[204]

Chapter VIII.

On imaginary functions and variables.

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[220] § V. — *Determination of continuous imaginary functions of a single variable that satisfy certain conditions.*

Let

$$\varpi(x) = \varphi(x) + \sqrt{-1}\chi(x)$$

be a continuous imaginary function of the variable x, where $\varphi(x)$ and $\chi(x)$ are two real continuous functions. The imaginary function $\varpi(x)$ is completely determined if for all the possible real values of the variables x and y, it is required to satisfy one of the equations

$$(1) \qquad \varpi(x+y) = \varpi(x) + \varpi(y) \quad \text{or}$$
$$(2) \qquad \varpi(x+y) = \varpi(x) \times \varpi(y),$$

or else, for all real positive values of the same variables, one of the following equations:

$$(3) \qquad \varpi(xy) = \varpi(x) + \varpi(y) \quad \text{or}$$
$$(4) \qquad \varpi(xy) = \varpi(x) \times \varpi(y).$$

We will solve these four equations successively, which will provide us with four problems analogous to those we have already treated in § I of Chapter V.

Problem I. — *To determine the imaginary function $\varpi(x)$ in such a manner that it remains continuous between any two real limits of the variable* [221] *x and so that for all real values of the variables x and y, we have*

$$(1) \qquad \varpi(x+y) = \varpi(x) + \varpi(y).$$

Solution. — If, with the aid of the formula

$$\varpi(x) = \varphi(x) + \chi(x)\sqrt{-1},$$

we replace the imaginary function ϖ in equation (1) with the real functions φ and χ, this equation becomes

$$\varphi(x+y) + \chi(x+y)\sqrt{-1} = \varphi(x) + \chi(x)\sqrt{-1} + \varphi(y) + \chi(y)\sqrt{-1},$$

then by equating the real parts and the coefficients of $\sqrt{-1}$ on both sides, we conclude

$$\varphi(x+y) = \varphi(x) + \varphi(y) \quad \text{and}$$
$$\chi(x+y) = \chi(x) + \chi(y).$$

From these last formulas (see Chapter V, § I, problem I), we get

$$\varphi(x) = x\varphi(1) \quad \text{and}$$
$$\chi(x) = x\chi(1).$$

Consequently

$$\varpi(x) = x\left[\varphi(1) + \chi(1)\sqrt{-1}\right], \tag{5}$$

or what amounts to the same thing,

$$\varpi(x) = x\varpi(1). \tag{6}$$

It follows from equation (5) that any value of $\varpi(x)$ that satisfies the given question is necessarily of the form

$$\varpi(x) = \left(a + b\sqrt{-1}\right)x, \tag{7}$$

where a and b denote two constant quantities. Moreover, it is easy to assure ourselves that any such value of $\varpi(x)$ satisfies equation (1), whatever the values of a and b. These quantities are thus two arbitrary constants.

[222] We could remark that to obtain the preceding value of $\varpi(x)$, it suffices to replace the arbitrary real constant a in the value of $\varphi(x)$ given by equation (7) of Chapter V (§ I) by the arbitrary but imaginary constant

$$a + b\sqrt{-1}.$$

Problem II. — *To determine the imaginary function $\varpi(x)$ in such a manner that it remains continuous between any two real limits of the variable x and so that for all real values of the variables x and y, we have*

$$\varpi(x+y) = \varpi(x)\,\varpi(y). \tag{2}$$

Solution.[1] — If we make $x = 0$ in equation (2), we get

$$\varpi(0) = 1,$$

or, because of the formula

$$\varpi(x) = \varphi(x) + \chi(x)\sqrt{-1},$$

we get what amounts to the same thing,

$$\varphi(0) + \chi(0)\sqrt{-1} = 1.$$

Consequently,

$$\varphi(0) = 1 \quad \text{and} \quad \chi(0) = 0.$$

The function $\varphi(x)$ reduces to one for the particular value 0 assigned to the variable x, and because it is assumed to be continuous between any limits, it is clear that in the neighborhood of this particular value, it is only very slightly different from one, and consequently it is positive. Thus, if α denotes a very small number, we can choose this number in such a way that the function $\varphi(x)$ remains constantly between the limits

