[98]

Chapter V.

Determination of continuous functions of a single variable that satisfy certain conditions.

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§ I. — *Research on a continuous function formed in such a manner that two such functions of variable quantities being added or multiplied together give for their sum or their product such a function of the sum or of the product of these variables.*

When, instead of integer functions we imagine any functions, so that we leave the form entirely arbitrary, we can no longer successfully determine them given a certain number of particular values, however large that number might be, but we can sometimes do so in the case where we assume certain general properties of these functions. For example, a continuous function of x, represented by $\varphi(x)$, can be completely determined when it is required to satisfy, for all possible values of the variables x and y, one of the equations

$$\varphi(x+y) = \varphi(x) + \varphi(y) \quad \text{or} \tag{1}$$

$$\varphi(x+y) = \varphi(x) \times \varphi(y), \tag{2}$$

as well as when, for all positive real values of the same variables, one of the following equations:

$$\varphi(xy) = \varphi(x) + \varphi(y) \quad \text{or} \tag{3}$$

$$\varphi(xy) = \varphi(x) \times \varphi(y). \tag{4}$$

The solution of these four equations presents four different problems, which we will treat one after another.

[99] **Problem I.** — *To determine the function $\varphi(x)$ in such a manner that it remains continuous between any two real limits of the variable x and so that for all real values of the variables x and y, we have*

$$\varphi(x+y) = \varphi(x) + \varphi(y). \tag{1}$$

Solution. — If in equation (1) we successively replace y by $y+z$, z by $z+u$, ..., we get

$$\varphi(x+y+z+u+\ldots) = \varphi(x) + \varphi(y) + \varphi(z) + \varphi(u) + \ldots,$$

however many variables x, y, z, u, ... there may be. Also, if we denote this number of variables by m and a positive constant by α, and then we make

$$x = y = z = u = \ldots = \alpha,$$

then the formula which we are have just found becomes

$$\varphi(m\alpha) = m\varphi(\alpha).$$

To extend this last equation to the case where the integer number m is replaced by a fractional number $\frac{m}{n}$, or even by an arbitrary number μ, we set, in the first case,

$$\beta = \frac{m}{n}\alpha,$$

where m and n denote integer numbers, and we conclude that

$$\begin{gathered} n\beta = m\alpha, \\ n\varphi(\beta) = m\varphi(\alpha) \quad \text{and} \\ \varphi(\beta) = \varphi\left(\tfrac{m}{n}\alpha\right) = \tfrac{m}{n}\varphi(\alpha). \end{gathered}$$

Then, by supposing that the fraction $\frac{m}{n}$ varies in such a way as to converge towards any number μ, and passing to the limit, we find that

$$\varphi(\mu\alpha) = \mu\varphi(\alpha).$$

[100] Now, if we now take $\alpha = 1$, then we have, for all positive values of μ,

$$\varphi(\mu) = \mu\varphi(1), \tag{5}$$

and consequently, by making μ converge towards the limit zero,

$$\varphi(0) = 0.$$

Moreover, if in equation (1) we set $x = \mu$ and $y = -\mu$, we conclude that

$$\varphi(-\mu) = \varphi(0) - \varphi(\mu) = -\mu\varphi(1).$$

Thus, equation (5) remains true when we change μ to $-\mu$. In other words, we have, for any values, positive or negative, of the variable x,

$$\varphi(x) = x\varphi(1). \tag{6}$$

It follows from formula (6) that any function $\varphi(x)$ which remains continuous between any limits of the variable and satisfies equation (1) is necessarily of the form

$$\varphi(x) = ax, \tag{7}$$

where a denotes a constant quantity. I add that the function ax enjoys the stated properties whatever the value of the constant a may be. Indeed, between any limits of the variable x, the product ax is a continuous function of that variable, and what's more, the assumption that $\varphi(x) = ax$ changes equation (1) into this other one,

$$a(x+y) = ax + ay,$$

which is evidently always an identity. Thus formula (7) gives a solution to the proposed question, whatever value is attributed to the constant a. The ability which we have to choose this constant arbitrarily has caused it to be given the name of an *arbitrary constant.*

