For protecting them, by a mock Trial, from punishment for any Murders which they should commit on the Inhabitants of these States: For cutting off our Trade with all parts of the world: For imposing Taxes on us without our Consent: For depriving us in many cases, of the benefits of Trial by Jury: For transporting us beyond Seas to be tried for pretended offences: For abolishing the free System of English Laws in a neighbouring Province, establishing therein an Arbitrary government, and enlarging its Boundaries so as to render it at once an example and fit instrument for introducing the same absolute rule into these Colonies: For taking away our Charters, abolishing our most valuable Laws, and altering fundamentally the Forms of our Governments: For suspending our own Legislatures and declaring themselves invested with power to legislate for us in all cases whatsoever. He has abdicated Government here, by declaring us out of his Protection and waging War against us. He has plundered our seas, ravaged our Coasts, burnt our towns, and destroyed the lives of our people. He is at this time transporting large Armies of foreign Mercenaries to compleat the works of death, desolation and tyranny, already begun with circumstances of Cruelty & perfidy scarcely paralleled in the most barbarous ages, and totally unworthy the Head of a civilized nation. He has constrained our fellow Citizens taken Captive on the high Seas to bear Arms against their Country, to become the executioners of their friends and Brethren, or to fall themselves by their Hands. He has excited domestic insurrections amongst us, and has endeavoured to bring on the inhabitants of our frontiers, the merciless Indian Savages, whose known rule of warfare, is an undistinguished destruction of all ages, sexes and conditions. In every stage of these Oppressions We have Petitioned for Redress in the most humble terms: Our repeated Petitions have been answered only by repeated injury. A Prince, whose character is thus marked by every act which may define a Tyrant, is unfit to be the ruler of a free people. Nor have We been wanting in attentions to our British brethren. We have warned them from time to time of attempts by their legislature to extend an unwarrantable jurisdiction over us. We have reminded them of the circumstances of our emigration and settlement here. We have appealed to their native justice and magnanimity, and we have conjured them by the ties of our common kindred to disavow these usurpations, which, would inevitably interrupt our connections and correspondence They too have been deaf to the voice of justice and of consanguinity. We must, therefore, acquiesce in the necessity, which denounces our Separation, and hold them, as we hold the rest of mankind, Enemies in War, in Peace Friends.

WE, THEREFORE, the Representatives of the united States of America, in General Congress, Assembled, appealing to the Supreme Judge of the world for the rectitude of our intentions, do, in the Name, and by Authority of the good People of these Colonies, solemnly publish and declare, That these United Colonies are, and of Right ought to be FREE AND INDEPENDENT STATES; that they are Absolved from all Allegiance to the British Crown, and that all political connection between them and the State of Great Britain, is and ought to be totally dissolved; and that as Free and Independent States, they have full Power to levy War, conclude Peace, contract Alliances, establish Commerce, and to do all other Acts and Things which Independent States may of right do. And for the support of this Declaration, with a firm reliance on the protection of divine Providence, we mutually pledge to each other our Lives, our Fortunes and our sacred Honor.

Questions for Discussion

The Declaration of Independence was set up by Jefferson in the form of a classical, Euclidean-style mathematical argument. It begins with axioms, then states a theorem and proceeds to give the proof. These questions should help you analyze the Declaration in this regard.

1. What are the axioms? Do you believe that these are, in fact, self-evident? How do they compare in this regard to Euclid's "self-evident" axioms and postulates?

2. What is the Theorem which Jefferson proposes to prove?

3. Outline the basic steps of the proof. Is this proof in the logical form familiar to you from proofs you have seen in your geometry course.

4. Has Jefferson proved his Theorem?

5. Find some biographical information about Jefferson's study of mathematics. What mathematics did he study as a young man? Why would he organize the Declaration of Independence in this form?

Introduction to Congruence Through Folding Activities

Teacher Notes

Level: This activity is suitable for middle school or high school geometry.

Materials: Notebook paper or graph paper, pencil or pen, ruler, protractor. Some teachers may want to use waxed paper for folding instead of ordinary paper.

Time Frame: One Class

Objective: To discover how to make proofs.

When to Use: This activity can be done early in the year to provide another introduction to proofs. Students should already know the meaning of the words congruent, midpoint, perpendicular, angle bisector, right angle

How to Use: This hands on series of folding activities is designed to allow the student to discover how to create a simple convincing argument by making creases in a piece of paper. Waxed paper also works well for the folding, but it is hard to write on. If you use waxed paper, use a pencil point or a pin to make small holes instead of drawing dots to mark points. A sharp pencil can also be used to make legible marks on waxed paper.

Although the tools are simple, it is an excellent way for students to begin their pilgrimage to the road to geometrical proofs. As with the checkerboard challenge, these "proofs" are a comfortable introduction. We suggest that you begin with a discussion of the following questions. Then move on to the handout on folding.

You may want to begin the lesson with the following introductory questions:

1. Suppose you have no measuring device and you wish to decide if two people are the same height, what might you do to decide? Why would your method work? Are there any problems with this method?

2. How could you decide if two empty egg cartons were identical in size and shape?

3. How could you determine, if two spoons were identical? If you were to measure instead, what would you have to measure?

In one of the world's most successful books, Euclid's Elements, congruent triangles are envisioned as two shapes that can be superimposed, or made to **coincide**. In these exercises you will find simple ways to show that two shapes are congruent (same shape and size) by actually making them **coincide** through folding.

Congruence Through Folding
Student Pages

Folding Activity 1: Find a midpoint

- On a sheet of paper draw a medium sized (about the length of your index finger, unmeasured!) line segment. Your problem is to find the midpoint of this segment without measuring, BUT you are allowed to fold and crease the paper.

  Why does your method work? Write an explanation using the ideas about **coinciding** and **congruent**.

- If you were to leave the segment folded and the visible (top) piece happened to be 5 cm., how long would the other (covered by the fold) piece be? Why?

- Suggest an alternate method of determining a midpoint if you are allowed to measure. Next suggest a new definition for congruence of segments using measurement.

- If you use measuring to decide about congruence, are you still using the notion of coinciding? Think about the segment on the ruler you are using and what happens when you use the ruler.

- If your "segment" had not been "straight" would your folding have been successful? Euclid's definition for straight line (segment) was:

  A **straight line** is a line which lies evenly with the points on itself.

  Discuss what Euclid probably meant when he defined a straight line (segment).

Legal Folds

Now that you have successfully used a folding procedure to find a midpoint by creating a point that splits its segment into two identical parts, you need to know what types of folding are considered legal. They are:

i) Two points can be made to coincide and the (vertical) crease between them considered a known line. This is probably the fold you just used.

ii) Two points can be used to determine a crease, or the line through them.

iii) A line may be folded at any point on it so that the parts of the line lie on one another.

iv) Two rays from the same point may be folded to coincide beginning at that point.

v) In a folded diagram, any geometric figure may be copied across the fold. You are allowed to peek in order to make a copy. For example, if you fold a point over a line, you may copy the point on the other side of the line to create new point that is the folded image of the original point. See the first diagram in the **Challenges** section at the end of this activity set.

Matching Activity

Here are some of Euclid's postulates. They tell us legal things to do in creating an object or supporting a claim about a relationship. Some people call the things you can do using these postulates "ruler and compass constructions."

1. A straight line can be drawn form any point to any point.
2. A finite straight line can be produced continuously in a straight line.
3. A circle may be described with any center and distance.
4. All right angles are equal to one another.

a) Which of the legal folds either duplicate one of the four postulates or can be used to verify it?

b) Rewrite one of the postulates as a legal folding procedure.

Folding Activity 2: Find an angle bisector

- Use your ruler to draw an angle, making the sides fairly dark. Your problem is to find the ray that bisects your angle without making any measurements at all. Again you are allowed to fold.

    Write a paragraph to explain your method and why you think it worked, based on the same ideas about coinciding geometric figures.

- Use your protractor to measure the three angles you have in your diagram, including the fold as a ray. Suggest an alternate definition for angle bisector. Now write a new definition for congruence of angles.

Folding Activity 3: Find a perpendicular to a line segment at its midpoint

- Draw a line segment. Use a fold to find a line that is perpendicular to the segment at its midpoint.

    Now explain why your procedure works. You have already proven that this finds the midpoint, but now you need mention two coinciding angles.

    If your protractor is placed so that its center is at your segment's midpoint what number should you expect to see corresponding to the ray at the fold? If the whole protractor is 180 degrees and your two angles are congruent, explain why the number you guessed must be correct.

- Draw a line segment and choose a point X on it. Use a fold to find a line perpendicular to the line segment at that point.

  Write a brief explanation for the correctness of your method.

- Draw a line segment. Now use a fold to find the perpendicular at one of the endpoints of the segment. You may wish to use Euclid's second postulate: to produce a finite straight line continuously in a straight line In other words line segments can be extended.

Summary: We now have used folding to create certain geometric configurations using the key idea of congruence as in Euclid's original concept of making shapes coincide.

How many different folds are there here?

How many different explanations (proofs) have been seen?

Thinking about what Euclid has said in his definition of a line (segment) and his in postulate for extending one, are there any other geometric shapes (curves) that seem to have both properties and for which your midpoint and perpendicular folds would work just as well?

Challenges

In each case write a brief explanation of the correctness of your method, based on the idea of coinciding figures and the methods you have already successfully justified above.

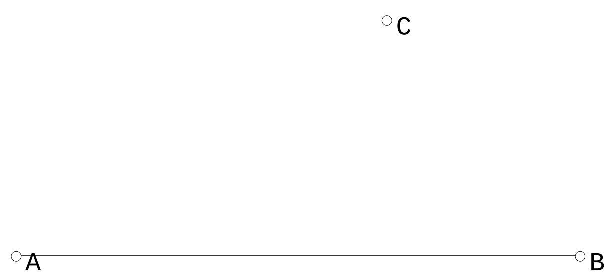
Can you fold a 45 degree angle?

Trisect a segment?

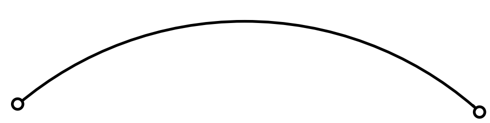
Fold a right triangle?

Fold a square?

Fold a perpendicular line from a point C to line segment AB (copy the diagram below)?



Copy or use the circular arc below. Can you fold its midpoint?



If you succeeded, do you see any curves that seem perpendicular? Why?

The circular arc above is part of a circle. Can you find the center of that circle?

Suggestions for student projects

1. Locate a book on origami folding (some are included in the Bibliography) and plan an introduction to some of those kinds of exercises that you present to the class.

2. Implement some of the folding exercises on computer software, such as Cabri Geometry or Geometer's Sketchpad. Plan a demonstration for your class in the computer lab or classroom if possible.

Historical Notes

1. Where did the ideas about congruence come from?

   As with many historical questions, we do not know the exact answer to this. However, there are strong analogies between the ideas of equality of counted sets and the congruence of geometric sets of points. For example, the word calculate has a Latin root in the word *calculus*, which meant pebble. Historians have conjectured that perhaps even before the invention of simple number systems, herdsmen kept track of their animals by placing pebbles in a leather pouch as the herd moved out to graze in the morning, then removed a pebble for each animal that returned in the evening. This way of assuring no animal has been lost uses a matching scheme without actually assigning a number to each animal or to the total size of the flock. The pebbles are made to correspond to the animals. A similar idea is present in the matching up of identically shaped objects that you probably suggested your responses to the introductory questions.

2. Did ancient mathematicians use folding?

   Probably not. Two thousand years ago in Greece the closest thing to paper was a writing material made of papyrus reeds, a relatively scarce and thus valuable commodity. Babylonian scribes and students wrote on clay tablets.

   In his first proofs (Book I of the Elements), Euclid creates equal lines (congruent segments) using circles, since he has set down a postulate that allows him To describe a circle with any center and distance He does not use the word congruent until he considers conditions that produce triangles that are the same shape and size. In that proposition he manages to copy parts of a first triangle onto a second and finally concludes thus the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it (Book I, Proposition 4)

3. Why did ancient Greek mathematicians need to give proofs?

   Ancient mathematics existed all over the globe and it is an open question as to when arithmetic and geometric applications began to become mathematics. In China, Mesopotamia, Egypt, and India, for example there are records of mathematical ideas about arithmetic and geometry. The civilizations that developed all had to solve some of the same problems: land use, water allocation, prediction of the seasons for planting and harvesting, storage capacity of granaries, to name a few. We have found no records that would indicate that the Egyptians or Babylonians required proofs for their results, so it is likely that their rules for computing volumes and areas were arrived at through trial and error. Unfortunately, several Babylonian formulae are incorrect. For example, to find the area of a four-sided polygon the Babylonian computation is: average the opposite sides and then multiply. You may wish to experiment on graph paper to see how well this calculation works.

The Greeks were said to travel in Mesopotamia and so were probably aware of much of this empirical mathematical knowledge. When the reasoning methods of the Greek philosophers were applied to the area of mathematics around 600 B.C.E. the nature of mathematics in western civilization was profoundly changed, many would say born. It is a consequence of correct reasoning from correct assumptions that incorrect computational formulas as in the Babylonian example can be eliminated (Eves & Newsome, 1958, p.2-10).

Another interesting explanation for the Greek emphasis on proof comes from a consideration of their society and economy. The geography of Greece prevented large-scale agriculture and the governments were small city-states ruled by law. Citizens were apt to learn how to reason and debate in such a community, since it would be in their interest to have such skills in a society governed by laws rather than by a king (Katz, 1998, p.46-7). As you study mathematics today, keep in mind that such skills are still essential to good citizenship and to intelligent economic habits.

Biographical Sketch: Euclid, author of the Elements , the most important mathematical text of Greek times, and probably of all time (Katz, 1998, p.58) taught and wrote about 2300 years ago at the Museum and Library at Alexandria. It is likely that he received his mathematical training in Athens from the students of Plato, since that is where the mathematicians who could have instructed him and whose work is collected, refined, and perhaps extended in the *Elements* (Heath, 1956, p. 2) were located.

Of Euclid the man, we know virtually nothing. Since his *Elements* was so influential, he is depicted in numerous works of art, but no one has any idea of his actual appearance. There are several stories of Euclid's teaching experiences, revealing his (or his biographers) humor when dealing with students. One student, after learning the first theorem in Euclid's text, asked the author, but what shall I get by learning these things Euclid summoned his slave and said, "Give him a three pence, since he must make gain out of what he learns" (Heath, 1956, p.3). The more famous anecdote is that of King Ptolemy, who after looking through Euclid's text, asked if there were not some shorter way to master the subject. Euclid is said to have replied that "In geometry there is no royal road!" (Aaboe, 1964, p.46)

Euclid wrote other works on mathematics, but the *Elements* is clearly the most significant. In it he collected and organized much of the earlier work of his predecessors into thirteen books or chapters. There are no exercises for the student, only the crisp sequence of theorems and their proofs that follow from his initial choices of postulates, common notions, and definitions. Thus, while influential the book makes for dry reading!

