[114]

Chapter VI.

On convergent and divergent series. Rules for the convergence of series. The summation of several convergent series.[1]

---

§ I. — *General considerations on series.*

We call a *series* an indefinite sequence of quantities,

$$u_0,\ u_1,\ u_2,\ u_3,\ \ldots,$$

which follow from one to another according to a determined law. These quantities themselves are the various *terms* of the series under consideration. Let

$$s_n = u_0 + u_1 + u_2 + \ldots + u_{n-1}$$

be the sum of the first n terms, where n denotes any integer number. If, for ever increasing values of n, the sum s_n indefinitely approaches a certain limit s, the series is said to be *convergent*, and the limit in question is called the *sum* of the series. On the contrary, if the sum s_n does not approach any fixed limit as n increases indefinitely, the series is *divergent*, and does not have a sum. In either case, the term which corresponds to the index n, that is u_n, is what we call the *general term*. For the series to be completely determined, it is enough that we give this general term as a function of the index n.

One of the simplest series is the geometric progression,

$$1,\ x,\ x^2,\ x^3,\ \ldots,$$

which has x^n for its general term, that is to say the nth power of the quantity [115] x. If we form the sum of the first n terms of this series, then we find

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x}.$$

[1]This chapter title is slightly different than what appears in the Table of Contents.

As the values of n increase, the numerical value of the fraction $\frac{x^n}{1-x}$ converges towards the limit zero, or increases beyond all limits, according to whether we suppose that the numerical value of x is less than or greater than one. Under the first hypothesis, we ought to conclude that the progression

$$1,\ x,\ x^2,\ x^3,\ \ldots$$

is a convergent series which has $\frac{1}{1-x}$ as its sum, whereas, under the second hypothesis, the same progression is a divergent series which does not have a sum.

Following the principles established above, in order that the series

$$u_0,\ u_1,\ u_2,\ \ldots,\ u_n,\ u_{n+1},\ \ldots \tag{1}$$

be convergent, it is necessary and it suffices that increasing values of n make the sum

$$s_n = u_0 + u_1 + u_2 + \ldots + u_{n-1}$$

converge indefinitely towards a fixed limit s. In other words, it is necessary and it suffices that, for infinitely large values of the number n, the sums

$$s_n,\ s_{n+1},\ s_{n+2},\ \ldots$$

differ from the limit s, and consequently from one another, by infinitely small quantities. Moreover, the successive differences between the first sum s_n and each of the following sums are determined respectively by the equations

$$\begin{aligned} s_{n+1} - s_n &= u_n, \\ s_{n+2} - s_n &= u_n + u_{n+1}, \\ s_{n+3} - s_n &= u_n + u_{n+1} + u_{n+2}, \\ \ldots\ldots\ldots &\ \ldots\ \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Hence, in order for series (1) to be convergent, it is first of all necessary [116] that the general term u_n decrease indefinitely as

n increases. But this condition does not suffice, and it is also necessary that, for increasing values of n, the different sums,

$$\begin{array}{l} u_n + u_{n+1}, \\ u_n + u_{n+1} + u_{n+2}, \\ \dots\dots\dots\dots\dots\dots, \end{array}$$

that is to say, the sums of as many of the quantities

$$u_n,\ u_{n+1},\ u_{n+2},\ \dots,$$

as we may wish, beginning with the first one, eventually constantly assume numerical values less than any assignable limit. Conversely, whenever these various conditions are fulfilled, the convergence of the series is guaranteed.[2]

Let us take, for example, the geometric progression

$$1,\ x,\ x^2,\ x^3,\ \dots. \tag{2}$$

If the numerical value of x is greater than one, that of the general term x^n increases indefinitely with n, and this remark alone suffices to establish the divergence of the series. The series is still divergent if we let $x = \pm 1$, because the numerical value of the general term x^n, which is one, does not decrease indefinitely for increasing values of n. However, if the numerical value of x is less than one, then the sums of any number of terms of the series, beginning with x^n, namely:

$$\begin{array}{rcl} x^n, & & \\ x^n + x^{n+1} & = & x^n \dfrac{1-x^2}{1-x}, \\ x^n + x^{n+1} + x^{n+2} & = & x^n \dfrac{1-x^3}{1-x}, \\ \dots\dots\dots\dots\dots\dots & \dots & \dots\dots\dots, \end{array}$$

are all contained between the limits

$$x^n \quad \text{and} \quad \frac{x^n}{1-x},$$

[2]This is the Cauchy convergence criterion. Even today this is one of the few necessary and sufficient conditions for convergence of series.

[117] each of which becomes infinitely small for infinitely large values of n. Consequently, the series is convergent, as we already knew.

As a second example, let us take the numerical series

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \frac{1}{n+1},\ \ldots. \tag{3}$$

The general term of this series, namely $\frac{1}{n+1}$, decreases indefinitely as n increases. Nevertheless, the series is not convergent, because the sum of the terms from $\frac{1}{n+1}$ up to $\frac{1}{2n}$ inclusive, namely

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n-1}+\frac{1}{2n},$$

is always greater than the product

$$n\frac{1}{2n}=\frac{1}{2},$$

whatever the value of n. As a consequence, this sum does not decrease indefinitely with increasing values of n, as would be the case if the series were convergent. Let us add that, if we denote the sum of the first n terms of series (3) by s_n and the highest power of 2 bounded by $n+1$ by 2^m, then we have

$$\begin{aligned} s_n &= 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1} \\ &> 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots \\ &\quad +\left(\frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+\ldots+\frac{1}{2^m}\right), \end{aligned}$$

and, *a fortiori*,

$$s_n > 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}=1+\frac{m}{2}.$$

We conclude from this that the sum s_n increases indefinitely with the integer number m, and consequently with n, which is a new proof of the divergence of the series.[3]

[3]Cauchy is probably not claiming originality for this "new" proof, which was first given by Oresme. Presumably he is simply observing that this is a different proof of divergence than the one he gave earlier in this paragraph.

