[106]

§ II. — *Research on a continuous function formed in such a manner that in multiplying two such functions of variable quantities, and then doubling the product, we get a result equal to that which we obtain by adding such functions of the sum and of the difference of these variables.*

In each of the problems of the preceding section, the equation to be solved contained, along with the unknown function $\varphi(x)$, two other similar functions, namely $\varphi(y)$ and $\varphi(x+y)$ or $\varphi(xy)$. Now we are going to propose a new problem of the same kind, but in which the equation of the condition that the function $\varphi(x)$ must satisfy contains four such functions in place of three. It consists of the following:

Problem. — *To determine the function $\varphi(x)$ in such a manner that it remains continuous between any two real limits of the variable x and so that for all real values of the variables x and y we have*

$$\varphi(y+x)+\varphi(y-x)=2\varphi(x)\,\varphi(y)\,. \tag{1}$$

Solution. — If we make $x=0$ in equation (1), we get[1]

$$\varphi(0)=1.$$

The function $\varphi(x)$ thus reduces to one for the particular value $x=0$, and because we suppose that it is continuous between any limits, it is clear that, in the neighborhood of this particular value, it is only very slightly different from one, and consequently is positive. Thus, by denoting a very small number by α, we can choose this number in such a way that the function $\varphi(x)$ remains constantly positive between the limits

$$x=0 \quad \text{and} \quad x=\alpha.$$

Given this, two things could happen: either the positive value of $\varphi(x)$ will be contained between the limits 0 and 1, or this value

[1] Cauchy ignores the trivial solution $\varphi(x)\equiv 0$.

will be [107] greater than one. We will examine successively these two hypotheses.

Now suppose that $\varphi(\alpha)$ has a value contained between the limits 0 and 1. We can represent this value by the cosine of a certain arc θ contained between the limits 0 and $\frac{\pi}{2}$, and as a consequence we can set

$$\varphi(\alpha) = \cos\theta.$$

Moreover, if in equation (1) put into the form

$$\varphi(y+x) = 2\varphi(x)\,\varphi(y) - \varphi(y-x)\,,$$

we successively make

$$\begin{array}{lll} x=\alpha & \text{and} & y=\alpha, \\ x=\alpha & \text{and} & y=2\alpha, \\ x=\alpha & \text{and} & y=3\alpha, \\ \cdots\cdots & & \cdots\cdots, \end{array}$$

then we deduce the formulas

$$\begin{aligned} \varphi(2\alpha) &= 2\cos^2\theta - 1 = \cos 2\theta, \\ \varphi(3\alpha) &= 2\cos\theta\cos 2\theta - \cos\theta = \cos 3\theta, \\ \varphi(4\alpha) &= 2\cos\theta\cos 3\theta - \cos 2\theta = \cos 4\theta, \end{aligned}$$

one after another and in general,

$$\varphi(m\alpha) = 2\cos\theta\cos(m-1)\,\theta - \cos(m-2)\,\theta = \cos m\theta,$$

where m denotes any integer number. I add that the formula

$$\varphi(m\alpha) = \cos m\theta$$

remains true even if we replace the integer number m by a fraction or even by any number μ. We will prove this easily as follows.

If we make $x = \frac{1}{2}\alpha$ and $y = \frac{1}{2}\alpha$ in equation (1), then we get

$$\left[\varphi\left(\frac{1}{2}\alpha\right)\right]^2 = \frac{\varphi(0)+\varphi(\alpha)}{2} = \frac{1+\cos\theta}{2} = \left(\cos\frac{1}{2}\theta\right)^2.$$

Then, by taking the positive roots of both sides and [108] observing that the two functions $\varphi(x)$ and $\cos x$ remain positive, the first between the limits $x = 0$ and $x = \alpha$ and the second between the limits $x = 0$ and $x = \theta$, we find

$$\varphi\left(\frac{1}{2}\alpha\right) = \cos\frac{1}{2}\theta.$$

Likewise, if we make

$$x = \frac{1}{4}\alpha \quad \text{and} \quad y = \frac{1}{4}\theta$$

in equation (1), then we get[2]

$$\left[\varphi\left(\frac{1}{4}\alpha\right)\right]^2 = \frac{\varphi(0) + \varphi\left(\frac{1}{2}\alpha\right)}{2} = \frac{1 + \cos\frac{1}{2}\theta}{2} = \left(\cos\frac{1}{4}\theta\right)^2.$$

