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The Sagacity of Circles: A History of the Isoperimetric Problem

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At the end of the third century A.D., a Hellenistic geometer named Pappus of Alexandria introduced Book V of his Mathematical Collections not with a discussion of mathematicians past or accomplishments to follow, but rather with a preface "On the Sagacity of Bees." By observing the near-perfect geometry of a bee's hexagonal comb structure, Pappus attributed to the insects "a certain geometrical forethought" (Thomas 591). "Bees," he wrote, "…know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each" (Thomas 593). Pappus' preface suggested much more than the natural efficiency of bees however. "We," he continued, "claiming a greater share in the wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them" (Thomas 593). With that, Pappus had undertaken the isoperimetric problem. Although isoperimetry contains many smaller problems with in it, the central goal is to discover which of all plane figures with the same perimeter has the largest area. The question of isoperimetry was several hundred years old when Pappus addressed it in the Collections, yet even generations later, it continued to fascinate the mathematical community. Appearing in both mathematical and literary texts and captivating the minds of mathematicians even in the modern age, the isoperimetric problem serves to illustrate both the perceptiveness of ancient mathematicians and the consistency of mathematical endeavor throughout history.

Although the isoperimetric problem is primarily mathematical in nature, it is unique in that poets and historians of both the ancient and medieval world incorporated it into their works. Most famously, Virgil made use of the concept in his Roman epic The Aeneid, written in the first century B.C.. In Book I of The Aeneid, Queen Dido flees her murderous brother Pygmalion to the shores of North Africa where she founds the city of Carthage. Virgil notes:

> "They sailed to the place where today you'll see
> Stone walls going higher and the citadel
> Of Carthage, the new town. They bought the land,
> Called Drumskin [Byrsa] from the bargain made, a tract
> They could enclose with one bull's hide" (Book I, 16).

According to legend, Dido made the hide given to her by the natives of Carthage into a long rope and, using the coast as part of her boundary, enclosed her lands in a semi-circle, thus using the fact that it was this shape which contains the greatest area (Nahin 45). It is from Virgil's tale that mathematicians give the name "Dido's Problem" to the isoperimetric problem. An earlier account of Carthaginian folklore compiled in the 3rd century A.D. by the Roman historian Marcus Junianus Justinus gives a more descriptive account of the legendary founding of Carthage by Dido, called Elissa by the Greeks:

> "Then [Elissa] bought some land, just as much as could be covered by
> a cow's hide, where she could give some recreation to her men…
> She next gave orders for the hide to be cut into very fine strips, and
> in this way she took possession of a greater area than she had apparently
> bargained for" (Book XVIII, 157).

Much later, the isoperimetric problem appeared in Geoffrey of Monmouth's Historia Regum Britanniæ (History of the Kings of England), an early account of the Arthurian legends written in the 12th century A.D.. In this tale, a German duke by the name of Hengist appeals to King Vortigern for land in return for military service:

> "'Grant,' saith [Vortigern], 'unto thy servant but so much only as
> may be compassed round about by a single thong within the land thou
> hast given me, that so I may build me a high place therein whereunto if
> need be I may betake me'...Straightaway…Hengist took a bull's hide, and
> wrought the same into a single thong throughout. He then compassed
> round with his thong a stony place that he had right cunningly chosen, and
> within the space thus meted out did begin to build a castle that was afterwards
> called in British , Kaercorrei, but in Saxon, Thongceaster, the which in

Latin's speech is called Castrum corrigae" (Monmouth 105-6).

The isoperimetric problem, therefore, held a particular appeal to not only the figures of the mythological past, but to the poets and historians who wrote of their deeds.

Despite its broad implications, the concept of isoperimetry was "naturally Greek." As Dunham notes, "…even before the time of Euclid the Greeks had enshrined the straight line and the circle as the two indispensable geometric figures, the two shapes constructible with geometric tools… The figure swept out by Euclid's compass enclosed the maximum area possible for the given perimeter. Is this not further indication of the ideal form the Greeks so admired?" (Dunham 112). Pappus himself introduced the notion of isoperimetry in three dimensions by commenting on philosophers "who considered that the creator gave the universe the form of a sphere because that was the most beautiful of all shapes also asserted that the sphere is the greatest of all solid figures which have their surfaces equal" (Heath 394). Together with Pappus' famous preface, isoperimetry was firmly established not as an isolated topic in ancient geometry, but rather as indicative of the grander scheme of the universe and the logic of the natural world.

