[121]

§ II. — *On series for which all the terms are positive.*

Whenever all the terms of the series

$$u_0,\ u_1,\ u_2,\ \ldots,\ u_n,\ \ldots \tag{1}$$

are positive, we may usually decide whether it is convergent or divergent by using the following theorem:

Theorem I.[1] — *Consider the limit or limits towards which the expression $(u_n)^{\frac{1}{n}}$ converges as n increases indefinitely, and let k denote the largest of these limits, or in other words, the limit of the largest values of the expression in question. Series* (1) *converges whenever $k < 1$ and diverges whenever $k > 1$.*

Proof. — First of all, suppose that $k < 1$ and choose an arbitrary third number U between the two numbers 1 and k, so that we have

$$k < U < 1.$$

As n increases beyond assignable limit, the largest values of $(u_n)^{\frac{1}{n}}$ cannot approach indefinitely the limit k without eventually being constantly less than U. Consequently, it is possible to assign an integer value to n large enough so that when n is greater than or equal to this value, we constantly have[2]

$$(u_n)^{\frac{1}{n}} < U, \quad \text{or} \quad u_n < U^n.$$

It follows that the terms of the series

$$u_0,\ u_1,\ u_2,\ \ldots,\ u_{n+1},\ u_{n+2},\ \ldots$$

[122] are eventually always smaller than the corresponding terms of the geometric progression

$$1,\ U,\ U^2,\ \ldots,\ U^n,\ U^{n+1},\ U^{n+2},\ \ldots.$$

[1] This theorem is now known as the Root Test. It is cited as the definition of upper and lower limits in [DSB Cauchy, p. 136].

[2] In the 1911 edition, the subscript is missing in the term $(u_n)^{\frac{1}{n}}$. It is present in the 1821 edition. (tr.)

As this progression is convergent (because $U < 1$) we may, by the previous remark, conclude *a fortiori* the convergence of series (1).

On the other hand, suppose that $k > 1$ and again pick a third number U between the two numbers 1 and k, so that we have

$$k > U > 1.$$

As n increases without limit, the largest values of $(u_n)^{\frac{1}{n}}$ in approaching k indefinitely, eventually become greater than U. We may therefore satisfy the condition

$$(u_n)^{\frac{1}{n}} > U$$

or, what amounts to the same thing, the following condition

$$u_n > U^n,$$

for values of n as large as we might wish. As a consequence, we find in the series

$$u_0,\ u_1,\ u_2,\ \ldots,\ u_n,\ u_{n+1},\ u_{n+2},\ \ldots$$

an indefinite number of terms greater than the corresponding terms of the geometric progression

$$1,\ U,\ U^2,\ \ldots,\ U^n,\ U^{n+1},\ U^{n+2},\ \ldots.$$

As this progression is divergent (because $U > 1$) and, as a consequence its various terms increase to infinity, the remark that we have just made suffices to establish the divergence of series (1).

In a great number of cases we may determine the values of the quantity k with the assistance of theorem IV (Chap. II, § III). Indeed, [123] by virtue of this theorem, any time the ratio $\frac{u_{n+1}}{u_n}$ converges towards a fixed limit, that limit is precisely the value of k. We may therefore state the following proposition:

Theorem II.[3] — *If, for increasing values of n, the ratio*

$$\frac{u_{n+1}}{u_n}$$

[3]This is the Ratio Test; see [DSB Cauchy, p. 136].

converges towards a fixed limit k, series (1) *converges whenever $k < 1$ and diverges whenever $k > 1$.*

For example, if we consider the series

$$1,\ \frac{1}{1},\ \frac{1}{1\cdot 2},\ \frac{1}{1\cdot 2\cdot 3},\ \ldots,\ \frac{1}{1\cdot 2\cdot 3\ldots n},\ \ldots,$$

then we find

$$\frac{u_{n+1}}{u_n} = \frac{1\cdot 2\cdot 3\ldots n}{1\cdot 2\cdot 3\ldots n(n+1)} = \frac{1}{n+1}, \quad \text{so } k = \frac{1}{\infty} = 0,$$

and consequently the series is convergent, as we already knew.

The first of the two theorems that we have just established leaves no doubt about the convergence or divergence of a series whose terms are positive, except in the particular case where the quantity k becomes equal to one. In this particular case, it is not always easy to answer the question of convergence. However, we will now prove two new propositions, which frequently help us to decide the issue.