Then, by extracting the positive roots of the first and last parts, we get

$$\varphi\left(\frac{1}{4}\alpha\right) = \cos\frac{1}{4}\theta.$$

By similar reasoning, we successively obtain the formulas

$$\begin{aligned} \varphi\left(\frac{1}{8}\alpha\right) &= \cos\frac{1}{8}\theta, \\ \varphi\left(\frac{1}{16}\alpha\right) &= \cos\frac{1}{16}\theta, \\ \cdots\cdots&\cdots\cdots, \end{aligned}$$

and in general

$$\varphi\left(\frac{1}{2^n}\alpha\right) = \cos\frac{1}{2^n}\theta,$$

where n denotes any integer number. If we operate on the preceding expression for $\varphi\left(\frac{1}{2^n}\alpha\right)$ to deduce that for $\varphi\left(\frac{m}{2^n}\alpha\right)$ as we operated on the expression for $\varphi(\alpha)$ to deduce that for $\varphi(m\alpha)$, then we find

$$\varphi\left(\frac{m}{2^n}\alpha\right) = \cos\frac{m}{2^n}\theta.$$

[2] In the 1911 edition, the numerator of the second part contains the expression $\varphi\left(\frac{1}{2}\right)\alpha$ in place of $\varphi\left(\frac{1}{2}\alpha\right)$. This error did not appear in the 1821 edition. (tr.)

Then, by supposing that the fraction $\frac{m}{2^n}$ varies in such a way as to approach [109] indefinitely the number μ, and passing to the limit, we obtain the equation

$$\varphi(\mu\alpha) = \cos\mu\theta. \tag{2}$$

Moreover, if we make[3]

$$x = \mu\alpha \quad \text{and} \quad y = 0$$

in formula (1), then we conclude that

$$\varphi(-\mu\alpha) = [2\varphi(0) - 1]\,\varphi(\mu\alpha) = \cos\mu\theta = \cos(-\mu\theta).$$

Thus, equation (2) remains true when we replace μ by $-\mu$. In other words, we have, for any values, positive or negative, of the variable x,

$$\varphi(\alpha x) = \cos\theta x. \tag{3}$$

If we change x to $\frac{x}{\alpha}$ in this last formula, we get

$$\varphi(x) = \cos\frac{\theta}{\alpha}x = \cos\left(-\frac{\theta}{\alpha}x\right). \tag{4}$$

The preceding value of $\varphi(x)$ corresponds to the case where the positive quantity $\varphi(x)$ remains contained between the limits 0 and 1. Now let us suppose that this same quantity is greater than one. It is easy to see that under this second hypothesis we can find a positive value of r that satisfies the equation

$$\varphi(\alpha) = \frac{1}{2}\left(r + \frac{1}{r}\right).$$

Indeed, it suffices to take

$$r = \varphi(\alpha) + \left\{[\varphi(\alpha)]^2 - 1\right\}^{\frac{1}{2}}.$$

[3] In the 1911 edition, this reads $x = \mu a$. It is correctly written $x = \mu\alpha$ in the 1821 edition. (tr.)

Given this, if we successively make

$$\begin{array}{lcl} x=\alpha & \text{and} & y=\alpha, \\ x=\alpha & \text{and} & y=2\alpha, \\ x=\alpha & \text{and} & y=3\alpha, \\ \cdots\cdots & & \cdots\cdots, \end{array}$$

in equation (1), [110] we then deduce the formulas

$$\varphi(2\alpha)=\tfrac{1}{2}\left(r+\tfrac{1}{r}\right)^2-1=\tfrac{1}{2}\left(r^2+\tfrac{1}{r^2}\right),$$

$$\varphi(3\alpha)=\tfrac{1}{2}\left(r+\tfrac{1}{r}\right)\left(r^2+\tfrac{1}{r^2}\right)-\tfrac{1}{2}\left(r+\tfrac{1}{r}\right)=\tfrac{1}{2}\left(r^3+\tfrac{1}{r^3}\right),$$

$$\varphi(4\alpha)=\tfrac{1}{2}\left(r+\tfrac{1}{r}\right)\left(r^3+\tfrac{1}{r^3}\right)-\tfrac{1}{2}\left(r^2+\tfrac{1}{r^2}\right)=\tfrac{1}{2}\left(r^4+\tfrac{1}{r^4}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

one after another. In general,

$$\begin{aligned}\varphi(m\alpha)&=\frac{1}{2}\left(r+\frac{1}{r}\right)\left(r^{m-1}+\frac{1}{r^{m-1}}\right)-\frac{1}{2}\left(r^{m-2}+\frac{1}{r^{m-2}}\right)\\&=\frac{1}{2}\left(r^m+\frac{1}{r^m}\right),\end{aligned}$$

where m denotes any integer number. I add that the formula

$$\varphi(m\alpha)=\frac{1}{2}\left(r^m+\frac{1}{r^m}\right)$$

remains true even if we replace the integer number m by a fraction or even by any number μ. We will prove this easily as follows.