In its most basic form, the isoperimetric problem asks the geometer "to determine, from among all curves of the same perimeter, the one enclosing the largest area" (Dunham 104). To many persons, the notion that the circle encloses the greatest area or even that different shapes of the same perimeter would have different areas seems paradoxical. At first glance, it may not be readily obvious that the last shape of the following four is the greatest in area.

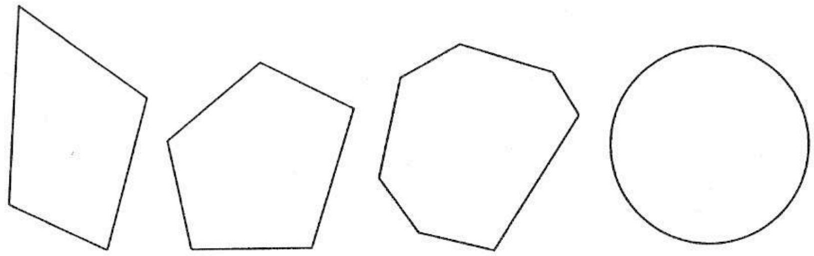

Fig. 1. Isoperimetric shapes, from David Wells, The Penguin Dictionary of Curious and Interesting Geometry (London: Penguin Books, 1991) 123.

Indeed, in his mathematical commentaries of the fifth century A.D., the philosopher Proclus described geographers "who [wrongly] inferred the size of cities from their perimeters" and of "members of communistic societies in his own time who cheated their fellow-members by giving them land of greater perimeter but less area than the plots which they took themselves" (Heath 206-7). It is quite simple to prove that of all rectangles of the same perimeter that which encloses the greatest area is a square (see Fig. 2), however for polygons of greater and greater numbers of sides, the problem becomes less straightforward. The development of the isoperimetric theorem, therefore, was a problem not for myth or history, but for successive generations of mathematicians and geometers.

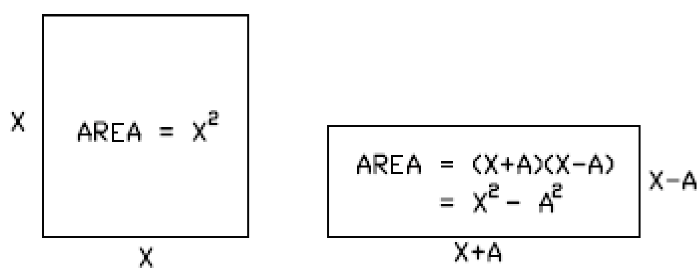

Fig. 2 Comparative areas of a square and rectangle, from William Dunham, The Mathematical Universe (New York: John Wiley & Sons, 1994) 105.

The isoperimetric problem as it is generally understood was first articulated by the Greek mathematician Zenodorus. Usually placed sometime after Archimedes in the 2nd century B.C., historians know of Zenodorus and his treatise On Isoperimetric Figures (now lost), through the

4^{th} century A.D. commentaries of Theon of Alexandria and Pappus (Nahin 47). Theon of Alexandria begins with the following assertion, taken from Book I of Ptolemy's Syntaxis:

> "'In the same way, since the greatest of various figures having an equal perimeter is that which has most angles, the circle is the greatest among plane figures and the sphere among the solid'" (Thomas 388).

He then develops this idea, with a summary of the proofs presented by Zenodorus in On Isoperimetric Figures. According to Theon, Zenodorus did not initiate his discussion of isoperimetry with the circle. Rather, he stated that "*Of all rectilinear figures having an equal perimeter – I mean equilateral and equiangular figures – the greatest is that which has the most angles*" (Thomas 388-89). In more modern language, the proposition is stated as follows: "Given two regular n-gons with the same perimeter, one with $n = n_1$, and the other with $n = n_2 > n_1$ then the regular n_2 –gon has the larger area" (Nahin 47). Following this, Zenodorus was able to arrive at the proposition that "*if a circle have an equal perimeter with an equilateral and equiangular rectilinear figure, the circle shall be the greater*" (Thomas 391). As Heath notes in his History of Greek Mathematics, Zenodorus chose to base his proof of this proposition on the theorem already established by Archimedes that "the area of a circle is equal to the right-angled triangle with perpendicular side equal to the radius and base equal to the perimeter of the circle" (209). From here, Zenodorus proceeded on the basis of two preliminary lemmas: first that "if there be two triangles on the same base and with the same perimeter, one being isosceles and the other scalene, the isosceles triangle has the greater area" (Heath 209); second that "given two isosceles triangles not similar to one another, if [one constructs] on the same bases two triangles similar to one another such that the sum of their perimeters is equal to the sum of the perimeters of the first two triangles, then the sum of the areas of the similar triangles is greater than the sum of the areas of the non-similar triangles" (Heath 210). It is at this point, however, that the difficulties of studying ancient mathematical texts comes to the forefront, for neither the text of Theon of Alexandria nor Pappus contains a clear indication of the direction of the remainder of Zenodorus' proof (Heath 211). Both commentators seem to hint that it will be covered in subsequent chapters, but as Heath bemoans "in the text as we have it the promise is not fulfilled" (212).