$$x = 0 \quad \text{and} \quad x = \alpha.$$

With this condition satisfied, because the quantity $\varphi(x)$ is itself positive, if we take

$$\rho = \sqrt{\varphi(\alpha)^2 + \chi(\alpha)^2} \quad \text{and} \quad \zeta = \arctan\frac{\chi(\alpha)}{\varphi(\alpha)},$$

[1] Note that this solution is very different from his solution to the corresponding problem II in Chapter V, § I. By contrast, problem I in this section followed as an easy corollary of problem I of Chapter V, § I.

we conclude that

$$\varpi(\alpha) = \varphi(\alpha) + \chi(\alpha)\sqrt{-1} = \rho\left(\cos\zeta + \sqrt{-1}\sin\zeta\right).$$

[223] Now imagine that in equation (2) we successively replace y by $y+z$, then z by $z+u$, ... We conclude that

$$\varpi(x+y+z+\ldots) = \varpi(x)\,\varpi(y)\,\varpi(z)\ldots,$$

however many variables, x, y, z, ... there may be. If we also donote by m the number of variables, and if we make

$$x = y = z = \ldots = \alpha,$$

then the equation we have just found becomes

$$\varpi(m\alpha) = [\varpi(\alpha)]^m = \rho^m\left(\cos m\zeta + \sqrt{-1}\sin m\zeta\right).$$

I add that the formula

$$\varpi(m\alpha) = \rho^m\left(\cos m\zeta + \sqrt{-1}\sin m\zeta\right)$$

remains true if we replace the integer number m by a fraction, or even by an arbitrary number μ. We will prove this easily in what follows.

If in equation (2) we make

$$x = \frac{1}{2}\alpha \quad \text{and} \quad y = \frac{1}{2}\alpha,$$

then we conclude

$$\left[\varpi\left(\frac{1}{2}\alpha\right)\right]^2 = \varpi(\alpha) = \rho\left[\cos\zeta + \sqrt{-1}\sin\zeta\right].$$

Then, by taking square roots of both sides in such a way that the real parts are positive, and by observing that the two functions $\varphi(x)$ and $\cos x$ remain positive, the first between the limits $x=0$ and $x=\alpha$, the second between the limits $x=0$ and $x=\zeta$, we find that

$$\varpi\left(\frac{1}{2}\alpha\right) = \varphi\left(\frac{1}{2}\alpha\right) + \chi\left(\frac{1}{2}\alpha\right)\sqrt{-1} = \rho^{\frac{1}{2}}\left(\cos\frac{\zeta}{2} + \sqrt{-1}\sin\frac{\zeta}{2}\right).$$

Likewise, if in equation (2) we make

$$x = \frac{1}{4}\alpha \quad \text{and} \quad y = \frac{1}{4}\alpha,$$

[224] then we conclude

$$\left[\varpi\left(\frac{1}{4}\alpha\right)\right]^2 = \varpi\left(\frac{1}{2}\alpha\right) = \rho^{\frac{1}{2}}\left(\cos\frac{\zeta}{2} + \sqrt{-1}\sin\frac{\zeta}{2}\right).$$

Then, by taking square roots of both sides so as to obtain positive real parts, we find

$$\varpi\left(\frac{1}{4}\alpha\right) = \rho^{\frac{1}{4}}\left(\cos\frac{\zeta}{4} + \sqrt{-1}\sin\frac{\zeta}{4}\right).$$