[118] Let us further consider the numerical series

$$1,\ \frac{1}{1},\ \frac{1}{1\cdot 2},\ \frac{1}{1\cdot 2\cdot 3},\ \dots,\ \frac{1}{1\cdot 2\cdot 3\dots n},\ \dots. \tag{4}$$

The terms of this series with index greater than n, namely

$$\frac{1}{1\cdot 2\cdot 3\dots n},\ \frac{1}{1\cdot 2\cdot 3\dots n(n+1)},\ \frac{1}{1\cdot 2\cdot 3\dots n(n+1)(n+2)},\ \dots,$$

are respectively less than the corresponding terms of the geometric progression

$$\frac{1}{1\cdot 2\cdot 3\dots n},\ \frac{1}{1\cdot 2\cdot 3\dots n}\frac{1}{n},\ \frac{1}{1\cdot 2\cdot 3\dots n}\frac{1}{n^2},\ \dots.$$

As a consequence, the sum of however many of the initial terms as we may wish is always be less than the sum of the corresponding terms of the geometric progression, which is a convergent series, and so *a fortiori*,[4] it is less than the sum of this series, which is to say

$$\frac{1}{1\cdot 2\cdot 3\dots n}\frac{1}{1-\frac{1}{n}} = \frac{1}{1\cdot 2\cdot 3\dots(n-1)}\frac{1}{n-1}.$$

Because this last sum decreases indefinitely as n increases, it follows that series (4) is itself convergent. We agree to denote the sum of this series by the letter e. By adding together the first n terms, we obtain an approximate value of the number e,

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\dots+\frac{1}{1\cdot 2\cdot 3\dots(n-1)}.$$

According to what we have just said, the error made will be smaller than the product of the nth term by $\frac{1}{n-1}$. Therefore, for example, if we let $n = 11$, we find as the approximate value of e

$$e = 2.7182818\dots, \tag{5}$$

and the error made in this approximation is less than the product [119] of the fraction $\frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}$ by $\frac{1}{10}$, that is $\frac{1}{36288000}$, so that it does not affect the seventh decimal place.

[4] This is an implicit use of the comparison test. Cauchy never states this test explicitly.

The number e, determined as we have just said, is often be used in the summation of series and in the infinitesimal Calculus. Logarithms taken in the system with this number as its base are called *Naperian*, for *Napier*, the inventor of logarithms, or *hyperbolic*, because they measure the various parts of the area between the equilateral hyperbola and its asymptotes.[5]

In general, we denote the sum of a convergent series by the sum of the first terms, followed by ellipses. Thus, when the series

$$u_0,\ u_1,\ u_2,\ u_3,\ \dots$$

is convergent, the sum of this series is denoted

$$u_0 + u_1 + u_2 + u_3 + \dots.$$

By virtue of this convention, the value of the number e is determined by the equation

$$e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \dots, \tag{6}$$

and, if one considers the geometric progression

$$1,\ x,\ x^2,\ x^3,\ \dots,$$

we have, for numerical values of x less than one,

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}. \tag{7}$$

Denoting the sum of the convergent series

$$u_0,\ u_1,\ u_2,\ u_3,\ \dots$$

by s and the sum of the first n terms by s_n, we have

$$\begin{aligned} s &= u_0 + u_1 + u_2 + \dots + u_{n-1} + u_n + u_{n+1} + \dots \\ &= s_n + u_n + u_{n+1} + \dots, \end{aligned}$$

and, as a consequence,

$$s - s_n = u_n + u_{n+1} + \dots.$$

[5] The curves with equations $y = \frac{1}{x}$ and $x^2 - y^2 = 1$ are examples of equilateral hyperbolas, also called rectangular or right hyperbolas.

[120] From this last equation, it follows that the quantities

$$u_n,\ u_{n+1},\ u_{n+2},\ \ldots$$

forms a new convergent series, the sum of which is equal to $s - s_n$. If we represent this sum by r_n, we have

$$s = s_n + r_n,$$

and r_n is called the *remainder* of series (1) beginning from the nth term.

Suppose the terms of series (1) involve some variable x. If the series is convergent and its various terms are continuous functions of x in a neighborhood of some particular value of this variable, then

$$s_n,\ r_n \text{ and } s$$

are also three functions of the variable x, the first of which is obviously continuous with repect to x in a neighborhood of the particular value in question. Given this, let us consider the increments in these three functions when we increase x by an infinitely small quantity α. For all possible values of n, the increment in s_n is an infinitely small quantity. The increment of r_n, as well as r_n itself, becomes infinitely small for very large values of n. Consequently, the increment in the function s must be infinitely small.[6] From this remark, we immediately deduce the following proposition:

Theorem I. — *When the various terms of series* (1) *are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value.*[7]

By virtue of this theorem, the sum of series (2) must be a continuous function of the variable x between the limits $x = -1$

[6] This passage is quoted in [Lutzen 2003, p. 168].

[7] This theorem as stated is incorrect. If we impose the additional condition of uniform convergence of the functions s_n, then it does hold. Some have argued that Cauchy really had uniform convergence in mind, although this seems unlikely. See [Lutzen 2003, pp. 168-169] for further discussion.

and $x = 1$, [121] as we may verify by considering the values of s given by the equation

$$s = \frac{1}{1 - x}.$$