Although the extant commentary of Theon of Alexandria gives only limited proof of Zenodorus' aforementioned proposition, it does contain Zenodorus' work on two other key aspects of the isoperimetric problem, namely the question of equilateral and equiangular polygons and the problem of the sphere. The commentary begins with the statement that "*of all rectilinear figures having an equal number of sides and equal perimeter, the greatest is that which is equilateral and equiangular*" (Thomas 395). The first portion of this proposition, namely that of all polygons of the same number of sides an equilateral polygon has the greatest area can be proven using the abovementioned lemma concerning the areas of isosceles and scalene triangles. First, assume that two sides of the "maximum polygon" ABCDE, shown here as AB and BC, are unequal (See Fig. 3). Draw the line AC and construct an isosceles triangle AFC such that AF + FC = AB + BC (Heath 212). By Zenodorus' first lemma, the area of the triangle AFC is greater than the area of he triangle ABC, and thus the area of the polygon was "increased by construction," creating a contradiction with the original hypothesis holding that the area was a maximum (Heath 212).

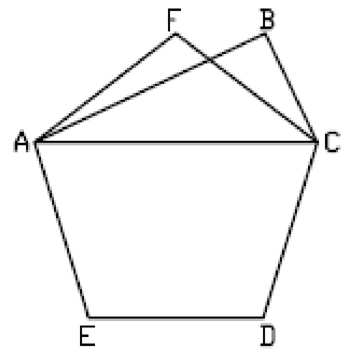

Fig. 3 "It is equilateral," from Sir Thomas Heath, A History of Greek Mathematics From Aristarchus to Diophantus (New York: Dover Publications, Inc., 1981) 212.

Similarly, the proof (See Fig. 4) that of all polygons of the same number of sides, an equiangular polygon has the greatest area makes use of Zenodorus' second lemma, concerning the areas of similar triangles. As Heath summarizes, "let the maximum polygon ABCDE [shown to be equilateral above] have the angle at B greater than the angle at D. Then BAC, DEC are non-similar isosceles triangles" (Heath 212). Construct two isosceles triangles FAC and GEC similar to one another on bases AC and CE respectively such that the sum of their perimeters is equal to the sum of the perimeters of triangles BAC and DEC. Then, concludes Zenodorus, "the sum of the areas of the two similar isosceles triangles is greater than the sum of the areas of the triangles BAC and DEC [by the second lemma]" and the area of the polygon has been increased by construction, once again contradicting the original maximum hypothesis (Heath 212).

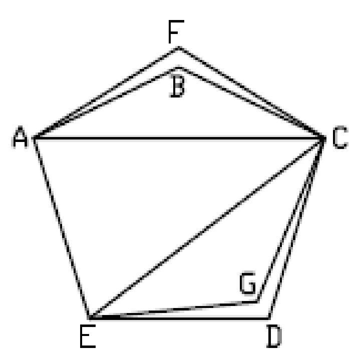

Fig. 4 "It is equiangular," from Sir Thomas Heath, A History of Greek Mathematics From Aristarchus to Diophantus (New York: Dover Publications, Inc., 1981) 212.

Beyond proving that "of all polygons of the same number of sides and equal perimeter the equilateral and equiangular polygon is the greatest in area" (Heath 207) as was demonstrated above, Zenodorus also dealt briefly with the notion of the sphere. As Theon quotes, "Now I say that, *of all solid figures having an equal surface, the sphere is the greatest*" (Thomas 395). Accordingly, Zenodorus made use of "theorems proved by Archimedes in his work On the Sphere and Cylinder" (Thomas 395), thus extending the isoperimetric problem to three dimensions.