Introduction to Congruence Through Folding Activities

Some final words to the teacher

These activities are designed with several uses in mind:

1. Introduction of very simple proofs based on the student's direct experience.

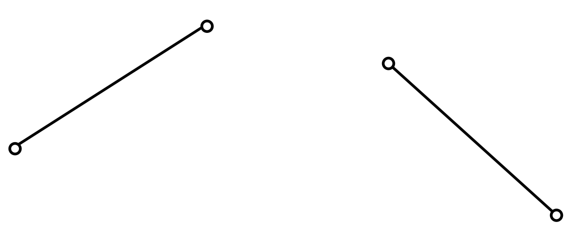
2. A constructive approach to definitions, undefined terms, postulates and theorems. Students understand the tasks and can accomplish them, then proceed to create their own definitions and theorems that will correspond well with those in your course.

3. A historical base for the ideas of proof and for the structure of geometry that motivates the activities and adds an extra dimension of interest and paths for outside research.

4. A clear connection to geometric transformations, which the teacher may introduce later.

5. A possibility of duplicating or extending these activities or ideas using geometry software.

Although none of the formal definitions from a standard modern text on geometry have been provided in the folding activities, students usually need no prompting to decide what the words midpoint, right angle, or angle bisector mean. You may wish to include a definition of perpendicular that will lead them toward a correct solution to that exercise. "Congruent adjacent angles whose exterior sides are opposite rays" is precise, but not in the spirit of these exercises. "Two lines that meet in four congruent angles" would probably do nicely and the exercise asks the student to try to decide why they must be right angles. Alternatively you may ask the students to create their own definitions for these words, based on the folding procedure, then propose equivalent definitions using measures of angles or lengths. You can point out, that while folding has worked well here it becomes impractical to require folding everything to decide about congruence. An example for a class that is curious would be to use folding to demonstrate the congruence of the two segments below in "general position".



If they try this, you might suggest the make the endpoints very dark or punch small holes in them to facilitate seeing the first and second folds. Your class will probably have several different correct solutions and you may wish to ask how many are possible.

A typical acceptable proof or explanation for Activity 1: When the segment is folded properly, the two endpoints coincide and the two smaller pieces of the segment are matched point for point up to the point at the fold. This means these two smaller segments are congruent, since they have been made to coincide. The point at the fold is the point that cuts the original into two identical parts, so it is the midpoint. Students who respond because its in the middle just need to be prodded to clarify that idea using the words coincide and congruent.

At some point, if you decide to discuss mathematical systems with your class, the various features in this set of folding exercises is a good place to start. The definitions for right angle (the only special angle Euclid mentions), perpendicular, and midpoint can be given correctly in several different ways, after your students have experimented enough to feel they understand the ideas. The undefined terms line and point are represented both by drawn lines and creases on the paper. Sometimes two points are sufficient for determining a line, sometimes a point and a right angle, as in the midpoint crease.

Euclid Through the Path of Tangrams

Teacher Notes

Level: This activity is suitable for middle school or high school geometry courses.

Materials: paper, pencil, tangram sheet, and scissors

Time Frame: One or two class periods.

Objective: To show students how to visualize the essence of a geometric proof.

When to Use: This activity can be used when proofs are being introduced.

How to Use: This activity shows students how geometry proofs involve visualization. It complements the Checkerboard Challenge activity by showing how you may come to believe something may be true so that you will know to look for a formal proof. It also gives good proofs of special cases of familiar geometrical theorems.

Ancient legend has it that Archimedes invented a set of tangrams which he used in his geometric demonstrations, but details of the tangrams and exactly how Archimedes used them have not survived. Tangrams are ways for students to observe that area may stay the same while perimeter may change. Proofs are like puzzles that you are trying to solve. What information is given to you? What are you to find? How are you going to accomplish the desired outcome? These activities have the "if" part and the "then" part. Students have to figure out the how.

Much that we do in life follows thinking processes that are required to do proofs. We relate proofs to geometry and mathematics. But it is a thinking process that allows us to reach our desired goal. Proof is a thinking process that allows us to have a better understanding of the problem, to explain it in different terms. The students will be able to visualize how to reach the desired outcome. The visualization is the creative step. This section relates to the NCTM standards for problem solving and mathematical reasoning. It consists of four activities. We also give Euclid's proofs of the propositions demonstrated with tangrams.

Activity 1: Fun with Tangrams

This in an introductory exercise to introduce the student to the relationship among the various tangram pieces.

Activity 2: Strange Shapes

This activity provides enticing puzzles that require the student to rearrange the tangram pieces. Students should notice that different shapes often have the same areas.

Activity 3: Using tangrams to demonstrate proofs of Euclid's Propositions

This activity utilizes tangrams to illustrate several of Euclid's proofs. This activity may be done in groups of students as projects and may be graded on a rubric scale. It is recommended that there be no more than three to a group. We suggest that teachers have the students make a journal entry on their tangram results or they may include their results in portfolio.

Activity 4: Using Geometer's Sketchpad or Cabri to demonstrate proofs of Euclid's propositions

This is an activity that uses Geometers Sketchpad or Cabri. We suggest no more than three students at a computer. Each student should turn in hard copies of each activity. You might choose to have each student evaluate the other students in the group. You might also include this project in your portfolio.

Lecture/Explanation: A Demonstration of How Euclid Proved These Propositions

The final option for this part of the module is for the teacher to choose one of the propositions and explain how Euclid actually wrote his proof. It may be most advantageous to select only one or two of these for demonstration purposes.

Euclid Through the Path of Tangrams
Student Pages

Activity 1

Fun With Tangrams

Tangrams are an ancient Chinese puzzle game. It consists of seven pieces. Both cl
may be entertained for hours by trying to form various shapes out of the individua
pieces are parts of a whole. Solve problems I -IV

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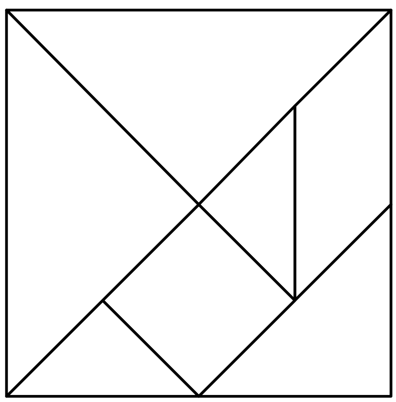
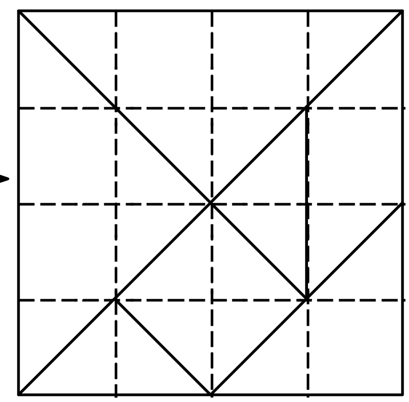


The seven tangram pieces

All squares have equal areas.



The seven tangram pieces on a grid.

I. Assume that the entire puzzle is 1 unit. Determine the fraction that each of the other pieces of the tangram puzzle represents.

Piece	Size
Large Triangle	
Small Triangle	

Piece	Size
Parallelogram	
Large Square	1

Piece	Size
Medium Triangle	
Small Square	

II. Find three different parts of the tangram puzzle that have the same area. Explain how you know that these pieces have the same area.

III. Assume that the large triangle piece of the tangram puzzle is one unit. Determine the fraction that each of the other pieces of the tangram puzzle represents.

Piece	Size
Large Triangle	1
Small Triangle	

Piece	Size
Parallelogram	
Large Square	

Piece	Size
Medium Triangle	
Small Square	

IV. Assume that the small square piece of the tangram puzzle is one unit. Determine the fraction that each of the other pieces of the tangram puzzle represents.

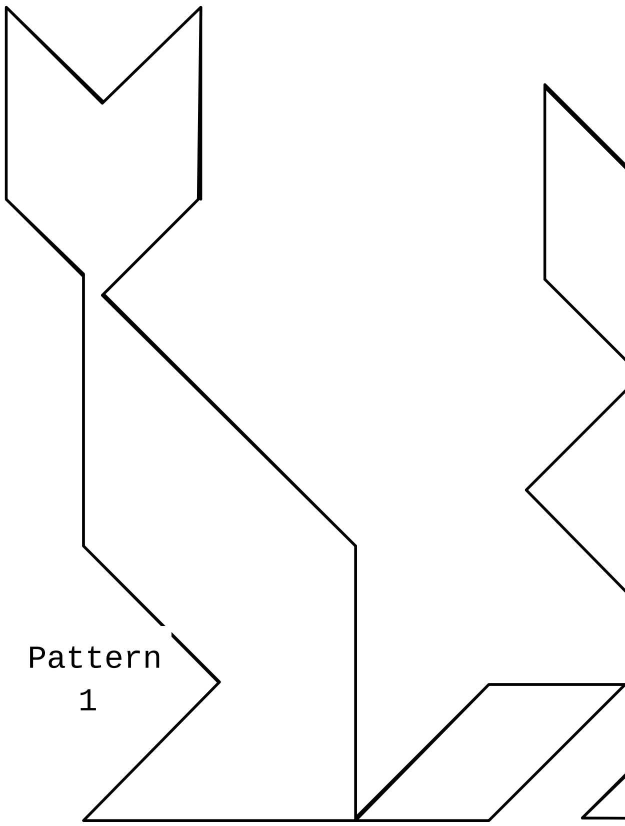
Piece	Size
Large Triangle	
Small Triangle	

Piece	Size
Parallelogram	
Large Square	

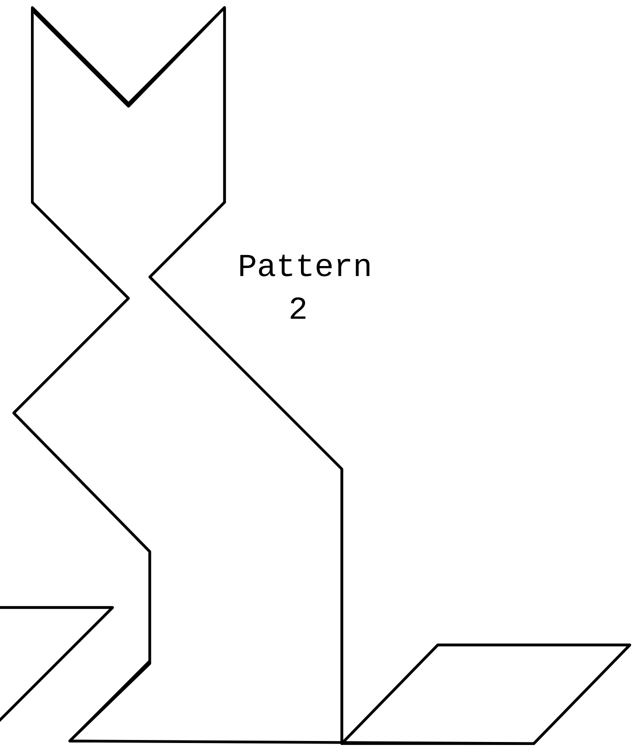
Piece	Size
Medium Triangle	
Small Square	1

Activity 2
Strange Shapes

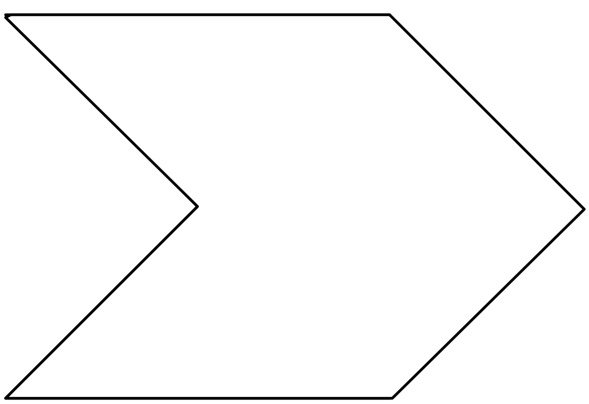
Can you use the entire set of seven tangrams to form the following patterns???



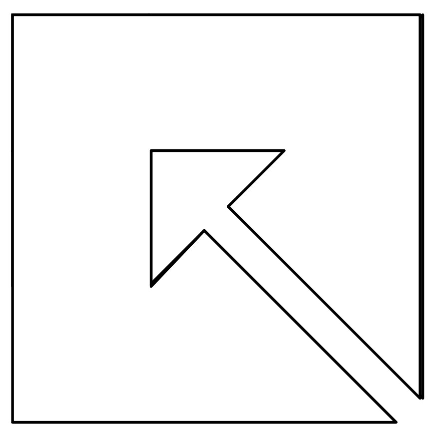
Pattern 1



Pattern 2



Pattern 3



Pattern 4

Activity 3

Demonstrations of selected propositions of Euclid using tangrams

Directions

This activity uses tangram pieces to visually demonstrate selected propositions that were formally proved by Euclid. You may use the enclosed master for tangram pieces. You may also make the activities as challenging as you want.

Archimedes is credited with designing a set of tangrams, though little is known of how his tangrams looked or exactly how he used them. Some people suggest that he used them to discover theorems about lengths and areas, or perhaps to help explain those theorems to other people.

In this activity, we follow in what may have been Archimedes' footsteps and demonstrate that certain of Euclid's theorems about areas are true in particular cases involving right angles and 45° angles. Though these demonstrations are not formal proofs, they do point towards what might be proved.

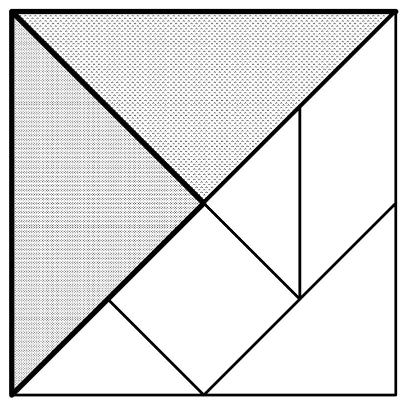
For each theorem, you will be given the statement of the theorem as it appears in Heath's translation of Euclid's *Elements*. We identify each theorem by its book in the thirteen books of the *Elements*, and by its theorem number. The translation is over a hundred years old, and grammar in English has changed a bit in the last century. When the statement of the theorem seems a bit confusing, we try to restate the theorem in a more modern style. Then you will be led to a way to use tangrams to illustrate what the theorem says and how it works. When you finish each theorem, you should understand what the theorem says, as well as how the tangrams illustrate the theorem.

Activity 3 (continued)

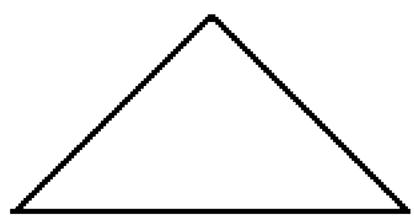
Book I, Proposition 41

If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

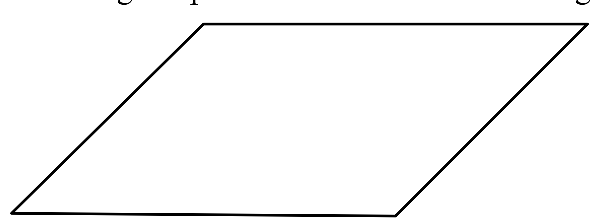
[In other words, the proposition states that if a triangle and parallelogram have the same base and same height, the area of the parallelogram is twice the area of the triangle.]