Problem II. — *To determine the function $\varphi(x)$ in such a manner that it remains continuous between any two real limits of the variable x and so that* [101] *for all real values of the variables x and y, we have*

$$\varphi(x+y) = \varphi(x)\varphi(y). \tag{2}$$

Solution. — First, it is easy to assure ourselves that the function $\varphi(x)$ required to satisfy equation (2) will admit only positive values. Indeed, if we make $y = x$ in equation (2), we find that

$$\varphi(2x) = [\varphi(x)]^2,$$

and then, writing $\frac{1}{2}x$ in place of x, we conclude that

$$\varphi(x) = \left[\varphi\left(\tfrac{1}{2}x\right)\right]^2.$$

Thus the function $\varphi(x)$ is always equal to a square, and consequently it is always positive. Given this, suppose that in equation (2) we successively replace y by $y+z$, z by $z+u$, ... We then get

$$\varphi(x+y+z+u+\ldots) = \varphi(x)\,\varphi(y)\,\varphi(z)\,\varphi(u)\ldots,$$

however many variables x, y, z, u, ... there may be. Also, if we denote this number of variables by m, and a positive constant by α, and then we make

$$x = y = z = u = \ldots = \alpha,$$

then the formula we have just found becomes

$$\varphi(m\alpha) = [\varphi(\alpha)]^m.$$

To extend this last formula to the case where the integer number m is replaced by a fractional number $\frac{m}{n}$, or even by an arbitrary number μ, we set, in the first case,

$$\beta = \frac{m}{n}\alpha,$$

where m and n denote two integer numbers, and we conclude that

$$n\beta = m\alpha,$$
$$[\varphi(\beta)]^n = [\varphi(\alpha)]^m \quad \text{and}$$
$$\varphi(\beta) = \varphi\left(\tfrac{m}{n}\alpha\right) = [\varphi(\alpha)]^{\frac{m}{n}}.$$

[102] Then, by supposing that the fraction $\frac{m}{n}$ varies in such a way as to converge towards any number μ, and passing to the limit, we find that

$$\varphi(\mu\alpha) = [\varphi(\alpha)]^\mu.$$

Now if we take $\alpha = 1$, we have for all positive values of μ

$$\varphi(\mu) = [\varphi(1)]^\mu, \tag{8}$$

and consequently, by making μ converge towards the limit zero,

$$\varphi(0) = 1.$$

Moreover, if in equation (2) we set $x = \mu$ and $y = -\mu$, we conclude that

$$\varphi(-\mu) = \frac{\varphi(0)}{\varphi(\mu)} = [\varphi(1)]^{-\mu}.$$

Thus, equation (8) remains true when we change μ to $-\mu$. In other words, we have, for any values, positive or negative, of the variable x,

$$\varphi(x) = [\varphi(1)]^{x}. \tag{9}$$

It follows from equation (9) that any function $\varphi(x)$ that solves the second problem is necessarily of the form

$$\varphi(x) = A^{x}, \tag{10}$$

where A denotes a positive constant. I add that we can attribute to this constant any value between the limits 0 and ∞. Indeed, for any positive value of A, the function A^x remains continuous from $x = -\infty$ to $x = +\infty$, and the equation

$$A^{x+y} = A^x A^y$$

is an identity. The quantity A is thus an arbitrary constant that admits only positive values.

[103] *Note.* — We can get equation (9) very simply in the following manner.

If we take logarithms of both sides of equation (2) in any system, we find that

$$\log\varphi(x+y) = \log\varphi(x) + \log\varphi(y),$$

and we conclude (see problem I) that

$$\log\varphi(x) = x\log\varphi(1),$$

then, by passing again from logarithms to numbers,

$$\varphi(x) = [\varphi(1)]^{x}.$$

Problem III. — *To determine the function $\varphi(x)$ in such a manner that it remains continuous between any two positive limits* *of the variable x and so that for all positive values of the variables x and y we have*

$$\varphi(xy) = \varphi(x) + \varphi(y). \tag{3}$$

Solution. — It would be easy to apply a method similar to the one we used to solve the first problem to the solution of problem III. However, we will arrive more promptly at the solution we seek by putting equation (3) into a form analogous to that of equation (1), as we are going to do.