The isoperimetric problem reached its greatest expression in the ancient world in the commentary of Pappus, a Hellenistic mathematician usually placed in the first half of the fourth century A.D. (Burton 221). The Mathematical Collections, Pappus' largest and most extensive work, was a summary and consolidation of the geometric theorems and formulae of the time. As Burton notes, it was designed to "give a synopsis of the contents of the great mathematical works of the past and then to clarify any obscure passages through various alternative proofs and supplementary lemmas" (Burton 221). Pappus' most famous contribution to the isoperimetric problem is also one of the most famous passages in mathematics, the brief preface to Book V entitled "On the Sagacity of Bees." As discussed earlier, Pappus' bees display a clear understanding of the isoperimetric problem. As Nahin notes, the insects address "the ancient question of how to tile the plane (how to divide an infinite two-dimensional surface into congruent n-gons)" (51). "They would necessarily think," writes Pappus,

> "that the figures must all be adjacent to one another and have their sides common, in that nothing else might fall into the interstices and so defile their work. Now there are only three rectilinear figures which would satisfy the condition, I mean regular figures which are equilateral and equiangular, inasmuch as irregular figures would be displeasing to the bees. For equilateral triangles, squares, and hexagons can lie adjacent

to one another and have their sides in common without irregular interstices" (Thomas 591).

Knowing, presumably through the proofs of Zenodorus, that "*of all rectilinear figures having an equal perimeter…the greatest is that which has most angles*" (Thomas 389), the bees choose the hexagon in order to achieve the maximum amount of honey storage.

Beyond merely collecting the proofs of Zenodorus and philosophizing on their significance in his preface, Pappus also contributed the notion of the circumference of semi-circles to the body of work on isoperimetry. He first proposed that "*of all circular segments having the same circumference the semi-circle is the greatest*" (Heath 390-91). Pappus based his proof on two preliminary lemmas, then proceeded to show that the area of a full semicircle ABC (see Fig. 5) is greater than the area of another circular segment DEF of equal circumference (Heath 393).

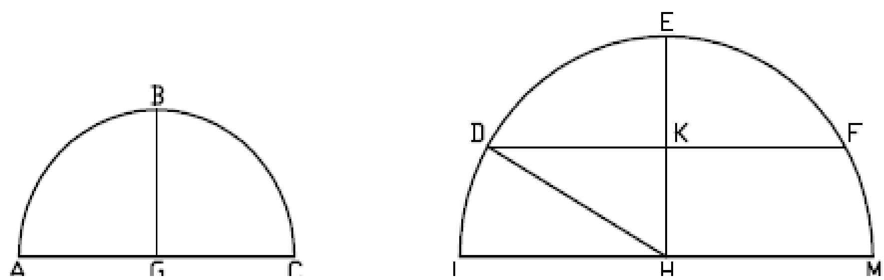

Fig. 5 "Semi-circle and circular segment" from Sir Thomas Heath, A History of Greek Mathematics From Aristarchus to Diophantus (New York: Dover Publications, Inc., 1981) 392.

One first constructs the semicircle ABC centered at G and a second circular segment DEF such that the circumference of ABC is equal to that of DEF. Construct H as the center of the circle DEF and draw EHK and BG perpendicular to DF and AC, respectively (Heath 392). Finally, draw DH and the line LHM parallel to DF. As summarized in Heath, Pappus arrives at the following conclusions:

$$LH : AG = (\text{arc } LE) : (\text{arc } AB)$$

$$= (\text{arc } LE) : (\text{arc } DE)$$

$$= (\text{sector } LHE) : (\text{sector } DHE)$$

and

$$LH^2 : AG^2 = (\text{sector } LHE) : (\text{sector } AGB).$$

Therefore,

$$(\text{sector } LHE) : (\text{sector } DHE) = (\text{sector } DHE) : (\text{sector } AGB).$$

By the preliminary lemmas, Pappus was able to conclude that:

$$(\text{sector } EDH) : (EDK) > R : (\angle DHE)$$

That is,

$$(\text{sector } EDH) : (EDK) > (\angle LHE) : (\angle DHE)$$

$$> (\text{sector } LHE) : (\text{sector } DHE)$$

$$> (\text{sector } EDH) : (\text{sector } AGB)$$

"Therefore," concludes Heath, "the half segment EDK is less than the half semicircle AGB, whence the semicircle BC is greater than the segment DEF" (Heath 393) and the semicircle is established as the figure of maximum area. Pappus' treatment of the isoperimetric problem was, therefore, an important step not only in developing ancient mathematical philosophy, but also in preserving and adding to the body of work already existing on Dido's now famous problem.