Note that one of the shaded tangram pieces is the following:



Use the two shaded tangram pieces that create the following parallelogram



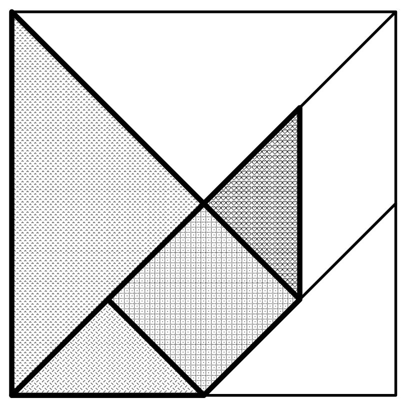
How is the area of the triangle related to the area of the parallelogram?

Can you demonstrate the same result with a different combination of tangram pieces?

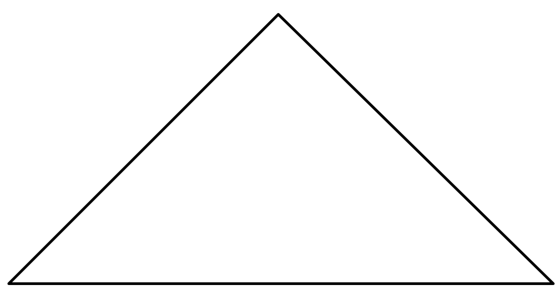
Book I, Proposition 42

To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

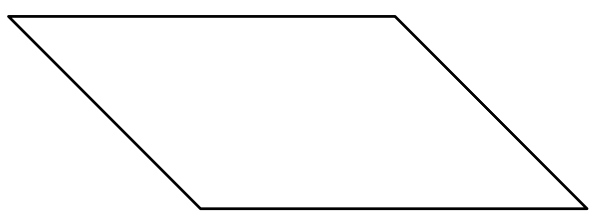
[Euclid shows in his construction that the parallelogram has the same area as the triangle. He starts by breaking the original triangle into two equal triangles, each having the same height as the original and each having half the base. The original triangle is twice the area of each of the smaller triangles. He then constructs a parallelogram with the same height as the original triangle, but with base the same as one of the smaller triangles. The parallelogram then has area double that of the smaller triangle, and therefore equal to that of the original triangle.]



Use the shaded tangram pieces to create the following figure.



Use the shaded tangram pieces to create the following parallelogram.



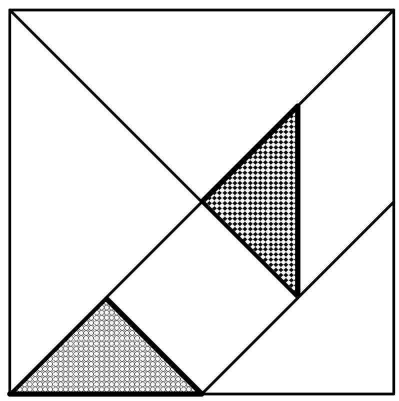
How can you tell if the two shapes have the same area?

How does this relate to Proposition 42?

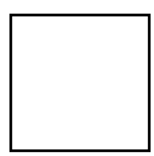
Book I, Proposition 45

To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.

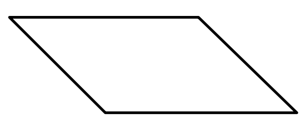
[Although he states that he will construct a parallelogram equal to any given rectilineal figure, in fact, Euclid use a quadrilateral and a given angle. He is to construct the parallelogram with that angle and with the same area as the quadrilateral. He divides the quadrilateral into two triangles and, using proposition I, 42, constructs a parallelogram equal to one of the triangles. Then he constructs a second parallelogram equal to the second triangle, using as side one of the sides of the first parallelogram. He shows that these two parallelograms form a single parallelogram equal in area to the original quadrilateral]



Use the shaded tangram pieces to create the following figure.



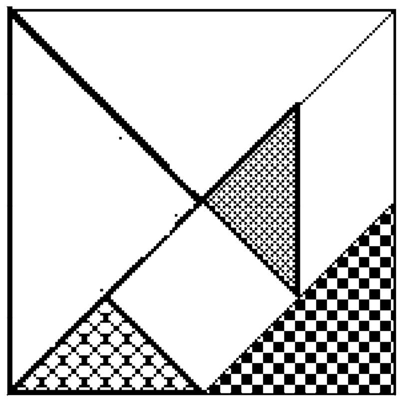
Use the shaded tangram pieces to create the following parallelogram.



Book II, Proposition 14

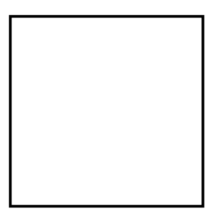
To construct a square equal to a given rectilinear figure.

[Euclid starts out with a polygon with area A. He then constructs a rectangle with the same area. Using the length and width of the rectangle, he constructs the geometric mean. The geometric mean is the length of the side for the square he is looking for.]



Use the shaded tangram pieces to create the following rectangle.

Use the shaded tangram pieces to create the following square.



Activity 4

In this activity, you will use Geometers' Sketchpad or Cabri to construct triangles and parallelograms and then formulate your own proofs of the same set of propositions from Euclid

1. Using the figures you constructed, now show an informal proof for **Book I, Proposition 41.** Show your construction.

If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

2. Repeat problem 2 for **Book I, Proposition 42.**

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

3. Repeat problem 2 for **Book I, Proposition 45.**

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

4. Repeat problem 2 for **Book II, Proposition 14.**

To construct a square equal to a given rectilinear figure.

5. There may be more than one way to show these propositions. You may include other ways for extra credit.

Extra Projects

1. Have the students write a report on Archimedes. See if they can find examples of some of his puzzles from his book *The Sandreckoner*.

2. Have students write a report on Euler. See if they can find examples of some of his puzzles or problems.

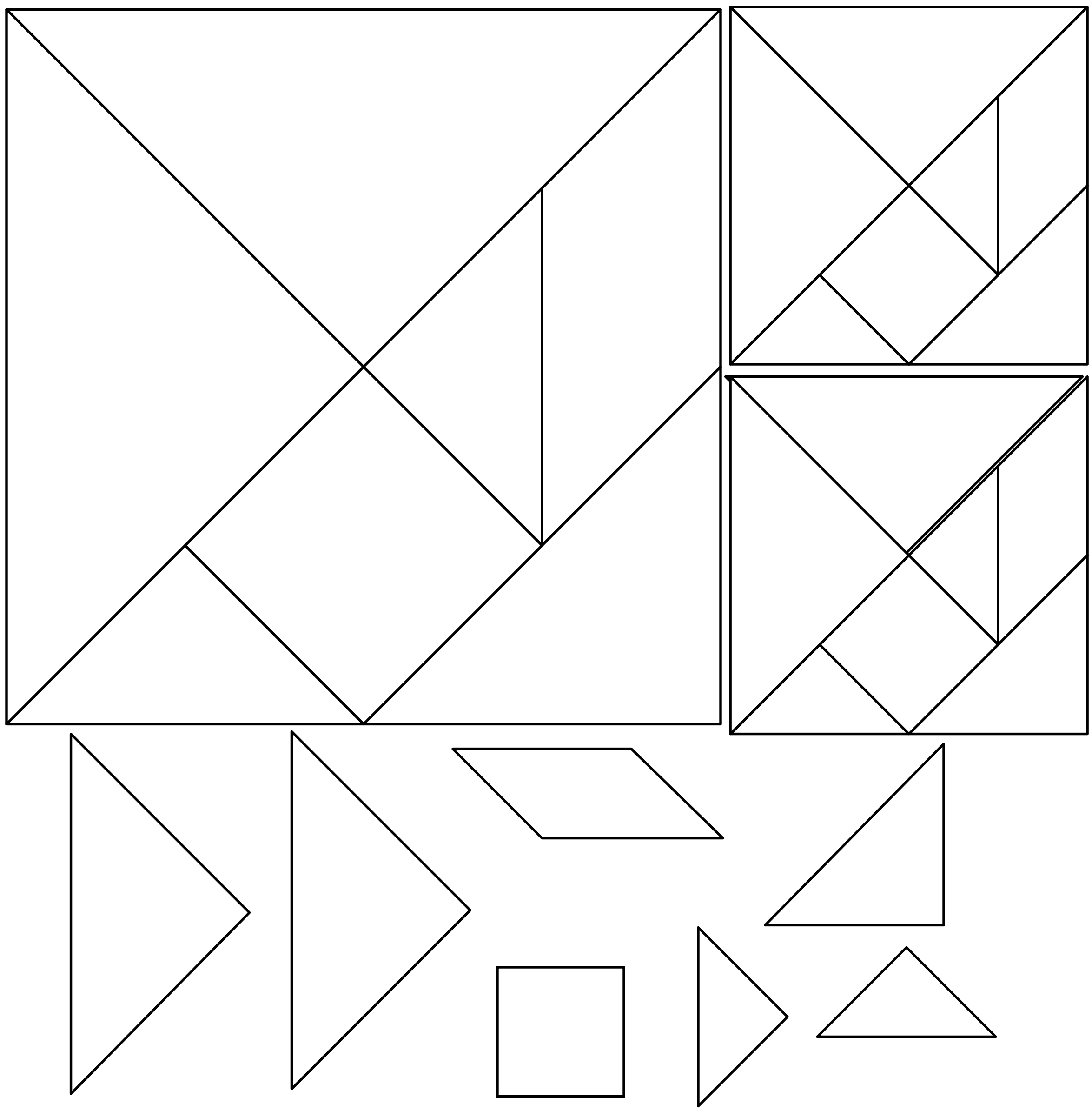
3. Have students find their own pattern with tangrams, or make a notebook of different shapes made from tangrams.

4. Have students look up other propositions from *Euclid's Elements*. See if they can find other propositions that could be shown by using tangrams.

5. Have students write a report about Euclid.

6. Have students do research about Greek mathematics of the time. Have them gather information about Ptolemy I, Alexandria, and its important to Greek mathematics.

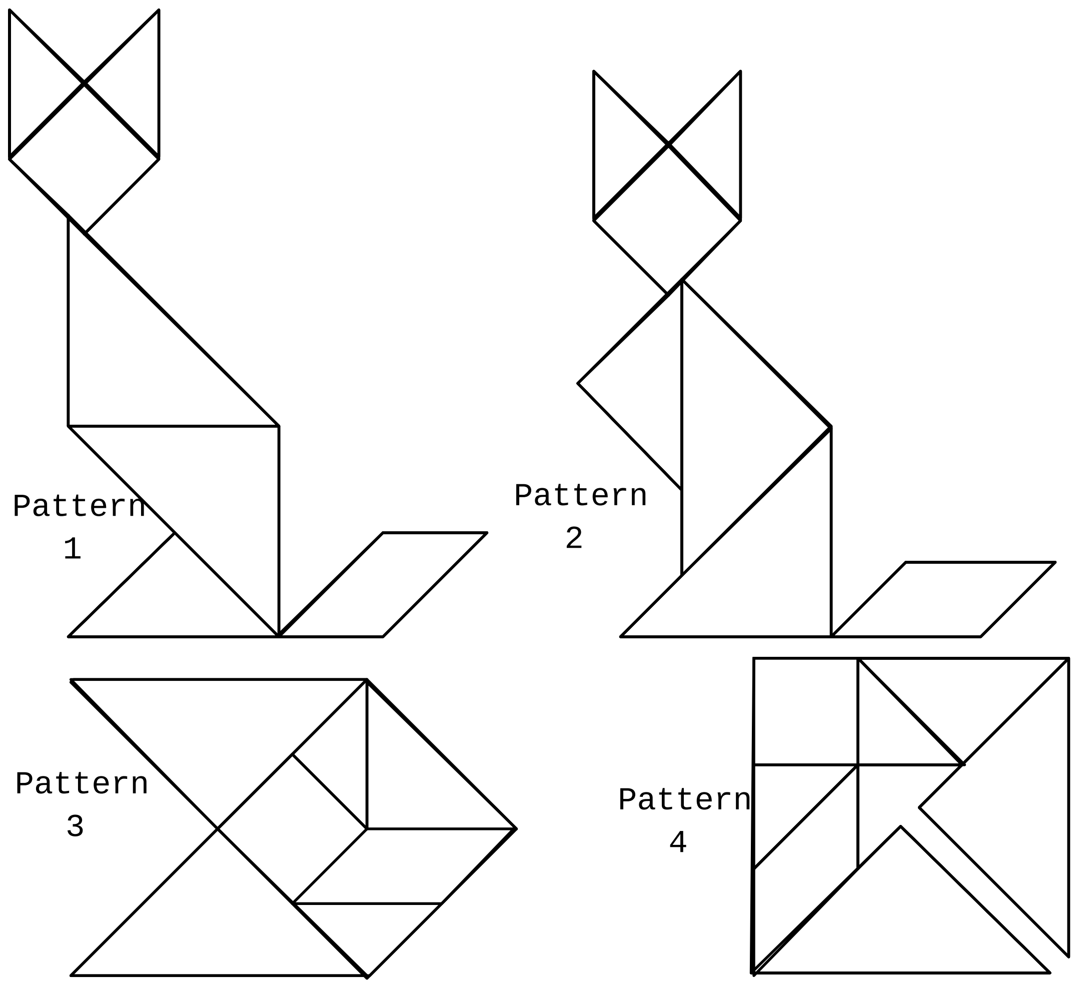
These tangrams and tangram pieces may be used to solve Activities 1, 2, and 3



5.