If we make $x=\frac{1}{2}\alpha$ and $y=\frac{1}{2}\alpha$ in equation (1), we get

$$\left[\varphi\left(\frac{1}{2}\alpha\right)\right]^2=\frac{\varphi(0)+\varphi(\alpha)}{2}=\frac{1+\frac{1}{2}\left(r+\frac{1}{r}\right)}{2}=\frac{1}{4}\left(r^{\frac{1}{2}}+r^{-\frac{1}{2}}\right).$$

Then, by taking the positive roots of both sides and observing that the function $\varphi(x)$ remains positive between the limits $x=0$ and $x=\alpha$, we find

$$\varphi\left(\frac{1}{2}\alpha\right)=\frac{1}{2}\left(r^{\frac{1}{2}}+r^{-\frac{1}{2}}\right).$$

Likewise, if we make

$$x = \frac{1}{4}\alpha \quad \text{and} \quad y = \frac{1}{4}\alpha$$

in equation (1), then we get[4]

$$\left[\varphi\left(\frac{1}{4}\alpha\right)\right]^2 = \frac{\varphi(0) + \varphi\left(\frac{1}{2}\alpha\right)}{2}$$

$$= \frac{1 + \frac{1}{2}\left(r^{\frac{1}{2}} + r^{-\frac{1}{2}}\right)}{2} = \frac{1}{4}\left(r^{\frac{1}{4}} + r^{-\frac{1}{4}}\right)^2.$$

[111] Then, by taking the positive roots of the first and the last parts, we get

$$\varphi\left(\frac{1}{4}\alpha\right) = \frac{1}{2}\left(r^{\frac{1}{4}} + r^{-\frac{1}{4}}\right).$$

By similar reasoning, we successively obtain the formulas

$$\varphi\left(\frac{1}{8}\alpha\right) = \frac{1}{2}\left(r^{\frac{1}{8}} + r^{-\frac{1}{8}}\right),$$
$$\varphi\left(\frac{1}{16}\alpha\right) = \frac{1}{2}\left(r^{\frac{1}{16}} + r^{-\frac{1}{16}}\right),$$
$$\dots\dots\dots\dots\dots\dots,$$

and in general

$$\varphi\left(\frac{1}{2^n}\alpha\right) = \frac{1}{2}\left(r^{\frac{1}{2^n}} + r^{-\frac{1}{2^n}}\right),$$

where n denotes any integer number. If we operate on the preceding expression for $\varphi\left(\frac{1}{2^n}\alpha\right)$ to deduce that for $\varphi\left(\frac{m}{2^n}\alpha\right)$ as we operated on the expression for $\varphi(\alpha)$ to deduce that for[5] $\varphi(m\alpha)$, then we find

$$\varphi\left(\frac{m}{2^n}\alpha\right) = \frac{1}{2}\left(r^{\frac{m}{2^n}} + r^{-\frac{m}{2^n}}\right).$$

[4]The negative signs were missing from the exponents $-\frac{1}{2}$ and $-\frac{1}{4}$ in the 1911 edition. They were present in the 1821 edition. (tr.)

[5]This word "for" is the translation of the word "de," which was present in the 1821 edition but absent in the 1911 edition. (tr.)

Then, by supposing that the fraction $\frac{m}{2^n}$ varies in such a way as to approach indefinitely the number μ, and passing to the limit, we obtain the equation

$$\varphi(\mu\alpha) = \frac{1}{2}\left(r^{\mu} + r^{-\mu}\right). \tag{5}$$

Moreover, if we make

$$x = \mu\alpha \quad \text{and} \quad y = 0$$

in formula (1), then we conclude that

$$\varphi(-\mu\alpha) = [2\varphi(0) - 1]\,\varphi(\mu\alpha) = \frac{1}{2}\left(r^{-\mu} + r^{\mu}\right).$$

[112] Thus, equation (5) remains true when we replace μ by $-\mu$. In other words, we have, for all values, positive or negative, of the variable x,

$$\varphi(\alpha x) = \frac{1}{2}\left(r^{x} + r^{-x}\right). \tag{6}$$

If we change x to $\frac{x}{\alpha}$ in this last formula, we get

$$\varphi(x) = \frac{1}{2}\left(r^{\frac{x}{\alpha}} + r^{-\frac{x}{\alpha}}\right). \tag{7}$$

When we make $\pm\frac{\theta}{\alpha} = a$ in equation (4) and $r^{\pm\frac{1}{\alpha}}A$ in equation (7), these equations give respectively the following forms:

$$\varphi(x) = \cos ax \quad \text{and} \tag{8}$$

$$\varphi(x) = \frac{1}{2}\left(A^{x} + A^{-x}\right). \tag{9}$$

Thus, if we denote a constant quantity by a and a constant number by A, then any function $\varphi(x)$ that remains continuous between any limits of the variable and that satisfies equation (1) is necessarily contained in one of the two forms that we have just described. Moreover, it is easy to assure ourselves that the values of $\varphi(x)$ given by equations (8) and (9) solve the proposed question, whatever values may be attributed to the quantity a and the number A. This number and this quantity are two arbitrary constants, of which one admits only positive quantities.

From what we have just said, the two functions

$$\cos ax \quad \text{and} \quad \frac{1}{2}\left(A^x + A^{-x}\right)$$

have the common property of satisfying equation (1), and this establishes a remarkable analogy between them. Both of these two [113] functions still reduce to one for $x = 0$. But one essential difference between the first and the second is that the numerical value of the first is constantly less than the limit 1, whenever it does not reach this limit, while, under the same hypothesis, the numerical value of the second is constantly above the limit 1.