Just as the problem of isoperimetry is associated with the work of Zenodorus and his commentator Pappus in the ancient world, it was the Swiss mathematician Jacob Steiner (1796 – 1863) who tackled the isoperimetric theorem in the modern world. Indeed, the problem of isoperimetry in the nineteenth century emerged at an important juncture in mathematical thought. Mathematicians working in all fields of inquiry struggled over the use of analytic (i.e. calculus) or synthetic (i.e. pure geometry) methods in solving problems ("Isoperimetric" 1). Although mathematicians have since identified flaws in the synthetic approach to the isoperimetric theorem, it is Steiner's 1842 geometrical proof which remains, as Nahin observes, a "model of mathematical ingenuity" (55). It is important to note that Steiner's proof of the isoperimetric theorem relied upon the duality (or logical equivalency) of the following statements:

> A. "Of all closed curves in a plane with equal perimeters, the circle bounds the largest area
> [and]
> B. Of all closed curves in a plane with equal areas, the circle has the smallest perimeter" (Nahin 55).

This can be shown as follows:

If A is true and B is false, "then for a given circle C there exists a figure F with the same area but with a perimeter shorter than that of C" ("Isoperimetric" 1). If C is then made into a smaller circle C' whose perimeter is equal to that of the figure F, then "the area of C' will clearly be smaller than that of C and, consequently, it will be smaller than the area of F" ("Isoperimetric" 1). It is here that the contradiction arises. According to the above statement, C' and F have equal perimeters, thus resulting in the area of C' being less than that of F. By the original statement A, however, the area of the circle C' must be greater than that of F. Thus A → B ("Isoperimetric" 1). A similar proof follows for the implication B → A ("Isoperimetric" 1).

With the logical equivalency of A and B duly established, Steiner then proceeded to the geometric proof of Dido's problem. He based his methods on two preliminary lemmas – first, that "any triangle inscribed in a circle, with a diameter as a side (the hypotenuse), is a right triangle"; and second, that "of all possible triangles with two sides of given length, the triangle of maximum area is the right triangle with the given sides as the perpendicular sides" (Nahin 56). With these two lemmas in place, Steiner then moved to an actual construction. As Nahin summarizes, “Let A and B be the two points on a given straight line L…and suppose the solution curve C is not a semicircle” (see Fig. 6) (Nahin 57). One can therefore conclude that there exists a point P contained on the line C such that $\angle$ APB $\neq$ 90° (Nahin 57). The area enclosed by the line L and the curve C is divided into three regions R_1, R_2, and R_3 by the lines AP and PB.

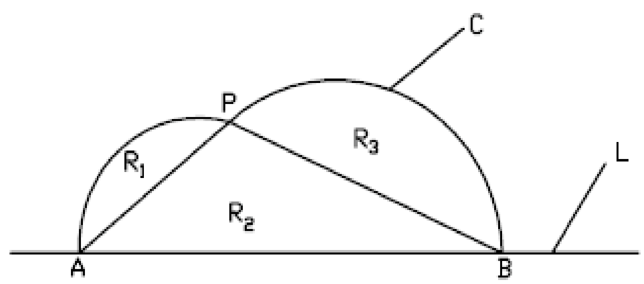

Fig. 6 “Steiner’s isoperimetric argument, part 1” from Paul J. Nahin, When Least is Best (Princeton, Princeton University Press, 2004) 57.

If the lines AP and PB are fixed at the point P and able to “slide” along the line L, one then moves A or B (or both) to A' and B' such that $\angle A'P'B'$ = 90°, where P' is adjusted such that the length of $A'P'$ is equal to that of AP and the length of $P'B'$ is equal to that of PB (see Fig. 7) (Nahin 57).