Euclid Through the Path of Tangrams
Solutions

Activities 1 & 2



Problem I

Piece	Size
Large Triangle	1/4
Small Triangle	1/16

Piece	Size
Parallelogram	1/8
Large Square	1

Piece	Size
Medium Triangle	1/8
Small Square	1/8

Problem II

[2 Large Triangles] [Two Small Triangles] [Medium Triangle, Small Square, Parallelogram]

Problem III

Piece	Size
Large Triangle	1
Small Triangle	1/4

Piece	Size
Parallelogram	½
Large Square	4

Piece	Size
Medium Triangle	1/2
Small Square	1/2

Problem IV

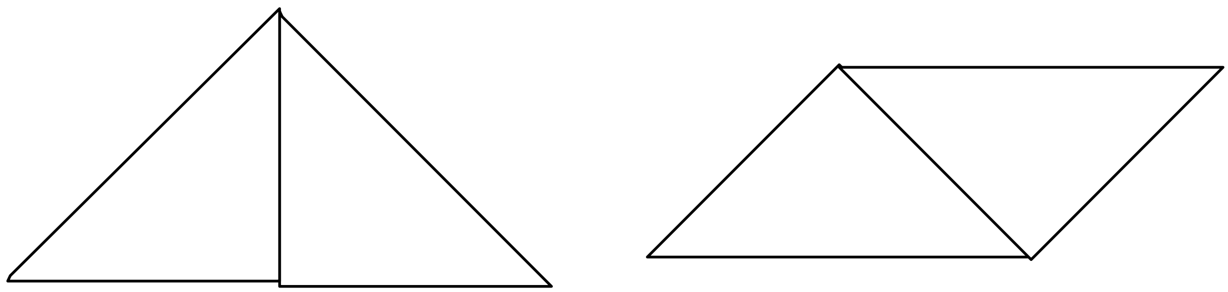
Piece	Size
Large Triangle	2
Small Triangle	1/2

Piece	Size
Parallelogram	1
Large Square	8

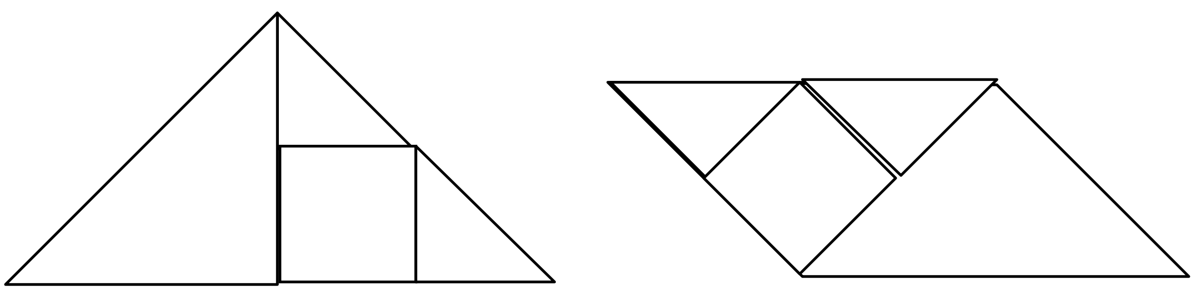
Piece	Size
Medium Triangle	1
Small Square	1

Activity 3

Book I, Proposition 41



Book I, Proposition 42



Book I, Proposition 45

Book II, Proposition 14

How Euclid Proved These Propositions

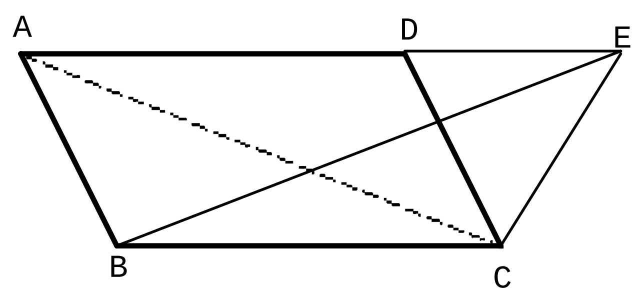
The following paragraphs first state the proposition as it appeared in Euclid's *Elements* and then proceed to show how Euclid proved the proposition or stepped through the construction process.

The Thirteen Books of Euclid's Elements translated with introduction and commentary by Sir Thomas Heath was used as the basis of the following paragraphs. The excerpts from Euclid's *Elements* are in Italics. Comments and references to other Propositions and Common Notions are enclosed in brackets. Euclid used "common notions" for what we call "axioms" and "propositions" for what we call "theorems"

Book I, Proposition 41

If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

Diagram used by Euclid



For let the parallelogram ABCD have the same base BC with the triangle EBC, and let it be in the same parallels BC, AE; I say that the parallelogram ABCD is double of the triangle BEC.

Proof

For let AC be joined. Then the triangle ABC is equal to triangle EBC; for it is on the same base BC with it and in the same parallels BC, AE.

[The above statement is based on **Book I, Proposition 37** which states: *Triangles which are on the same base and in the same parallels are equal to one another.*]

But the parallelogram ABCD is double of the triangle ABC; for the diameter AC bisects it;

[The above statement is based on **Book I, Proposition 34** which states: *In parallelogrammic areas the opposite sides and the angles are equal to one another, and the diameter bisects the areas.*]

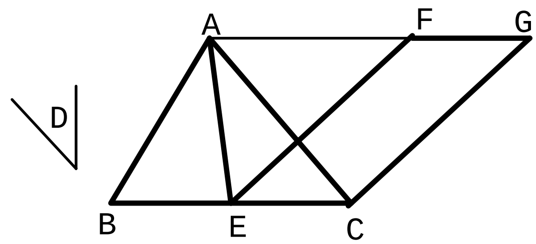
so that the parallelogram ABCD is also double of the triangle EBC.

Therefore: *Q.E.D.*

Book I, Proposition 42

To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

Diagram used by Euclid



Let ABC be the given triangle, and D the given rectilineal angle;, it is required to construct in the rectilineal angle D, a parallelogram equal to the triangle ABC.

Construction Procedure

Let BC be bisected at E, and let AE be joined; on the straight line EC, and at the point E on it, let the angle CEF be constructed equal to the angle D;

[The above construction used **Book I, Proposition 23** which states: On *a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.*]

through A let AG be drawn parallel to EC,

[The above construction used **Book I, Proposition 31** which states: Through *a given point to draw a straight line parallel to a given straight line.*]

and through C let CG be drawn parallel to EF.

Then FECG is a parallelogram. And, since BE is equal to EC, the triangle ABE is also equal to triangle AEC, for they are on equal bases BE, EC and in the same parallels BC, AG;

[The above statement is based on **Book I, Proposition 38** which states: *Triangles which are on equal bases and in the same parallels are equal to one another.*]

therefore the triangle ABC is double of the triangle AEC.

But the parallelogram FECG is also double of the triangle AEC, for it has the same base with it and is in the same parallels with it.

[The above statement is based on **Book I, Proposition 41** (previously stated)

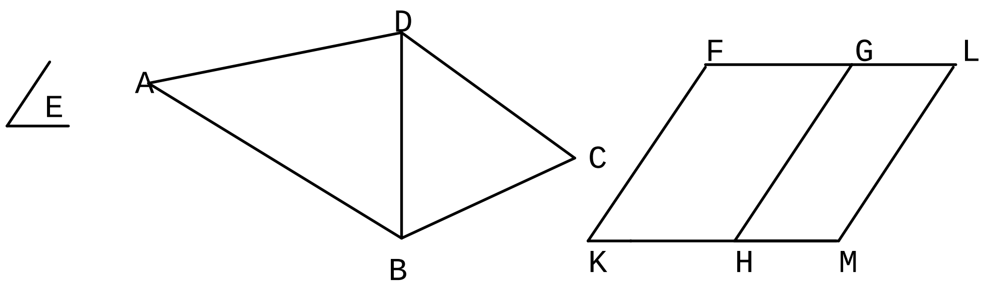
Therefore the parallelogram FECG is equal to the triangle ABC. And it has the angle CEF equal to the given angle D. Therefore the parallelogram FECG has been constructed equal to the given triangle ABC, in the angle CEF which is equal to D.

Q.E.F.

Book I, Proposition 45

To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.

Diagram used by Euclid



Let ABCD be the given rectilinear figure and E the given rectilineal angle ; thus it is required to construct, in the given angle E, a parallelogram equal to the rectilineal figure ABCD.

Construction Procedure

Let DB be joined, and let the parallelogram FH be constructed equal to triangle ABD, in the angle HKF which is equal to E;

[The above construction used **Book I, Proposition 42** (see previous page) to construct parallelogram *FH* equal to triangle *ABD* with angle *HKF* equal to angle *E*.]

let the parallelogram GM equal to triangle DBC be applied to straight line GH, in the angle GHM which is equal to E.

[The above construction used **Book I, Proposition 44** which states: *To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle*.]

Then, since the angle E is equal to each of the angles HKF, GHM, the angle HKF is also equal to the angle GHM.

[The above statement is based on **Common Notion 1** which states: *Things which are equal to the same thing are also equal to one another.*]

Let the angle KHG be added to each; therefore angles FKH, KHG are equal to angles KHG, GHM.

But the angles FKH, KHG are equal to two right angles;

[The above statement is based on **Book I, Proposition 29** which states: *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles*]

therefore the angles KHG, GHM are also equal to two right angles. Thus, with a straight line GH, and at the point H on it, two straight lines KH, HM not lying on the same side make the adjacent angles equal to two right angles; therefore KH is in a straight line with HM.

[The above statement is based on **Book I, Proposition 14** which states: *If with any straight line, and a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.*]

And, since the straight line HG falls upon the parallels KM, FG, the alternate angles MHG, HGF are equal to one another.

[The above statement is based on **Book I, Proposition 29** (previously stated)]

Let the angle HGL be added to each; therefore the angles MHG, HGL are equal to HGF, HGL.

[The above statement is based on **Common Notion 2** which states: *If equals are added to equals, the wholes are equal.*]

But the MHG, HGL are equal to two right angles;

[The above statement is based on **Book I, Proposition 29** (previously stated)]

therefore the angles HGF, HGL are also equal to two right angles.

[The above statement is based on **Common Notion 1** (previously stated)]

Therefore FG is in a straight line with GL.

[The above statement is based on **Book I, Proposition 14** (previously stated)]

And, since FK is equal and parallel to HG, and HG to ML also.

[The above statement is based on **Book I, Proposition 34** (previously stated)]

KF is also equal and parallel to ML.

[The above statement is based on **Common Notion 1** (previously stated) and **Book I, Proposition 30** which states: *Straight lines parallel to the same straight line are also parallel to one another.*]

and the straight lines KM, FL join them (at their extremities); therefore KM, FL are also, equal and parallel.

[The above statement is based on **Book I, Proposition 33** which states: *The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel.*]

Therefore KFLM is a parallelogram. And, since the triangle ABD is equal to the parallelogram FH, and DBC to GM, the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM.

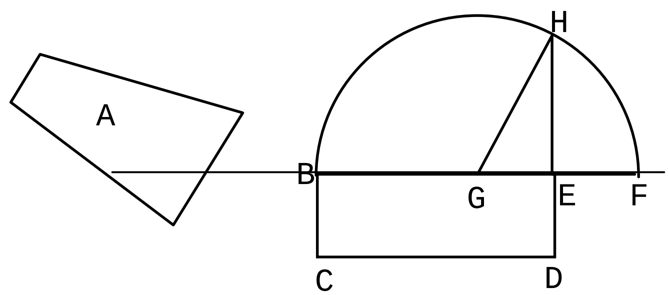
Therefore the parallelogram KFLM has been constructed equal to the given rectilinear figure ABCD, in the angle FKM which is equal to the given angle E.

Q.E.F.

Book II, Proposition 14

To construct a square equal to a given rectilinear figure.

Diagram used by Euclid



Let A be the given rectilinear figure; thus it is required to construct a square equal to the rectilinear figure A.

Construction Procedure

For let there be constructed the rectangular parallelogram BD equal to the rectilineal figure A.

[The above statement is based on **Book I, Proposition 45** which was previously described.]

Then, If BE is equal to ED, that which was enjoined will have been done [would have been done]; *for a square BD has been constructed equal to the rectilineal figure A.*

But, if not, one of the straight lines BE, ED is greater. Let BE be greater, and let it be produced to F; let EF be made equal to ED, and let BF be bisected at G. With centre G and distance one of the straight lines GB, GF let the semicircle BHF be described; let DE be produced to H, and let GH be joined.

Then, since the straight line BF has been cut into equal segment at G, and into unequal segments at E, the rectangle contained by BE, EF together with the square on EG is equal to the square on GF.

[The above statement is based on **Book II, Proposition 5** which states: *If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.* See below for more on this proposition.]

But GF is equal to GH; therefore the rectangle BE, EF together with the square on GE is equal to the square on GH.

But the squares on HE, EG are equal to the square on GH;

[The above statement is based on **Book I, Proposition 47** which states: *In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*]

therefore the rectangle BE, EF together with the square on GE is equal to the squares on HE, EG. Let the square on GE be subtracted from each;

therefore the rectangle contained by BE, EF which remains is equal to the square on EH.

but the rectangle BE, EF is BD, for EF is equal to ED; therefore the parallelogram BD is equal to the square on HE. And BD is equal to the rectilineal figure A.

Therefore the rectilineal figure A is also equal to the square which can be described on EH.

Therefore a square, namely that which can be described on EH, has been constructed equal to the given rectilinear figure on A.

Q.E.D.

Geometric Algebra and Quadratic Equations

Teacher Notes

Level: This activity is suitable in an introductory high school geometry class, assuming that the students have previous studied quadratic equations.

Materials: Paper and pencil.

Time Frame: At least two class periods.

Objective: To relate the solution of quadratic equations to the geometry upon which those equations are originally based.

When to Use: This can be studied at any time after the tangram lesson. In particular, it allows you to review the study of quadratic equations, material which students probably had in the previous year. It also allows you to connect the study of quadratic equations to geometry.

How to Use: It may be worthwhile to have the students work through the second half of this lesson (the part dealing with Proposition II-6) in small groups, once you have taught the first half of the lesson (the part dealing with II-5). During the teaching of that first part, you need to go through the translation from words to geometry and from words to algebra very carefully. It may be worthwhile to prepare copies of the diagrams so that students can cut out the various squares and rectangles and manipulate them as they try to prove the proposition. In any case, it is important to emphasize that a quadratic equation is an equation about squares and rectangles – about geometric objects. It is not just an algebraic statement. The derivation here helps make that clear.

The students may need help deriving the general form of the quadratic formula, particularly in simplifying algebraic expressions under radical signs. But this may well be a good opportunity to review that idea. Then the students can practice the use of the quadratic formula with numerous examples which can be found in any algebra text.

Answers to Exercises:

Set 1:

1. 3, 5
2. 1, 3
3. 3, 8
4. $5 \pm \sqrt{20}$
5. $9/2 \pm \sqrt{53}/2$
6. 2, 4
7. No real solutions. This occurs when $(b/2)^2 - c < 0$.

Set 2:

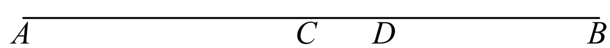
1. 3, –9
2. 1, –9
3. 3, –8
4. $-3 \pm \sqrt{27}$
5. $-2 \pm \sqrt{7}$

Geometric Algebra and Quadratic Equations
Student Pages

Proposition II-5 of Euclid's *Elements* appears to be very complicated:

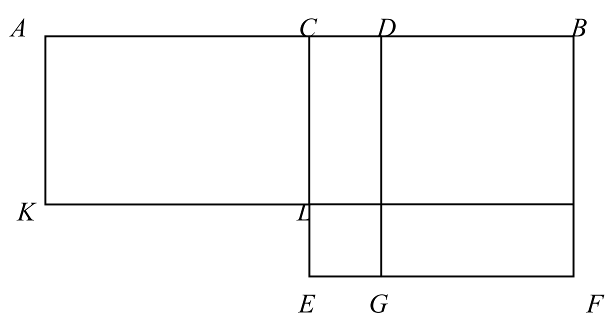
If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

Let us try to translate this result into understandable terms. Consider the following diagram in which C is the midpoint of the line segment AB and where D is some point between C and B.



Then the straight line AB has been cut into equal segments at C and into unequal segments at D. The theorem then deals with three geometrical quantities. First, we have the "rectangle contained by the unequal segments of the whole," that is, the rectangle whose sides are AD and DB. Second, there is the "square on the straight line between the points of section," that is, the square on the segment CD, since C and D are the two "points of section." Finally, we have the "square on the half," or the square on the segment BC.