By similar reasoning, we can establish successively the formulas

$$\begin{aligned}
\varpi\left(\frac{1}{8}\alpha\right) &= \rho^{\frac{1}{8}}\left(\cos\frac{\zeta}{8} + \sqrt{-1}\sin\frac{\zeta}{8}\right),\\
\varpi\left(\frac{1}{16}\alpha\right) &= \rho^{\frac{1}{16}}\left(\cos\frac{\zeta}{16} + \sqrt{-1}\sin\frac{\zeta}{16}\right),\\
\cdots\cdots&\cdots\cdots\cdots\cdots,
\end{aligned}$$

and in general, where n denotes any integer number,

$$\varpi\left(\frac{1}{2^n}\alpha\right) = \rho^{\frac{1}{2^n}}\left[\cos\left(\frac{1}{2^n}\zeta\right) + \sqrt{-1}\sin\left(\frac{1}{2^n}\zeta\right)\right].$$

If we operate on the preceding value of $\varpi\left(\frac{1}{2^n}\alpha\right)$ to derive the value of $\varpi\left(\frac{m}{2^n}\alpha\right)$ the same way we operate on the value of $\varpi(\alpha)$ to derive that of $\varpi(m\alpha)$, we find that

$$\varpi\left(\frac{m}{2^n}\alpha\right) = \rho^{\frac{m}{2^n}}\left[\cos\left(\frac{m}{2^n}\zeta\right) + \sqrt{-1}\sin\left(\frac{m}{2^n}\zeta\right)\right],$$

or what amounts to the same thing,

$$\varphi\left(\frac{m}{2^n\alpha}\right) + \chi\left(\frac{m}{2^n\alpha}\right)\sqrt{-1} = \rho^{\frac{m}{2^n}}\left[\cos\left(\frac{m}{2^n}\zeta\right) + \sqrt{-1}\sin\left(\frac{m}{2^n}\zeta\right)\right].$$

Consequently,

$$\begin{aligned}
\varphi\left(\frac{m}{2^n}\alpha\right) &= \rho^{\frac{m}{2^n}}\cos\left(\frac{m}{2^n}\zeta\right) \quad \text{and}\\
\chi\left(\frac{m}{2^n}\alpha\right) &= \rho^{\frac{m}{2^n}}\sin\left(\frac{m}{2^n}\zeta\right).
\end{aligned}$$

[225] Then, by supposing that the fraction $\frac{m}{2^n}$ varies in such a way as to approach indefinitely the number μ and passing to the limit, we get the equations

$$\varphi(\mu\alpha) = \rho^\mu \cos\mu\zeta \quad \text{and} \quad \chi(\mu\alpha) = \rho^\mu \sin\mu\zeta,$$

from which we conclude that

$$\varpi(\mu\alpha) = \rho^\mu \left(\cos\mu\zeta + \sqrt{-1}\sin\mu\zeta\right). \tag{8}$$

Moreover, if in equation (2) we set

$$x = \mu\alpha \quad \text{and} \quad y = -\mu\alpha,$$

we get

$$\varpi(-\mu\alpha) = \frac{\varpi(0)}{\varpi(\mu\alpha)} = \rho^{-\mu}\left[\cos(-\mu\zeta) + \sqrt{-1}\sin(-\mu\zeta)\right].$$

Thus, formula (8) remains true when we replace μ by $-\mu$. In other words, for all real values of the variable x, both positive and negative, we have

$$\varpi(\alpha x) = \rho^x\left[\cos\zeta x + \sqrt{-1}\sin\zeta x\right] = [\varpi(\alpha)]^x. \tag{9}$$

In this last formula, if we write $\frac{x}{\alpha}$ instead of x, it becomes

$$\varpi(x) = \rho^{\frac{x}{\alpha}}\left[\cos\left(\frac{\zeta}{\alpha}x\right) + \sqrt{-1}\sin\left(\frac{\zeta}{\alpha}x\right)\right] = [\varpi(\alpha)]^{\frac{x}{\alpha}}. \tag{10}$$