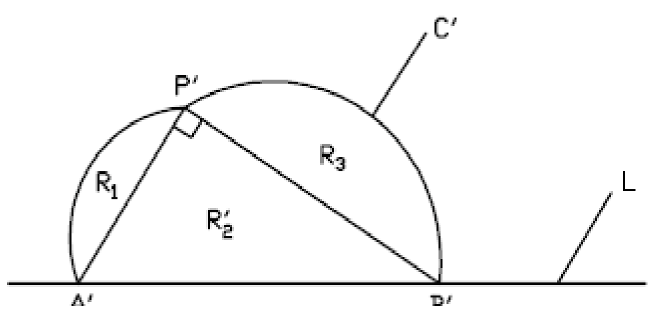

Fig. 7 “Steiner’s isoperimetric argument, part 2” from Paul J. Nahin, When Least is Best (Princeton, Princeton University Press, 2004) 58.

The areas of the regions R_1, R_2, and R_3 under the new curve then transfer as R_1, R_2', and R_3 (R_1 and R_3 remaining unchanged because of the construction AP = $A'P'$ and PB = $P'B'$) (Nahin 58). By the preliminary lemma, the area of R_2' is larger than that of R_2, thus demonstrating the fact that "an arbitrary curve C [has been transformed] into a curve C' with the same perimeter that encloses (with L) an area greater than that enclosed by C and L" (Nahin 58). Accordingly Steiner concluded that "the only curve C that does not allow such an area-increasing, perimeter-preserving transformation is the semicircle" in which, by the preliminary lemma, there does not exist a point P contained on C such that $\angle$ APB $\neq$ 90° (Nahin 58).

Steiner thus believed he had proved the circle to be the solution to the isoperimetric problem. As later scholars, including the German mathematician Peter Dirichlet (1805-1859), noted, Steiner had made an underlying assumption not explicitly addressed in his proof, namely that a solution exists (Nahin 59). The lemmas and final proof can only function if it is assumed (or known) that the isoperimetric problem is indeed solvable. As Nahin notes, this is not quite as obvious as it might first appear. Numerous geometrical problems, including the problem of "finding that convex figure of greatest area among all convex figures with a perimeter less than one" and the so-called Kakeya problem (finding "the smallest area in which a line segment of unit length can be rotated through 360°") are, in fact, without solution (Nahin 59-60).

Other mathematicians attempted to tackle the isoperimetric problem from the analytic or calculus-based perspective. Using calculus, the isoperimetric problem can be stated as follows:

> "Find an arc with parametric equations $x = x(t), y = y(t)$ for $t \in [t_1, t_2]$ such that $x(t_1) = x(t_2), y(t_1) = y(t_2)$ (where no further intersections occur) constrained by
>
> $$l = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2}\, dt$$
>
> such that

$$A = \frac{1}{2}\int_{t_1}^{t_2} (xy' - x'y)dt$$

is a maximum"
(Weisstein 1).

As Dunham observes in The Mathematical Universe, even Steiner's elegant proof and the application of calculus did not end work on the isoperimetric problem. Although the circle is indeed the solution to the question as Zenodorus considered it, "it is conceivable," writes Dunham, "that [one] might exceed a circle's area by assembling not regular polygons…but parabolas, ellipses, or some other irregular curves" (112). Indeed, it was over an aspect of isoperimetry that the mathematician brothers Jakob and Johann Bernoulli came to blows, a struggle which eventually contributed to the development of the calculus of variations (Dunham 14-15, 112-13).

Although the isoperimetric problem has its origins in the ancient world, it is nonetheless a problem for the ages. Despite the fact that he was a member of a mathematical culture as yet unacquainted with calculus or the modern notions of extrema, Zenodorus developed a geometrical proof of a concept which proved so significant as to find its way into literary, philosophical, and, most importantly, modern mathematical texts. Ancient problems, such as that of isoperimetry, are not relics of an obsolete mathematical past, but rather important steps towards the development of contemporary mathematics. Problems posed by the ancients not only speeded the progression towards more rigorous, complete systems of mathematics, but also prompted later innovators to develop new systems to deal with these early questions. The isoperimetric problem thus demonstrates an important continuity in mathematical thought. From Zenodorus to Pappus and from Steiner to the mathematicians of the twenty-first century, isoperimetry has transcended its origins in ancient geometry to become a building block of more modern analytic systems of mathematics. In his preface "On the Sagacity of Bees," Pappus observed that God had given to man "the best and most perfect understanding of wisdom and mathematics" (Thomas 589). Indeed, it is this wisdom which led geometers not only to the solution of the isoperimetric problem but more importantly to a greater appreciation of the beauty and logic of the mathematical world.

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