The theorem then says that the sum of the areas of the rectangle and the first square (that is, the rectangle whose sides are AD and DB and the square whose side is CD) is equal to the area of the second square (the square whose side is BC). To demonstrate this, we need to add the squares and rectangles to our line:



To make the rectangle whose sides are AD and DB, we draw AK so that it is equal to DB. Then the rectangle we want is the rectangle $ADHK$. To draw the square whose side is CD, we extend DH to G, where $HG = CD$. Since $LH = CD$, the square we want is the square $LHEG$. Finally, since $BC = CD + DB$, we also have $CE = CL + LE = AK + HG = DB + CD = BC$. Therefore, the square whose side is BC is the square $BCEF$.

To check that Euclid is correct, that the sum of the areas of the rectangle and first square is equal to the area of the second square, it is easiest to number the five rectangles in the diagram. That is, number the three rectangles across the top 1, 2, and 3, and the two rectangles at the bottom, 4 and 5. Then the rectangle whose sides are AD and DB consists of the regions numbered 1 and 2; the square whose side is CD is region 4, and the square whose side is BC consists of the four regions 2, 3, 4, and 5. Thus, to prove the result, we must show that the regions numbered 1, 2, and 4 are together equal to the regions 2, 3, 4, and 5. Since regions 2 and 4 are common, we must show that region 1 is equal to regions 3 and 5. Region 1 is a rectangle with sides AC and AK. The two regions 3 and 5 together form a rectangle with sides DB and BF. But we know that $AK = DB$. Also, AC is half of AB and $BF = BC$, where BC is also half of AB. It follows that the two rectangles are in fact equal and Euclid's result has been proved.

So what is the point of this complicated result? First of all, the proof of it has enabled us to complete the proof of Book II, Proposition 14 above. But more importantly, this result is critical in the algebraic process of solving quadratic equations. To demonstrate this, we will translate Proposition II-5 into algebra. Let us suppose that the length AB is b, that $AC = CB = b/2$, and that $DB = x$. Then we also have $AK = x$, $AD = b - x$, and $CD = b/2 - x$. The area of rectangle $ADHK$ is then $(b - x)x$, the area of square $LHEG$ is $(b/2 - x)^2$, and the area of square $BCEF$ is equal to $(b/2)^2$. The theorem therefore says, in algebra, that

$$(b-x)x+\left(\frac{b}{2}-x\right)^2=\left(\frac{b}{2}\right)^2.$$

As an exercise, you can multiply out all the terms and check that this equation is correct algebraically. Of course, it has to be, because we already checked it geometrically.

So let us see how to use this equation to solve a quadratic equation. We try the quadratic equation $x^2 + 21 = 10x$. In order to use Proposition II-5, we rewrite this as $10x - x^2 = 21$, or, factoring the left side, as $(10 - x)x = 21$. Comparing this to our formula above, we note that $b = 10$ and therefore we have

$$21 + (5 - x)^2 = 5^2.$$

We can rewrite this as $(5 - x)^2 = 25 - 21$, or as $(5 - x)^2 = 4$, or finally, by taking square roots, as $5 - x = 2$. It follows that $x = 3$ is the solution to our quadratic equation.

Note further that when we took square roots, we only took the positive square root of 4. But 4 has a negative square root, namely, -2. So we can get a second solution to the equation by solving $5 - x = -2$, which gives us $x = 7$. Our equation thus has two solutions, $x = 3$ and $x = 7$.

We can now apply this method to the general case $x^2 + c = bx$. As before, we first rewrite this as $bx - x^2 = c$, or, factoring the left side, as $(b - x)x = c$. We can now plug the value of $(b - x)x$ into the formula of Proposition II-5. We get

$$c+\left(\frac{b}{2}-x\right)^2=\left(\frac{b}{2}\right)^2.$$

We can rewrite this as

$$\left(\frac{b}{2}-x\right)^2=\left(\frac{b}{2}\right)^2-c.$$

Taking square roots – and we might as well take both the positive and negative ones – we get

$$\frac{b}{2}-x=\pm\sqrt{\left(\frac{b}{2}\right)^2-c}$$

or, finally,

$$x=\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c}.$$

This formula is a version of the quadratic formula, which you may have already learned.

Exercise Set 1:

Use the formula just derived to solve the following quadratic equations:

1. $x^2 + 15 = 8x$
2. $x^2 + 3 = 4x$
3. $x^2 + 24 = 11x$
4. $x^2 + 5 = 10x$ (Note that the answers here are irrational numbers; just express them in terms of radicals.)
5. $x^2 + 7 = 9x$
6. $x^2 - 6x + 8 = 0$
7. $x^2 + 6 = 2x$ (What happens in this case? Can you generalize this?)

Proposition II-5 was in fact used, a thousand years after Euclid wrote it down, to justify the Islamic method of solving one type of quadratic equation. The other types were solved by the use of Proposition II-6. We therefore state this proposition and ask you to fill in the details on its geometric meaning and its use in the solution of quadratic equations.

If a straight line is bisected and a straight line is added to it, the rectangle contained by the whole with the added straight line and the added straight line together with the square of the half is equal to the square on the straight line made up of the half and the added straight line.

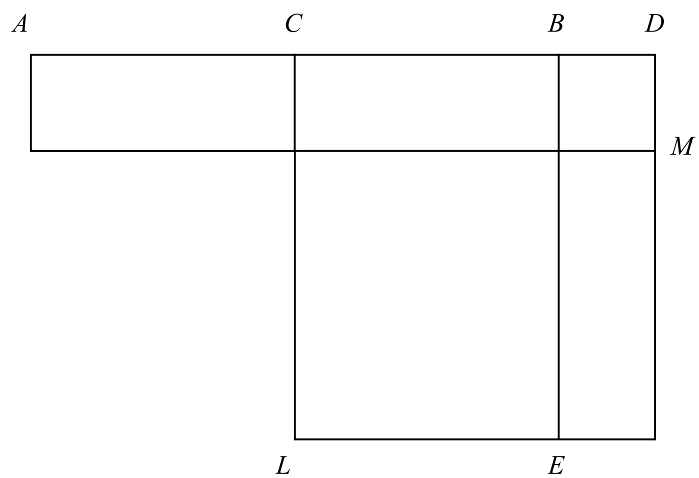
We draw a straight line to get you started on this:

In this case, the original straight line is *AB*. That line is bisected at *C*. Then a new straight line *BD* is added at the end. We again are dealing with three geometrical quantities in the statement of the proposition. First, we have the rectangle "contained by the whole with the added straight line" and "the added straight line." So this is a rectangle whose two sides are what?

Next, we have the "square on the half." This is the square whose side is which segment?

Finally, we have the "square on the straight line made up of the half and the added straight line." This is the square on which line segment?

The proposition states that the sum of the areas of the first two of these regions equals the area of the last one. To see this, we use the following diagram:



In this diagram, we have $LE = BC$ and $DM = BD$. Then the rectangle in the theorem is the rectangle with which four corners? The first square, the "square on the half," is which square? Finally, the last square is which square?

To prove the theorem, we again number the regions 1 through 5. The sum of the areas of the first two regions stated in the proposition is the sum of which of the five regions? The last square consists of which of the five regions? Now, prove the proposition.

As before, we can turn this proposition into an algebraic result. We set *AB* equal to *b*, *AC* and *BC* equal to $b/2$, and *DB* equal to *x*. Then this proposition becomes the algebraic formula

$$(b+x)x+\left(\frac{b}{2}\right)^2=\left(\frac{b}{2}+x\right)^2.$$

We can now solve another type of quadratic equation. In fact, use this result to solve $x^2 + 10x = 39$, an equation which we can rewrite as $(10 + x)x = 39$. Taking $b = 10$, substitute that value for *b* and the value 39 for $(10 + x)x$ into the algebraic formula for proposition II-6. Solve the resulting equation for x by taking square roots. Use both the positive and negative square roots. The two solutions are then ?

Now, let us rewrite your method into a formula for solving the quadratic equation $x^2 + bx = c$. We first rewrite this in the form $(b + x)x = c$ and then substitute into the formula from II-6. We get

$$c + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2} + x\right)^2 .$$

We take square roots to get

$$\frac{b}{2} + x = \pm\sqrt{\left(\frac{b}{2}\right)^2 + c}$$

or, finally,

$$x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + c} ,$$

another version of the quadratic formula.

Exercise Set 2:

Use this new version of the quadratic formula to solve the following quadratic equations:

1. $x^2 + 4x = 45$
2. $x^2 + 8x = 9$
3. $x^2 + 5x = 24$
4. $x^2 + 6x = 18$
5. $x^2 + 4x = 3$

How could you apply either of the two versions of the quadratic formula which we have considered to solve the equation $x^2 = 5x + 14$?

Recall that there were cases of the equation $x^2 + c = bx$ which had no solutions. Do all cases of the equation $x^2 + bx = c$ have solutions, where we are always assuming that b and c are positive numbers?

Derive the general quadratic formula by starting with either of the two versions we have and applying them to the equation $x^2 + bx + c = 0$, the standard quadratic equation, where we can now allow the coefficients b and c to be negative. Also, generalize this further to the most general form of the quadratic equation: $ax^2 + bx + c = 0$.

Figurate Numbers
Teacher Notes

Level: This activity is suitable for middle and high school students.

Materials: Paper and Pencil

Time Frame: 20 minutes

Objective: To demonstrate an ancient connection between arithmetic and geometry.

When to Use: Almost anywhere before the Pythagorean Project, which follows.

How to Use: This brief activity demonstrates some of the connections between geometry and numbers. Students can do the activity either individually or in groups. The activity leads the students through the following chain of observations:

1. The nth triangular number is the sum of the numbers from 1 to n. The sequence goes 1, 3, 6, 10, 15, 21, 28, ... At this point, they won't have a formula for the nth triangular number, just a procedure for finding it.

2. The sum of two consecutive triangular numbers is a square. In symbols, $T_{n-1} + T_n = n^2$.

3. Oblong numbers are the areas of rectangles that are one unit longer than they are wide. The nth oblong number is equal to $n(n + 1)$. The sequence goes 2, 6, 12, 20, 30, 42, 56, ...

4. The double of the nth triangular number is the nth oblong number.

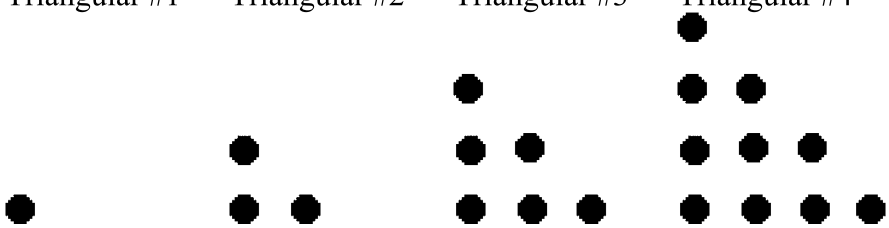
5. Since they have discovered a formula for the nth oblong number, this leads to a formula for the nth triangular number: $T_n = n(n + 1)/2$.

Background: Pythagoras and the Pythagoreans were interested in the figurate numbers (triangular, square, oblong, pentagonal, and so on), as part of their general interest in numbers. The Pythagoreans knew the famous theorem which bears Pythagoras' name, although it was certainly well-known a thousand years before his time. They also explained the musical scale with simple whole number ratios. Since the Greeks at that time considered numbers to be lengths, they had found the connection between number and the right angle and number and musical tones. It should be no surprise that they then declared "All is number." They believed that all things should be associated with their correct counting number. In geometry the point was paired with 1, the line (which was finite, in other words a segment with two endpoints) 2, the plane with 3 (representing the 3 vertices of a triangle), and space with 4 (pebbles stacked to form a tetrahedral shaped pile). Now the various regular polygons needed numbers and they chose to arrange pebbles in regular triangular arrays to represent successive sizes of equilateral triangles {1,3, 6, …}, squares {1,4,9,…}, regular pentagons, regular hexagons, and so on. For a more complete tour of Pythagorean number mysticism, see Bell [2].

Figurate Numbers
Student Pages

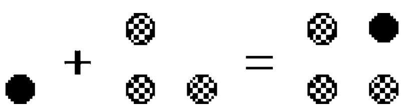
The numbers which we count with are the basic building blocks of all of mathematics. For this project we will examine numbers that are in the form of simple geometrical shapes such as triangles and rectangles. First, we will look at the sequence of triangular numbers, numbers which form a special triangular shape. The nth triangular number is created by constructing a triangle in the form $1 + 2 + 3 + . . . + n$. Examine the first four triangular numbers in the chart below. Then complete the chart for the fifth and sixth triangular numbers, for $n = 5$ and $n = 6$.

Triangular #1	Triangular #2	Triangular #3	Triangular #4	Triangular #5	Triangular #6



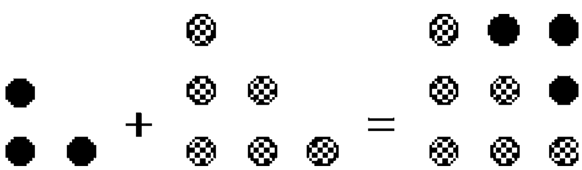
$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
1=1	1+2=3	1+2+3=6	1+2+3+4=10		

Let's look at what happens when we add the first two triangular numbers, $n = 1$ and $n = 2$.



$1 + 3 = 4 = 2^2$

Now add the next two consecutive triangular numbers, $n = 2$ and $n = 3$.



$3 + 6 = 9 = 3^2$

In mathematics, a conjecture is an educated guess. Make a conjecture about the sum of two consecutive triangular numbers.

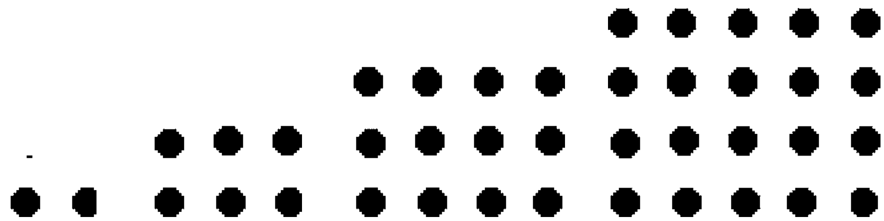
Conjecture

Use pictures to show that your conjecture is correct for three more pairs of consecutive triangular numbers.

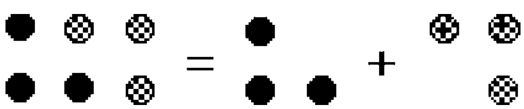
Use your conjecture to find the sum of the tenth triangular number and the eleventh triangular number.

Use your conjecture to find the sum of the 100^{th} triangular number and the 101^{st} triangular number.

Oblong numbers might also be called rectangular numbers, because they are in the shape of a special rectangle. Each rectangle has exactly one more column than it has rows. The first oblong number has 1 row and 2 columns. The next oblong number has exactly 2 rows and 3 columns. Examine the first four oblong numbers in the chart below. Then complete the chart for the fifth and sixth oblong numbers, for $n = 5$ and $n = 6$.

Oblong #1	Oblong #2	Oblong #3	Oblong #4	Oblong #5	Oblong #6
					
$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
1●2=2	2●3=6	3●4=12	4●5=20		

Let's look carefully a picture of the second oblong number and the a relationship between it and triangular numbers.