Theorem III.[4] — *Whenever each term of series* (1) *is smaller than the one preceding it, that series and the following one*

$$(2) \qquad u_0,\ 2u_1,\ 4u_3,\ 8u_7,\ 16u_{15}, \ldots$$

are either both convergent or both divergent.

Proof. — First of all, suppose that series (1) is convergent and [124] let s denote its sum. Then

$$\begin{aligned} u_0 &= u_0, \\ 2u_1 &= 2u_1, \\ 4u_3 &< 2u_2 + 2u_3, \\ 8u_7 &< 2u_4 + 2u_5 + 2u_6 + 2u_7, \\ \ldots \quad &\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

[4]This is known as the Cauchy Condensation Test.

and consequently, the sum of as many of the terms of series (2) as we may wish is smaller than

$$u_0 + 2u_1 + 2u_2 + 2u_3 + 2u_4 + \ldots = 2s - u_0.$$

It follows that series (2) converges.

On the other hand, suppose that series (1) diverges. The sum of its terms, taken in great number, eventually surpasses any assignable limit. Because we have

$$\begin{array}{rcl} u_0 & = & u_0, \\ 2u_1 & > & u_1 + u_2, \\ 4u_3 & > & u_3 + u_4 + u_5 + u_6, \\ 8u_7 & > & u_7 + u_8 + u_9 + u_{10} + u_{11} + u_{12} + u_{13} + u_{14}, \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

we must conclude that the sum of the quantities

$$u_0,\ 2u_1,\ 4u_3,\ 8u_7,\ \ldots,$$

taken in great number, is itself eventually greater than any given quantity. Series (2) is therefore divergent, conforming to the stated theorem.

Corollary. — Let μ be an any quantity. If series (1) is

$$1,\ \frac{1}{2^\mu},\ \frac{1}{3^\mu},\ \frac{1}{4^\mu},\ \ldots, \tag{3}$$

then series (2) becomes

$$1,\ 2^{1-\mu},\ 4^{1-\mu},\ 8^{1-\mu},\ \ldots.$$

[125] This last series is a geometric progression, convergent whenever we have $\mu > 1$ and divergent in the opposite case. As a consequence, series (3) is itself convergent if μ is a number greater than one, and divergent if $\mu = 1$ or $\mu < 1$. For example, of the three series

$$1,\ \frac{1}{2^2},\ \frac{1}{3^2},\ \frac{1}{4^2},\ \ldots, \tag{4}$$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots, \tag{5}$$

$$1,\ \frac{1}{2^{\frac{1}{2}}},\ \frac{1}{3^{\frac{1}{2}}},\ \frac{1}{4^{\frac{1}{2}}},\ \ldots, \tag{6}$$

the first is convergent and the other two divergent.

Theorem IV.[5] — *Suppose that* log *denotes the characteristic of the logarithm in any system and that the ratio*

$$\frac{\log(u_n)}{\log\left(\frac{1}{n}\right)}$$

converges towards a finite limit h *for increasing values of* n*. Series* (1) *is convergent if* $h > 1$ *and divergent if* $h < 1$.

Proof. — First of all, suppose $h > 1$ and choose any third quantity a between the two quantities 1 and h, so that we have

$$h > a > 1.$$

The ratio $\frac{\log(u_n)}{\log\left(\frac{1}{n}\right)}$, or its equivalent

$$\frac{\log\left(\frac{1}{u_n}\right)}{\log(n)},$$

eventually, for very large values of n, is constantly greater than a. In other words, if n increases beyond [126] a certain limit, we always have

$$\frac{\log\left(\frac{1}{u_n}\right)}{\log(n)} > a,$$

or what amounts to the same thing,

$$\log\left(\frac{1}{u_n}\right) > a\log(n),$$

and, as a consequence,

$$\frac{1}{u_n} > n^a, \quad \text{so} \quad u_n < \frac{1}{n^a}.$$

It follows that the terms of series (1) eventually are constantly smaller than the corresponding terms of the following series

$$1,\ \frac{1}{2^a},\ \frac{1}{3^a},\ \frac{1}{4^a},\ \ldots,\ \frac{1}{n^a},\ \frac{1}{(n+1)^a},\ \ldots.$$

[5]This is the Logarithmic Convergence Test.

As this last series is convergent (because $a > 1$), we may, by the previous remark, conclude *a fortiori* the convergence of series (1).