If, for brevity, we make

$$\rho^{\frac{1}{\alpha}} = A \quad \text{and} \quad \frac{\zeta}{\alpha} = b, \tag{11}$$

we find

$$\varpi(x) = A^x\left(\cos bx + \sqrt{-1}\sin bx\right). \tag{12}$$

Thus any value of $\varpi(x)$ that satisfies the given question is necessarily of the form

$$A^x\left(\cos bx + \sqrt{-1}\sin bx\right),$$

where A and b denote two real quantities, of which the first must be [226] positive. Moreover, it is easy to assure ourselves that such a value of $\varpi(x)$ satisfies equation (2), whatever the values of the number A and the quantity b may be. This number and this quantity are thus arbitrary constants.

Corollary. — In the particular case where the function $\varphi(x)$ remains positive between the limits $x = 0$ and $x = 1$, we can, instead of supposing that α is very small, set $\alpha = 1$. Then we conclude immediately from equations (9) and (10) that

$$\varpi(x) = [\varpi(1)]^x . \tag{13}$$

Problem III. — *To determine the imaginary function $\varpi(x)$ in such a manner that it remains continuous between any two positive limits of the variable x and so that for all positive values of the variables x and y,*

$$\varpi(xy) = \varpi(x) + \varpi(y) . \tag{3}$$

Solution. — If with the aid of the formula

$$\varpi(x) = \varphi(x) + \chi(x)\sqrt{-1},$$

we replace the imaginary function ϖ in equation (3) by the real functions φ and χ, then we equate the real parts and the coefficients of $\sqrt{-1}$ on both sides, we find

$$\begin{aligned} \varphi(xy) &= \varphi(x)\,\varphi(y) \quad \text{and} \\ \chi(xy) &= \chi(x)\,\chi(y) . \end{aligned}$$

Moreover, if A denotes any number and log denotes the characteristic of logarithms in the system for which the base is A, we get from the preceding equations (see Chapter V, § I, problem III)

$$\begin{aligned} \varphi(x) &= \varphi(A)\log(x) \quad \text{and} \\ \chi(x) &= \chi(A)\log(x) . \end{aligned}$$

We conclude that

$$\varpi(x) = \left[\varphi(A) + \chi(A)\sqrt{-1}\right]\log(x) , \tag{14}$$

[227] or what amounts to the same thing,

$$\varpi(x) = \varpi(A)\log(x). \tag{15}$$

It follows from formula (14) that any value of $\varpi(x)$ that satisfies the given question is necessarily of the form

$$\varpi(x) = \left(a + b\sqrt{-1}\right)\log(x), \tag{16}$$

where a and b denote constant quantities. Moreover, it is easy to assure ourselves that such a value of $\varpi(x)$ satisfies equation (3), whatever the quantities a and b may be. Thus these quantities are arbitrary constants.

We could remark that to obtain the preceding value of $\varpi(x)$, it suffices to replace the arbitrary real constant a in the value of $\varphi(x)$ given by equation (12) of Chapter V (§ I) by the arbitrary but imaginary constant

$$a + b\sqrt{-1}.$$

Note. — We could arrive very simply at equation (15) in the following manner.

By virtue of the identities

$$x = A^{\log x} \quad \text{and} \quad y = A^{\log y},$$

equation (3) becomes

$$\varpi\left(A^{\log x + \log y}\right) = \varpi\left(A^{\log x}\right) + \varpi\left(A^{\log y}\right).$$

Because in this last formula the variable quantities $\log x$ and $\log y$ take on all real values, both positive and negative, as a result we have, for all possible real values of the variables x and y,

$$\varpi\left(A^{x+y}\right) = \varpi\left(A^{x}\right) + \varpi\left(A^{y}\right).$$