We see that the second oblong number is exactly twice the second triangular number.

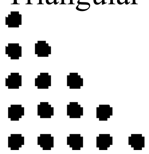
Try to find a relationship between each oblong number in the chart and the triangular number associated with it. Then make a conjecture about the relationship of the nth oblong number and the nth triangular number.

Conjecture

Use your conjecture to find a formula for the sum of the first n positive integers. Then, use your result to find a formula for the sum of the first n positive integers. Finally, determine the relationship between triangular numbers and the sums of consecutive positive integers?

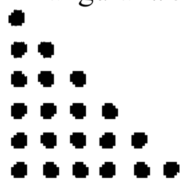
Figurate Numbers
Solutions

Triangular #5



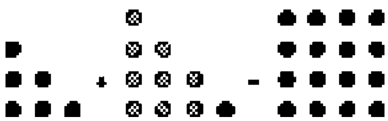
$n = 5$
1+2+3+4+5=15

Triangular #6

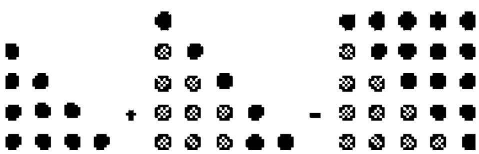


$n = 6$
1+2+3+4+5+6=21

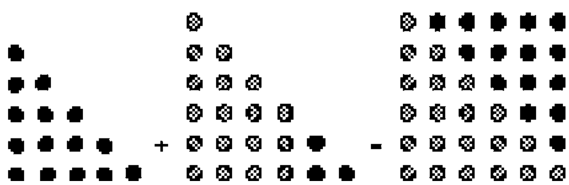
Conjecture: The sum of two consecutive triangular numbers is a square.



Triangular #3 + Triangular #4 = 16 = 4^2



Triangular #4 + Triangular #5 = 25 = 5^2

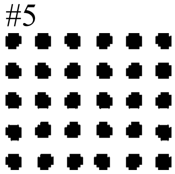


Triangular #5 + Triangular #6 = 36 = 6^2

Triangular #10 + Triangular #11 = 11^2 = 121

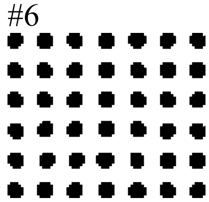
Triangular #100 + Triangular #101 = 101^2 = 10201

Oblong
#5



$n = 5$
5•6=30

Oblong
#6



$n = 6$
6•7=42

The 3rd oblong number is exactly two times the 3rd triangular number.
The 4th oblong number is exactly two times the 4th triangular number.
The 5th oblong number is exactly two times the 5th triangular number.

The 6th oblong number is exactly two times the 6th triangular number.

Conjecture	The *n*th oblong number is exactly two times the *n*th triangular number.

The *n*th triangular number T_n is given by the formula

$$T_n = \frac{n(n+1)}{2}.$$

Thus, the *n*th triangular number is half of the *n*th oblong number. Also, the sum of the first n positive integers is equal to the *n*th triangular number and therefore to T_n.

Heron's Formula and the Euler Line
Teacher Notes

Level: This activity is suitable for a high school geometry class.

Materials: The exercises in this activity require the use of Geometer's Sketchpad. If the class is not familiar with that program, the alternative is to have students perform the requested constructions and measurements by hand. But Geometer's Sketchpad allows repetition on a scale that would not be feasible otherwise. Nevertheless, the proofs themselves are purely geometric and do not require any software.

Time Frame: This activity works best if the students have time to accomplish the various explorations in the exercises before the proofs are discussed. Assuming that these explorations must be done in class time, this activity will take a minimum of three class periods.

Objective: To make a conjecture and then prove certain results in Euclidean geometry that were not known to Euclid.

When to Use: This activity can be used in a geometry class which has already discussed the constructions of the perpendicular bisector of a line, the perpendicular to a line from a point, and the bisector of an angle. The class needs also to have had experience in using the triangle congruence theorems and the basic results on similar triangles. Given that the ideas here are not present in all standard geometry texts – and if they are, it is often without proof – this activity should be considered a supplemental activity, but one which can generate a lot of interest.

How to Use: This activity is set up for presentation by the teacher. The first part of the activity is the statement and proof of Heron's theorem. The proof is not difficult, but it needs a careful presentation. Since the proof involves finding the center of the inscribed circle, you might want to ask the students actually to construct this point for a given triangle. There are no exercises in applying Heron's formula, but these can easily be supplied. You can make up any triangles you want to give the students practice in using it. The first exercise after the presentation of the theorem requires the use of Heron's theorem. Students should make a drawing of the situation. The drawing will be just like Figure 1, where here we are given that $x = 6$, $y = 8$, and $r = 4$. We need to find z in order to answer the question. But we know that the area of the triangle is equal to $rs = 4(6+8+z) = 4(z+14)$. By Heron's formula, the area is also equal to $\sqrt{(z+14)6\cdot 8\cdot z} = \sqrt{48z(z+14)} = 4\sqrt{3z(z+14)}$. Since these two expressions must be equal, we have $4(z+14) = 4\sqrt{3z(z+14)}$, or $z+14 = \sqrt{3z(z+14)}$. If we square both sides of this equation, we get $(z+14)^2 = 3z(z+14)$, which reduces to $z + 14 = 3z$. The solution is therefore $z = 7$. Therefore, the two remaining sides of the triangle have lengths 13 and 15.

The trigonometric proof of Heron's theorem should only be assigned if the students are familiar with trigonometry and, in particular, with the tangent function. Part a of that problem comes from the definition of the tangent, while the sum formula for the tangent can be derived from the sum formulas for the sine and the cosine with some manipulation. The remainder of the proof just amounts to using the definition of the tangent function, keeping in mind that the sum of the three half angles is 90°, and some algebraic manipulation. If the students can handle this theorem, it might be worthwhile at this point to discuss the differences between the geometric and the trigonometric proofs. Which one is easier to understand? Which one makes the theorem more understandable? Neither of these proofs is the one given originally by Heron. His proof is somewhat more difficult than Euler's in that it doesn't use any algebraic manipulation at all. If you are interested, you can find a discussion of that proof in [Dunham, *Journeys*, chapter 5].

If Geometer's Sketchpad is available, students should certainly use it to do exercises 3 and 4 in this exercise set. In particular, they should do exercise 4 before you give the next proof. This proof assumes that the intersection of the two perpendicular bisectors is inside the triangle. But you can easily do examples where it is on one of the sides or is outside the triangle. In the first of these cases, as in exercise 1 in the next set, the triangle is a right triangle. In this case, the result can be rewritten to say that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices. You should certainly ask students to model a proof of the result in the second case after the one already presented.

Again, you should have the students use Geometer's Sketchpad to do the following two exercises. In particular, for exercise 3 of this set, they should be able to discover the fact that the centroid of a triangle, the intersection of the medians, is two-thirds of the way from any vertex to the midpoint of the opposite side. You can now do the proofs of the two results. If the students are used to doing proofs in two-column form, it may be worthwhile to have them rewrite one or both of these in that format.

The final exercise in this activity should again be done before the theorem following it is proved. It is a very surprising result, one which although it could have been proved by the Greeks, was not in fact discovered until the middle of the eighteenth century. As noted in the text, Euler's proof of this theorem used analytic geometry. You can see the proof in [Dunham, *Euler*, 133–141]. It would be very tedious to go through this proof, even if your class was expert in analytic geometry. On the other hand, if you let the given triangle be a right triangle, and put its vertex with the right angle at the origin, it is not difficult to prove Euler's result in this case by analytic geometry. You could ask students to do this for a special project.

Background: This activity deals with some interesting concepts of geometry that are often not explored in a standard geometry course. Most of the ideas here come from Greek times, but the final part, the Euler line, is due to Euler in the eighteenth century. You should discuss with your class the fact that not all geometric facts are known. New ones are being discovered and proved. And the explorations possible through such software as Geometer's Sketchpad enable even out students to come up with conjectures which may never have been found before.

Sources: For more details on Euler, the book by Dunham already referred to is excellent. For more geometric results which can be explored with your students, see [Heilbron]. This latter book has lots of interesting geometric results, along with the history of them. Many of these are probably not in your textbook. There is little biographical information on Heron, and it was only in recent times that historians were able to determine that he lived in the first century of our era. More information on his work can be found in [Heath].

Heron's Formula and the Euler Line
Student Pages

You have studied a good deal of geometry, including the procedures for construction of perpendiculars to lines and bisectors of angles. We want to look at these constructions in interesting ways that perhaps you have not seen. Some of what we will do was known in ancient Greece and is recorded in the *Elements* of Euclid (c. 300 B.C.E.), the most important mathematical work of ancient times. But even Euclid could not consider everything. And so we will also look at some geometry that was first written down by Heron, a mathematician from Alexandria who lived in the first century C.E. Finally, we will look at a piece of plane geometry that, although it contains nothing that the Greeks would not have understood, was not discovered until the eighteenth century by Leonhard Euler (1707–1783), the most prolific mathematician who ever lived.

We begin with a triangle. The side-side-side triangle congruence theorem tells us that there can be essentially only one triangle with sides equal to three given lengths, as long as for these lengths, the sum of any two is greater than the third. Thus, since the sides determine the triangle, that triangle has a definite area. It is useful to be able to calculate that area, just given the lengths of the sides. It was Heron of Alexandria who first wrote down a procedure for doing this, although it is probable that the result was due to Archimedes some three hundred years earlier.

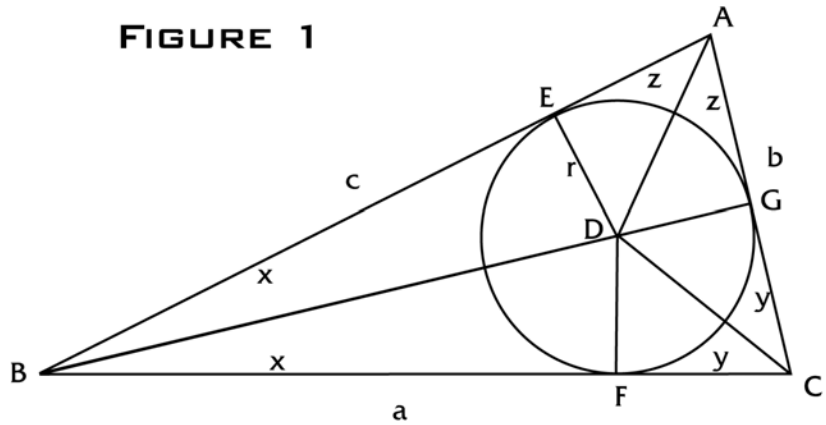
Heron wrote several works dealing with the applications of mathematics. Unlike Euclid, he was very interested in calculating – and so he developed rules, some exact and some approximate, for finding areas and volumes. He is best known, however, for his exact formula for the area of a triangle. If the lengths of the three sides are a, b, and c, and if the semi-perimeter s of the triangle is defined to be $s = \frac{1}{2}(a+b+c)$, then Heron's formula says that the area of the triangle is given by

$$\text{Area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

Now this is a very strange formula to come to us from the Greek world. The formula is strange because it involves the product of four lengths. This kind of product would not, in general, make sense to the Greeks because it is not geometrical. There are only three dimensions, so a product of two lengths could represent an area, a product of three lengths could represent a volume, but there is no geometrical entity which could be represented by the product of four lengths. Nevertheless, Heron wrote down a proof of this formula, one which is quite long, but extraordinarily clever. Since Heron's time, other people have given simpler proofs. We will look here at a proof due to Leonhard Euler and then outline an even simpler trigonometric proof.

Euler's proof depends first on the construction of a circle inscribed in the given triangle ABC (Figure 1). This construction is in Book IV, proposition 4 of Euclid's *Elements*. We construct the bisectors of the angles B and C and let their intersection point be D. Now drop perpendiculars from DE, DF, and DG to the three sides AB, BC, and AC, respectively, of the

FIGURE 1



triangle. Now consider triangles *EDB* and *FDB*. They are each right triangles; they have a common side *BD*; and they have equal angles *EBD* and *FBD*. So the triangles are congruent. (Why?) It follows that *DE* = *DF*. Similarly, *DG* = *DF*. Therefore, *DE* = *DF* = *DG* and the circle drawn with center *D* and radius *DE* will pass through each of the points *E*, *F*, and *G*. And since *DE*, *DF*, and *DG* are perpendicular to the three sides, respectively, of the triangle, those sides are tangent to the circle at the points *E*, *F*, and *G*. Thus, we have inscribed a circle *EFG* in the triangle *ABC*.

For future reference, we note that *AD* is the bisector of angle *A* (because triangles *AED* and *AGD* are congruent). Thus the three angle bisectors of a triangle intersect at a point *D*. Since this point is the center of the inscribed circle, it is called the **incenter** of the triangle.

We also want to label some of the lengths involved in the diagram above to use in the proof of Heron's theorem. We will let r be the radius *DE* of the inscribed circle. Let $a = BC$, $b = AC$, and $c = AB$. Also, let $x = BF = BE$, $y = FC = CG$, and $z = AE = AG$. Then the perimeter of the triangle is $a+b+c = 2x+2y+2z$, so the semi-perimeter s is given by $s = x + y + z$. Then $s-a = s-(x+y) = z$. Similarly $s-b = x$ and $s-c = y$..

The area of triangle *ABC* is equal to the sums of the areas of triangles *BDC*, *ADB*, and *ADC*. The area of triangle *BDC* is $\frac{1}{2}ra = \frac{1}{2}r(x+y)$. Similarly, the area of triangle *ADB* is $\frac{1}{2}r(x+z)$ and the area of triangle *ADC* is $\frac{1}{2}r(y+z)$. It follows that the area of triangle *ABC* is $\frac{1}{2}r(x+y+x+z+y+z) = \frac{1}{2}r(2s) = rs$. In words, we have found that the area of a triangle is the product of the radius of its inscribed circle by its semi-perimeter.

We are now ready to present Euler's proof of Heron's formula. We begin by inscribing a circle in triangle *ABC* centered at *D*, according to our previous construction (Figure 2). We connect *DA*, *DB*, *DC* and recall that those lines bisect angles *A*, *B*, and *C* respectively. Let us call these half-angles $\alpha/2, \beta/2$, and $\gamma/2$. We next drop perpendiculars from *D* to each of the three sides and label the segments of the sides as above. Again, as before, the line segments *DE*, *DF*, and *DG* are all equal. Now, we drop a perpendicular from *B* to *CD* extended, meeting that line at *V*. Finally, we extend *BV* to the point *N* where the line meets the extension of line *FD*.