On the other hand, suppose that $h < 1$, and again pick a third quantity a between 1 and h, so that we have

$$h < a < 1.$$

Eventually, for very large values of n, we constantly have

$$\frac{\log\left(\frac{1}{u_n}\right)}{\log(n)} < a,$$

or what amounts to the same thing,

$$\log\left(\frac{1}{u_n}\right) < a\log(n),$$

and, as a consequence,

$$\frac{1}{u_n} < n^a, \quad \text{so} \quad u_n > \frac{1}{n^a}.$$

It follows that the terms of series (1) eventually are constantly [127] greater than the corresponding terms of the following series

$$1,\ \frac{1}{2^a},\ \frac{1}{3^a},\ \frac{1}{4^a},\ \ldots,\ \frac{1}{n^a},\ \frac{1}{(n+1)^a},\ \ldots.$$

As this last series is convergent (because $a < 1$), we may, by the remark we have just made, conclude *a fortiori* the divergence of series (1).

Given two convergent series, the terms of which are positive, we may, by adding or multiplying these same terms, form a new series, the sum of which results from the addition or the multiplication of the sums of the first two. On this subject, we establish the two following theorems:

Theorem V. — *Let*

$$(7) \qquad \begin{cases} u_0, & u_1, & u_2, & \ldots, & u_n, & \ldots, \\ v_0, & v_1, & v_2, & \ldots, & v_n, & \ldots \end{cases}$$

be two convergent series composed only of positive terms, having s and s', respectively, as sums. Then

$$(8) \qquad u_0+v_0,\ u_1+v_1,\ u_2+v_2,\ \dots,\ u_n+v_n,\ \dots$$

is a new convergent series, which has $s+s'$ as its sum.

Proof. — If we let

$$\begin{aligned} s_n &= u_0+u_1+u_2+\ldots+u_{n-1} \quad \text{and} \\ s'_n &= v_0+v_1+v_2+\ldots+v_{n-1}, \end{aligned}$$

then s_n and s'_n converge, for increasing values of n, towards the limits s and s', respectively. As a consequence, $s_n+s'_n$, that is the sum of the first n terms of series (8), converges towards the limit $s+s'$, which suffices to establish the stated theorem.

Theorem VI. — *Under the same hypotheses as the previous theorem,*

$$(9) \qquad \begin{cases} u_0v_0,\ u_0v_1+u_1v_0,\ u_0v_2+u_1v_1+u_2v_0,\ \dots \\ \dots,\ u_0v_n+u_1v_{n-1}+\ldots+u_{n-1}v_1+u_nv_0,\ \dots \end{cases}$$

is a new convergent series, which has ss' as its sum.

[128] *Proof.* — Once again, let s_n and s'_n be the sums of the first n terms of the two series (7), and additionally denote the sum of the first n terms of series (9) by s''_n. If we denote by m the greatest integer included in $\frac{n-1}{2}$, that is to say $\frac{n-1}{2}$ when n is odd and $\frac{n-2}{2}$ otherwise, we clearly have[6]

$$\begin{aligned} &u_0v_0+(u_0v_1+u_1v_0)+\ldots \\ &\quad +\ (u_0v_{n-1}+u_1v_{n-2}+\ldots+u_{n-2}v_1+u_{n-1}v_0) \\ &\quad\quad <\ (u_0+u_1+\ldots+u_{n-1})(v_0+v_1+\ldots+v_{n-1}) \end{aligned}$$

and

$$>\ (u_0+u_1+\ldots+u_m)(v_0+v_1+\ldots+v_m).$$

[6]The following formula for s''_n contained some subscripting errors in the 1821 edition, which were not included in the *Errata* of that edition. The editors of the 1911 edition corrected these.

In other words,

$$s''_n < s_n s'_n \quad \text{and} \quad > s_{m+1} s'_{m+1}.$$

Now suppose that we make n increase beyond all limit. The number

$$m = \frac{n - \frac{3}{2} \pm \frac{1}{2}}{2}$$

itself increases indefinitely, and the two sums s_n and s_{m+1} converge towards the limit s, while s'_n and s'_{m+1} converge towards the limit s'. As a consequence, the two products $s_n s'_n$ and $s_{m+1} s'_{m+1}$, as well as the sum s''_n contained between these two products, converge towards the limit ss', which suffices to establish theorem VI.[7]

[7] This is another implicit application of the Squeeze Theorem.