We conclude [see problem I, equation (6)] that

$$\varpi\left(A^{x}\right) = x\varpi\left(A^{1}\right) = x\varpi(A),$$

and consequently

$$\varpi\left(A^{\log x}\right) = \varpi(A)\log x,$$

[228] or what amounts to the same thing,

$$\varpi(x) = \varpi(A) \log x.$$

Problem IV. — *To determine the imaginary function $\varpi(x)$ in such a manner that it remains continuous between any two positive limits of the variable x and so that for all positive values of the variables x and y, we have*

$$\varpi(xy) = \varpi(x)\varpi(y) \tag{4}$$

Solution. — It would be easy to apply a method similar to that which we used to solve the second problem to the solution of this problem. However, we will arrive more promptly at the solution we seek if we observe that, by denoting by log the characteristic of logarithms in the system for which the base is A, we can put equation (4) into the form

$$\varpi\left(A^{\log x+\log y}\right) = \varpi\left(A^{\log x}\right)\varpi\left(A^{\log y}\right).$$

Because in this last equation the variable quantities $\log x$ and $\log y$ admit any real values, positive and negative, it follows that we have, for all possible real values of the variables x and y,

$$\varpi\left(A^{x+y}\right) = \varpi(A^x)\,\varpi(A^y).$$

If α represents a very small number and if we replace $\varpi(x)$ with $\varpi(A^x)$ in equation (10) of the second problem, we conclude that

$$\varpi(A^x) = [\varpi(A^\alpha)]^{\frac{x}{\alpha}}.$$

Consequently, we find that

$$\varpi\left(A^{\log x}\right) = [\varpi(A^\alpha)]^{\frac{\log x}{\alpha}},$$

or what amounts to the same thing,

$$\varpi(x) = [\varpi(A^\alpha)]^{\frac{\log x}{\alpha}}. \tag{17}$$

It is essential to observe that the imaginary function $\varpi(A^x)$, and consequently its real part $\varphi(A^x)$, reduce to one for $x = 0$,

[229] or in other words, that the imaginary function $\varpi(x)$ and its real part φx reduce to one for $x = 1$. We can prove this directly by taking

$$x = A^0 = 1$$

in equation (4). As for the number α, it need only to be small enough that the real part of the imaginary function $\varpi\left(A^x\right)$ remain constantly positive between the limits $x = 0$ and $x = \alpha$. When this condition is satisfied, the real part of the imaginary expression

$$\varpi\left(A^\alpha\right) = \varphi\left(A^\alpha\right) + \chi\left(A^\alpha\right)\sqrt{-1}$$

is itself positive. Consequently, if we make

$$\rho = \sqrt{\left[\varphi\left(A^\alpha\right)\right]^2 + \left[\chi\left(A^\alpha\right)\right]^2} \quad \text{and} \quad \zeta = \arctan\frac{\chi\left(A^\alpha\right)}{\varphi\left(A^\alpha\right)},$$

we have

$$\varpi\left(A^\alpha\right) = \rho\left(\cos\zeta + \sqrt{-1}\sin\zeta\right).$$

Given this, equation (17) becomes

$$(18) \qquad \begin{cases} \varpi(x) = \rho^{\frac{\log x}{\alpha}}\left[\cos\left(\frac{\zeta}{\alpha}\log x\right) + \sqrt{-1}\sin\left(\frac{\zeta}{\alpha}\log x\right)\right] \\ \qquad\;\; = x^{\frac{\log\rho}{\alpha}}\left[\cos\left(\frac{\zeta}{\alpha}\log x\right) + \sqrt{-1}\sin\left(\frac{\zeta}{\alpha}\log x\right)\right]. \end{cases}$$

By virtue of this last equation, any value of that satisfies the given question is necessarily of the form

$$(19) \qquad \varpi(x) = x^a\left[\cos\left(b\log x\right) + \sqrt{-1}\sin\left(b\log x\right)\right],$$

where a and b denote two constant quantities. Moreover, it is easy to assure ourselves that these two constant quantities ought to remain entirely arbitrary.