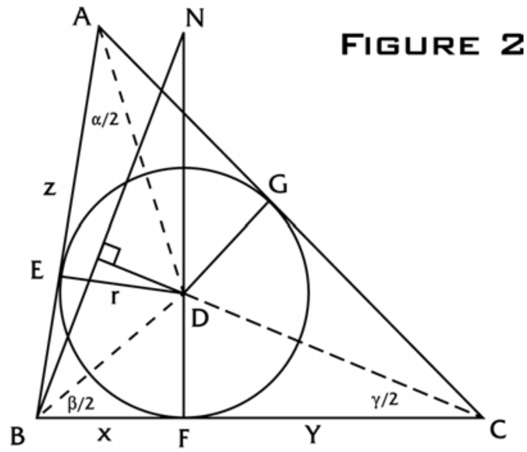


FIGURE 2

Now angle BDV is an exterior angle to triangle BDC. Therefore, $\angle BDV = \gamma/2 + \beta/2$. (Why?) But $\alpha/2 + \beta/2 + \gamma/2 = 90^\circ$. So $\angle BDV = 90^\circ - \alpha/2 = \angle ADE$. Therefore, right triangles VDB and ADE are similar. It follows that $BV/VD = AE/DE = z/r$.

There are lots of other similar triangles in this diagram that we need. For example, ΔNFB is similar to ΔCVB, and ΔNFB is similar to ΔNDV. (Why?) Thus ΔNDV is similar to ΔCVB, and therefore $VD/DN = BV/BC$. Rearranging this proportion gives us $BV/VD = BC/DN$. But the left side of this proportion is equal to z/r. So we get

$$\frac{z}{r} = \frac{BC}{DN} = \frac{x+y}{FN-r}.$$

To determine FN, we note that since triangles CVB and CDF are similar (why?), so are triangles NFB and CDF. It follows that $FN/BF = CF/FD$. So $FN/x = y/r$ and $FN = xy/r$. Substituting this value for FN into the previous equation gives us

$$\frac{z}{r} = \frac{x+y}{xy/r - r}.$$

This equation simplifies to

$$\frac{xyz}{r} - zr = xr + yr \text{ or } xyz = (x+y+z)r^2 \text{ or } xyz = r^2 s.$$

We can now calculate the area of triangle ABC, which we recall is equal to rs. We have

$$\text{Area}(\Delta ABC) = rs = \sqrt{r^2 s^2} = \sqrt{s(r^2 s)} = \sqrt{sxyz} = \sqrt{szxy} = \sqrt{s(s-a)(s-b)(s-c)},$$

as claimed.

Exercises:

1. A circle of radius 4 units is inscribed in a triangle. A point of tangency of the circle with a side of the triangle divides the side into lengths of 6 and 8 units. What are the lengths of the remaining two sides of the triangle?

2. Here is a trigonometric proof of Heron's formula:

   a. Show that for any angle δ, we have $\tan\delta = \dfrac{1}{\tan(90-\delta)}$ or $\tan\delta\,\tan(90-\delta) = 1$.

   b. Show that for any two angles δ, ρ, we have

$$\tan(\delta+\rho) = \frac{\tan\delta + \tan\rho}{1-\tan\delta\,\tan\rho}.$$

   c. With the notation for the angles of triangle *ABC* as above, use a. and b. to show that

$$1 = \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}+\frac{\gamma}{2}\right) = \tan\left(\frac{\alpha}{2}\right)\frac{\tan(\beta/2)+\tan(\gamma/2)}{1-\tan(\beta/2)\tan(\gamma/2)}.$$

   d. Conclude from c. that

$$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)+\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\gamma}{2}\right)+\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right)=1.$$

   e. Using the notation in our diagram above, substitute into the equation of d. to get

$$\frac{r}{z}\frac{r}{x}+\frac{r}{z}\frac{r}{y}+\frac{r}{x}\frac{r}{y}=1 \text{ or } xyz = r^2(x+y+z) = r^2 s.$$

   f. Starting with the final equation from e., use the argument at the end of the original proof of the theorem to prove Heron's formula.

3. Use Geometer's Sketchpad to show that the three angle bisectors of a triangle meet at a point and that this point is equidistant from the three sides. Do this with several different triangles.

4. Use Geometer's Sketchpad to show that the three perpendicular bisectors of the sides of a triangle meet at a point and that this point is equidistant from the three vertices. Do this with several different triangles.

We showed in the course of our discussion above that the three angle bisectors of a triangle met at the point called the incenter, the center of the circle inscribed in the triangle. Euclid shows similarly in proposition 5 of Book IV of the *Elements* that the perpendicular bisectors of the three sides meet at a point called the **circumcenter**, the center of the circle which is circumscribed about the triangle.

Let the triangle be *ABC* and let *D* be the intersection of the perpendicular bisectors *DE* and *DF* of sides *AB* and *AC* (Figure 3).

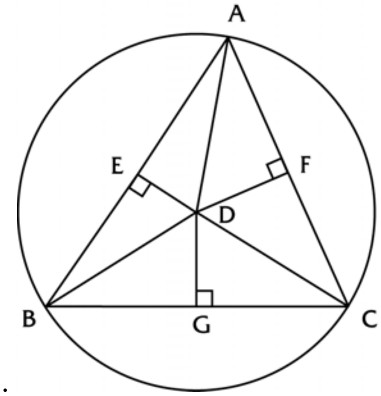


FIGURE 3

We will assume that *D* is inside the triangle. Connect *DA*, *DB*, and *DC*. Then triangles *ADE* and *BDE* are congruent as are triangles *ADF* and *CDF*. (Why?) Therefore, *AD* = *BD* = *CD*, so triangle *BDC* is isosceles. Thus the perpendicular from *D* to *BC* bisects *BC* (why?) and the three perpendicular bisectors meet at *D*. What Euclid notes here is that the circle with center *D* and radius *AD* passes through the three vertices of the triangle and therefore circumscribes the triangle.

Exercises:

1. In the case where the intersection *D* of the two perpendicular bisectors is on the line *BC*, show that that point is still the intersection of all three perpendicular bisectors and is equidistant from the three vertices. What kind of triangle is *ABC* in this case?

2. In the case where the intersection *D* of the two perpendicular bisectors is outside the triangle, show that *D* is still the intersection of all three perpendicular bisectors and is equidistant from the three vertices.

3. Use Geometer's Sketchpad to show that the three medians of a triangle (the lines from the vertices to the midpoints of the opposite sides) all meet in one point. Using the measuring capabilities of Sketchpad, determine the ratio of the distance from the vertex to that intersection point to the distance from the intersection point to the midpoint of the opposite side. Perform this measurement for all the medians of a several given triangles. Conjecture a theorem about the position of the intersection points of the medians.

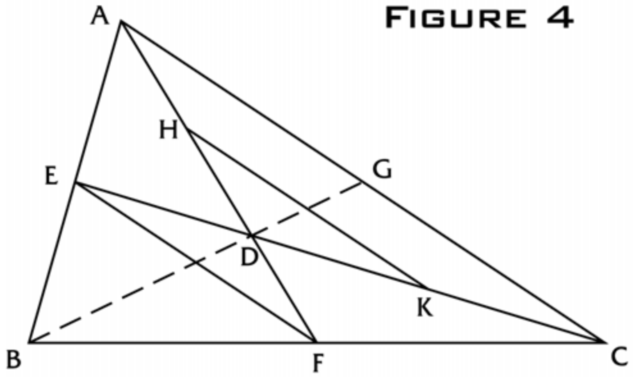
4. Use Geometer's Sketchpad to show that the three altitudes of a triangle (the lines from the vertices perpendicular to the opposite sides) all intersect at a point. Do this with several different triangles.

Another important point in a circle is the intersection of the three medians. This point is called the **centroid** of the circle, or the center of gravity. Recall that a median is a line from a vertex to the midpoint of the opposite side. The basic idea here is that the two triangles into which the median divides the given triangle have the same area and thus the same "weight."

Therefore, the center of gravity of any triangle is on each of the medians and thus is located at their intersection. It was Archimedes who first conducted a detailed study of centers of gravity and who first proved this result.

It remains to show that the three medians in fact intersect at a point. We will also show that this point divides each median into two segments the ratio of whose lengths is 2:1. So let *ABC* be the given triangle, *E* and *F* the midpoints of sides *AB* and *BC* respectively (Figure 4). Let *D* be the intersection of the two medians *AF* and *CE*. Let *H* be the midpoint of *AD*, *K* be the midpoint of *CD*, and connect *HK* and *EF*. Then the three line segments *EF*, *HK*, and *AC* are all parallel. (Why?)

FIGURE 4

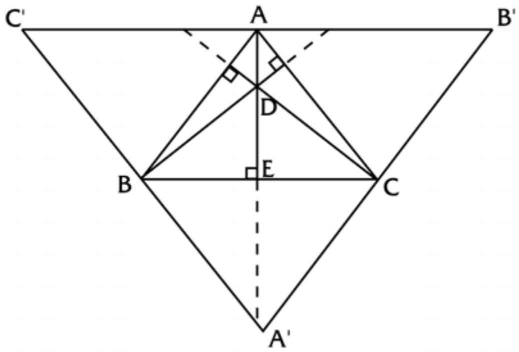


Because triangles *BEF* and *AC* are similar, and since *BE* is half of *BA*, it follows that *EF* is half of *AC*. Also, because triangles *DHK* and *DAC* are similar and *DH* is half of *DA*, it follows that *HK* is half of *AC*. Therefore *HK* = *EF* and triangles *DEF* and *DHK* are congruent. (Why?) Therefore *ED* = *DF* = *FC*, and the intersection *D* of the two medians is two-thirds of the distance from the vertex *C* to the midpoint *E* of the opposite side. The point *D* is also two-thirds of the distance from the vertex *A* to the midpoint *F* of its opposite side. Since *AF* and *CE* were any pair of medians, it follows that if we draw the third median *BG* of the triangle, it must intersect *AF* at a point two-thirds of the way from *A* to *F*, namely, at the point *D*. Our result is therefore proved.

We now show that the three altitudes of triangle *ABC* also intersect at one point. This point is called **orthocenter** of the triangle. The idea of this proof is to find another triangle *A'B'C'* such that the altitudes of the original triangle are the perpendicular bisectors of the sides of the new triangle. Then, since we already know that the perpendicular bisectors meet in a point, the same will be true of the altitudes. To make the altitudes into perpendicular bisectors in a new triangle, we need to have the points *A*, *B*, and *C* be the midpoints of those sides. So that is how we begin (Figure 5).

We first draw the line *C'AB'* through *A* parallel to *BC* and such that *C'A* = *AB'* = *BC*. Then connect *B'C* and *C'B* and extend these lines until they meet at *A'*. By construction, *A* is the midpoint of *B'C'*. We want to show that, in addition, *C* is the midpoint of *A'B'* and *B* is the midpoint of *A'C'*. First, note that triangles *AB'C* and *ABC* are congruent by side-angle-side, since $\angle B'AC = \angle BCA$.. It follows that *B'C'* = *AB* and therefore that *AB'CB* is a parallelogram. Similarly *C'ACB* is a parallelogram. So *B'C'* = *BA* and *C'B* = *AC*. Also, $\angle BCA' = \angle AB'C = \angle ABC$ (why?) and $\angle A'BC = \angle BC'A = \angle ACB$. Therefore, triangles *ABC* and $A'BC$ are congruent and $A'C = AB = B'C$. Thus C is the midpoint of $A'B'$ and, similarly, B is the midpoint of $A'C'$.

FIGURE 5



The perpendicular bisectors of the sides of the large triangle $A'B'C'$ pass through the points A, B, and C, since those points are the midpoints of the sides of that triangle. We know from above that those three perpendicular bisectors intersect at a point D. But if we extend AD to the point E on BC, then AE is perpendicular to BC, so it is the altitude to BC. Similarly, the extensions of BD and CD are the altitudes to the sides AC and AB respectively of the original triangle. It follows that these three altitudes also intersect at the point D as claimed.

Exercise:

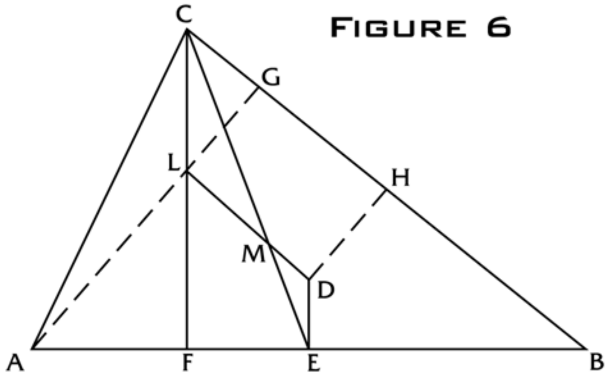
Draw an arbitrary triangle in Geometer's Sketchpad and find its centroid M, orthocenter L, and circumcenter D as before. Draw the line connecting L and D and show that M lies on this line. Then, using the measuring capabilities of the program, calculate the ratio of DM to ML. Repeat this exercise for several triangles. Conjecture a theorem on the relationship of DM to ML.

Although the various points that we have discussed above were all known to the Greeks, there is no evidence from Greek times that anyone noticed that the centroid, the orthocenter, and the circumcenter all lay on a single straight line. Apparently, this was first noticed by Euler, who wrote about it in 1767. Not only did he show that these points lay on a straight line, with the centroid between the other two points, but he also showed that the distance from the centroid to the orthocenter was always twice that of the distance from the centroid to the circumcenter. In other words, he proved

Theorem. *In any triangle, the orthocenter L, the circumcenter D, and the centroid M are collinear, with* $ML = 2MD$.

Euler's proof of the results about what is now called the ``Euler line'' used analytic geometry. Namely, he put the triangle on a coordinate system, calculated the coordinates of the three relevant points, and then calculated, using the distance formula, the relevant distances. His argument is very long and involved, but it is straightforward. One does not have to think hard about what step to take next; one only has to calculate and be very careful with the long and detailed calculations. But there is a geometric proof of Euler's result, a very clever proof which involves no calculation. It is this proof that we will consider here.

In the triangle *ABC*, let *CE* be the median from *C* to the midpoint *E* of side *AB* (Figure 6). Let *M* on *CE* be the centroid and *D* the circumcenter of the triangle. Since *D* is the circumcenter, we know that *DE* is perpendicular to *AB*. Connect *DM* and extend it to *L* so that $LM = 2MD$. We will show that *L* is the orthocenter, the intersection of the altitudes.



Join *C* and *L* and extend the line to its intersection *F* with *AB*. We want to show that *CF* is perpendicular to *AB*, or, what amounts to the same thing, that *CF* is parallel to *DE*. To do this, we show that triangles *CML* and *EMD* are similar. This is true because $\angle CML = \angle DME$, and the ratios $CM : ME$ and $LM : MD$ are both equal to 2:1. The similarity of the triangles then implies that $\angle CLM = \angle EDM$. These are alternate interior angles of the transversal *LD* across the two lines *CF* and *DE*. The equality of these angles then implies that the lines *CF* and *DE* are parallel.

So we know that *CF* is an altitude to *AB*. But how do we know that *L* is the intersection of the three altitudes? We simply repeat the construction by connecting *AL* and extending it to *G* on *BC*. We then show just as before that *AG* is an altitude by showing that it is parallel to *DF*, the perpendicular bisector of *BC*. Since *L* is the intersection of two altitudes, it must be the intersection of all three, and the theorem is proved.

It is interesting that, although one generally thinks that ``Euclidean'' geometry was totally worked out in Greek times, in the eighteenth and nineteenth centuries there was a revival of interest in the subject. Many mathematicians were able to prove new results about triangles, parallelograms, and circles, results which would have been accessible to the Greeks but for which there is no evidence that the Greeks discovered them. There is certainly no reason to believe that the supply of geometric theorems has been exhausted even as we enter the twenty-first century. With new exploration tools available, such as Geometer's Sketchpad, it is certain that new theorems will be discovered and proved, maybe even by members of this class.

The Pythagorean Project
Teacher Notes

Level: This activity is suitable for middle school or high school.

Materials: Paper, scissors, toga

Time Frame: 20 - 30 minutes

Objective: To understand the meaning and a proof of the Pythagorean Theorem.

When to Use: The most appropriate time for this is after discussing the statement of the theorem, but *before* a formal proof is covered. This is also an ideal time for the student to use geometry software to explore the theorem as well.

How to Use: Have the students work through the activity in small groups. To add a touch of drama to the classroom, use a sheet to dress up in a toga and be "Pythagoras for a Day." Hand each student a **PYTHAGOREAN BROTHERHOOD MEMBERSHIP CERTIFICATE** as they enter the room. The activities in this section help the student to visualize why the Pythagorean Theorem is true.

The Pythagorean Project
Student Pages

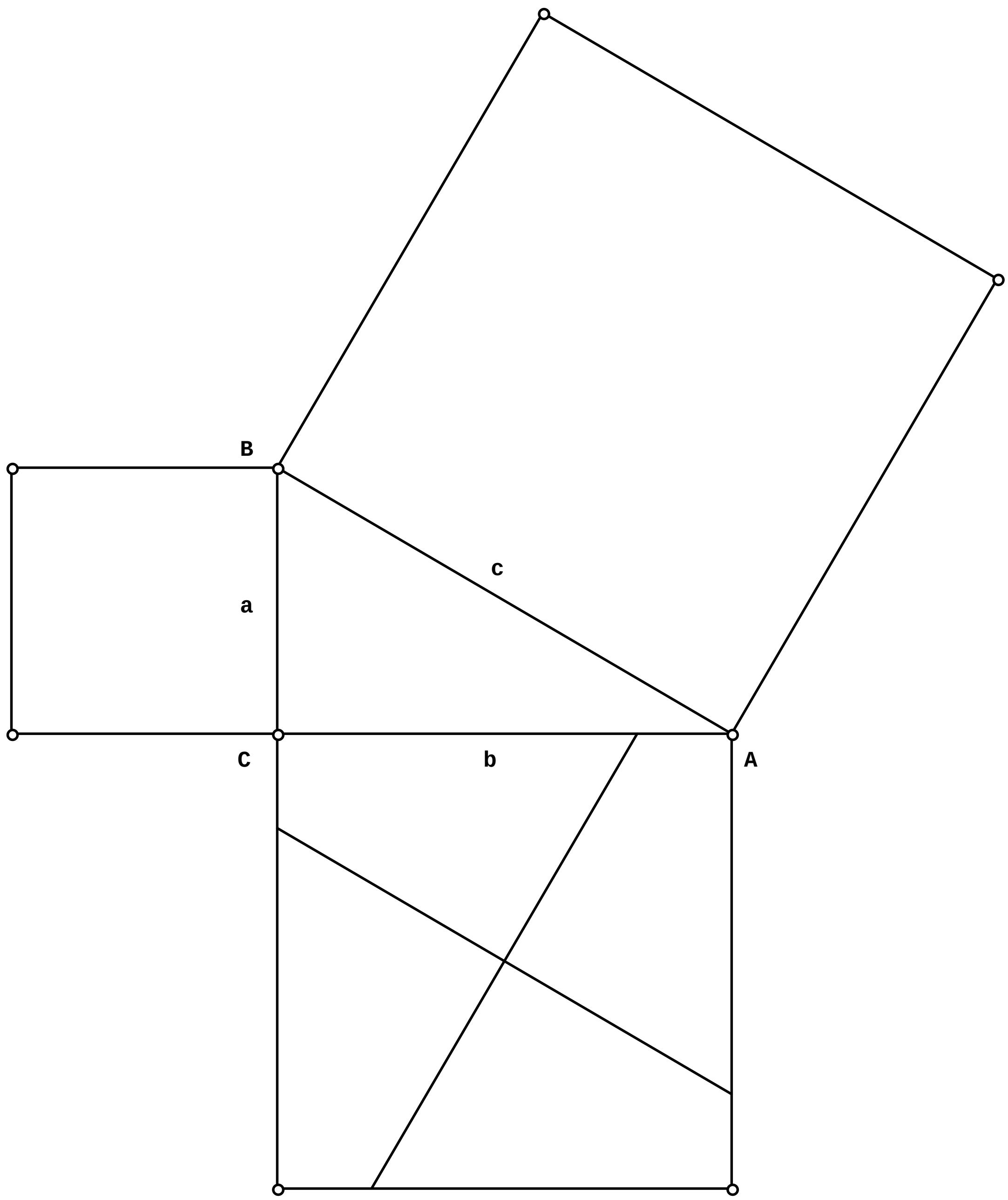
All is Number

Pythagoras was a mathematician from Samos, an island off the coast of Asia Minor, between modern day Turkey and Greece. He lived about 2500 years ago, from approximately 582 to 500 BC. Pythagoras studied mathematics in Egypt and Babylonia. When he moved to Italy, Pythagoras started a brotherhood of individuals, the Pythagoreans, who were interested in like things. This was a secret society of individuals who were vegetarians, and believed in reincarnation and the transmigration of souls. They were also interested in music and developed a theory of harmony. They particularly studied properties of numbers, hence their motto, "All is number." To the Greeks numbers were represented by physical objects. Odd and even numbers, square numbers, perfect number, prime numbers, triangular numbers were all important to the Pythagoreans.

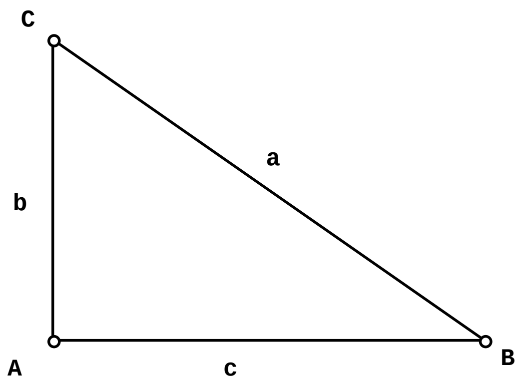
A number is composed of units. The Pythagorean Theorem about right triangles says that the square on the hypotenuse is equal to the sum of the squares on the legs. In particular, if the right triangle is isosceles, it says that the square on the hypotenuse is twice the square on one of the legs. If we assume that the length of the side is equal to 1, we conclude that the square on the hypotenuse is equal to 2 and therefore that the hypotenuse itself must have length equal to the square root of 2. Mathematicians were then able to show that there is no way to find a length of any size of which both the side and the hypotenuse were multiples. Therefore, they were forced to conclude that if the leg of this triangle was a "number," that is, a collection of units, then the hypotenuse was not, and conversely. So, it turned out, "All is number," is wrong!

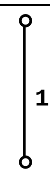
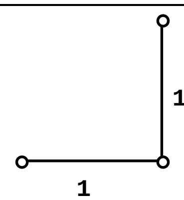
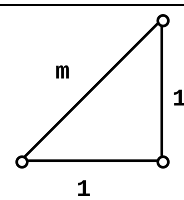
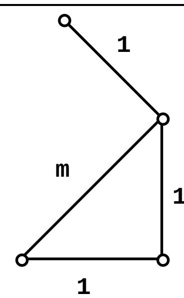
The Pythagorean Theorem, which we now consider, is thought by many to be the most important theorem in elementary mathematics. It is actually ironic that the theorem is called the Pythagorean Theorem because evidence suggests that it was known at least 1000 years before Pythagoras.

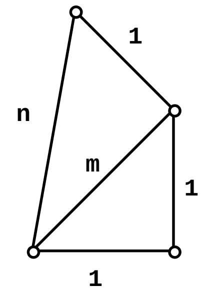
One way to prove this theorem is by beginning with the diagram on the next page, in which triangle *ABC* is a right triangle. On each side of the triangle a square is constructed. We are to show that the area of the square constructed on side **c** is equal to the sum of the areas of the squares constructed on sides **a** and **b**. We can do this by cutting out the shapes and rearranging. Use the large diagram on the next page and cut out the small square on the left and the four quadrilaterals that form the square on the bottom. Can you fit these five pieces into the top square?


B
a
c
C
b
A

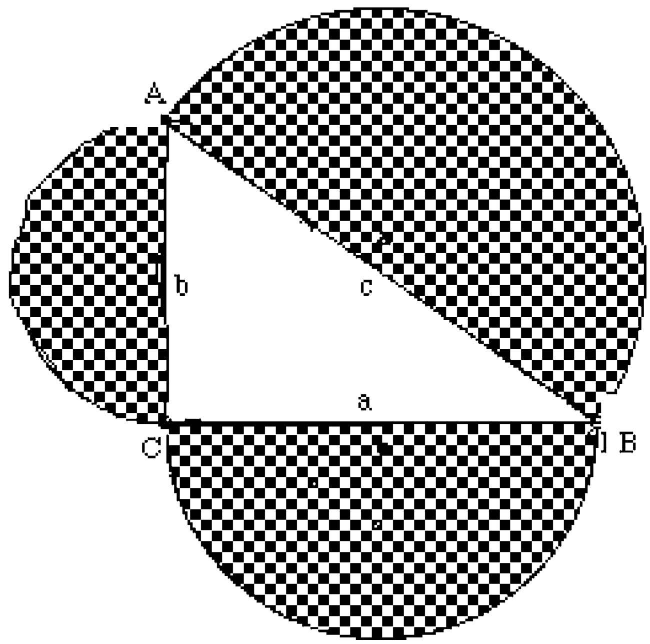
There are many problems that can be solved using the Pythagorean Theorem. For each exercise, remember that for any right triangle ABC, $a^2 + b^2 = c^2$.



For the Greeks one represented a single unit. Suppose we assume the line segment measures one unit.	
Construct another line segment of unit length perpendicular to this one. In other words, we are rotating this segment 90°.	
Connect the endpoints of these two line segments to form a right triangle. Use the Pythagorean Theorem to find the length of segment *m*.	
Let's continue this process. This time construct a line segment of length one that is perpendicular to line *m*.	

Connect the two endpoints to make a new triangle. What is the length of line segment n?	

Follow this pattern and explain how to construct a line segment of length $\sqrt{4} = 2$ and then of $\sqrt{5}$. How many triangles need to be drawn in this manner to construct a line segment of length $\sqrt{101}$.



Let's try a challenging problem.

Construct a semicircle using side **a** as a diameter. The radius of this semicircle is _______. The area of this semicircle is _______.	Construct a semicircle using **b** as a diameter. The radius of this semicircle is _______. The area of this semicircle is _______.	Construct a semicircle using **c** as a diameter. The radius of this semicircle is _______. The area of this semicircle is _______.

Use the Pythagorean Theorem to find an equation relating the areas of the three semicircles.

PYTHAGOREAN BROTHERHOOD MEMBERSHIP CERTIFICATE

THIS CERTIFICATE IS HEREBY PRESENTED TO

____

FOR THE MYRIAD OF NUMERICAL QUESTIONS YOU HAVE MASTERED.
THEY HAVE CHALLENGED YOUR GENIUS, CREATIVITY, UNDERSTANDING, AND PATIENCE.
TO KEEP YOUR MEMBERSHIP IN GOOD STANDING COMPLETE THE PROBLEMS IN CLASS TODAY.

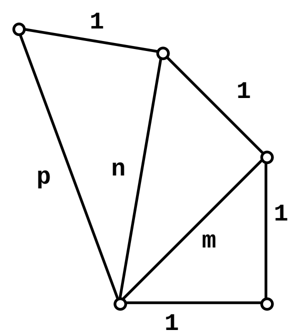
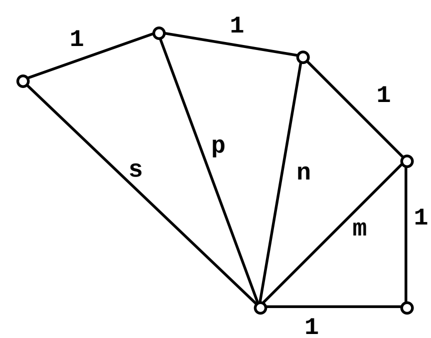
OFFICIAL DOCUMENT OF THE SOCIETY OF PYTHAGORAS

ΠΨΤΗΑΓΟΡΑΣ

PYTHAGORAS, FOUNDER

The Pythagorean Project
Solutions

The length of segment m can be found from the equation $m^2 = 1^2 + 1^2 = 2$. So $m = \sqrt{2}$ Then the length of segment n is found from $n^2 = 1^2 + m^2 = 1^2 + \left(\sqrt{2}\right)^2 = 1 + 2 = 3$. So $n = \sqrt{3}$. The next two analogous line segments are found as follows:

$p^2 = 1^2 + n^2$ $p^2 = 1^2 + \left(\sqrt{3}\right)^2$ $p^2 = 1 + 3 = 4$ $p = \sqrt{4} = 2$	
$s^2 = 1^2 + p^2$ $s^2 = 1^2 + 2^2$ $s^2 = 1 + 4 = 5$ $s = \sqrt{5}$	

To construct a line segment of length $\sqrt{101}$, note that the first segment drawn has length $\sqrt{2}$; the second segment drawn has length $\sqrt{3}$; the third segment drawn has length $\sqrt{4}$; the fourth segment drawn has length $\sqrt{5}$. Following this pattern, we expect the nth segment to have length $\sqrt{n+1}$; hence, the 100th segment drawn has length $\sqrt{101}$.

The radius of the semicircle with side **a** as diameter is $\frac{a}{2}$. The area of this semicircle is $\frac{1}{2} \bullet \pi \left(\frac{a}{2}\right)^2 = \frac{1}{2} \bullet \pi \bullet \frac{a^2}{4} = \frac{\pi a^2}{8}$.	The radius of the semicircle with side **b** as diameter is $\frac{b}{2}$. The area of this semicircle is $\frac{1}{2} \bullet \pi \left(\frac{b}{2}\right)^2 = \frac{1}{2} \bullet \pi \bullet \frac{b^2}{4} = \frac{\pi b^2}{8}$.	The radius of the semicircle with side **c** as diameter is $\frac{c}{2}$. The area of this semicircle is $\frac{1}{2} \bullet \pi \left(\frac{c}{2}\right)^2 = \frac{1}{2} \bullet \pi \bullet \frac{c^2}{4} = \frac{\pi c^2}{8}$.

Since the Pythagorean Theorem tells us that $a^2 + b^2 = c^2$, we also have $\frac{\pi}{8}\left(a^2 + b^2\right) = \frac{\pi}{8}c^2$. Therefore, $\frac{\pi}{8}a^2 + \frac{\pi}{8}b^2 = \frac{\pi}{8}c^2$, so the area of the semicircle constructed on **a** plus the area of the semicircle constructed on **b** equals the area of the semicircle constructed on **c**.

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