If A denotes any number and log denotes the characteristic of logarithms in the system for which the base is A, then for all positive values of the variables x and y we have

$$x = A^{\log x} \quad \text{and} \quad y = A^{\log y},$$

so that equation (3) becomes

$$\varphi\left(A^{\log x + \log y}\right) = \varphi\left(A^{\log x}\right) + \varphi\left(A^{\log y}\right).$$

Because in this last formula the variable quantities $\log x$ and $\log y$ admit any values, positive or negative, it follows [104] that we have, for all possible real values of x and y,

$$\varphi\left(A^{x+y}\right) = \varphi\left(A^{x}\right) + \varphi\left(A^{y}\right).$$

We conclude that [see problem I, eqn. (6)]

$$\varphi\left(A^{x}\right) = x\varphi\left(A^{1}\right) = x\varphi(A),$$

and consequently

$$\varphi\left(A^{\log x}\right) = \varphi(A)\log x,$$

or what amounts to the same thing

$$\varphi(x) = \varphi(A)\log x. \tag{11}$$

It follows from formula (11) that every function $\varphi(x)$ that solves problem III is necessarily of the form

$$\varphi(x) = a\log(x), \tag{12}$$

where a denotes a constant. Moreover, it is easy to assure ourselves: 1° that the constant a remains entirely arbitrary, and 2° that by choosing the number A suitably, which is itself arbitrary, we can reduce the constant a to one.

Problem IV. — *To determine the function $\varphi(x)$ in such a manner that it remains continuous between any two positive limits of the variable x and so that for all positive values of the variables x and y we have*

$$\varphi(xy) = \varphi(x)\,\varphi(y)\,. \tag{4}$$

Solution. — It would be easy to apply a method similar to that which we used to solve the second problem to the solution of problem IV. However, we will arrive more promptly at the solution we seek if we observe that, by denoting by log the characteristic of logarithms in the system for which the base is A, we can put equation (4) into the form

$$\varphi\left(A^{\log x+\log y}\right) = \varphi\left(A^{\log x}\right)\varphi\left(A^{\log y}\right).$$

Because in this last equation the variable quantities $\log x$ [105] and $\log y$ admit any values, positive or negative, it follows that we have, for all possible real values of the variables x and y,

$$\varphi\left(A^{x+y}\right) = \varphi\left(A^{x}\right)\varphi\left(A^{y}\right).$$

We conclude that [see problem II, eqn. (9)]

$$\varphi\left(A^{x}\right) = \left[\varphi(A)\right]^{x}$$

and consequently

$$\varphi\left(A^{\log x}\right) = \left[\varphi(A)\right]^{\log x} = x^{\log \varphi(A)},$$

or what amounts to the same thing,

$$\varphi(x) = x^{\log \varphi(A)} \tag{13}$$

It follows from equation (13) that any function $\varphi(x)$ that solves problem IV is necessarily of the form

$$\varphi(x) = x^{a}, \tag{14}$$

where a denotes a constant. Moreover it is easy to assure ourselves that this constant ought to remain entirely arbitrary.

The four values of $\varphi(x)$ which respectively satisfy equations (1), (2), (3) and (4), namely

$$ax, \quad A^x, \quad a\log x \quad \text{and} \quad x^a,$$

have this much in common, that each of them contains an arbitrary constant, a or A. Thus we ought to conclude that there is a great difference between the questions where it is a matter of calculating the unknown values of certain quantities and the questions in which we propose to discover the unknown nature of certain functions that have given properties. Indeed, in the first case, the values of unknown quantities are ultimately expressed by means of other known and determined quantities, while in the second case the unknown functions can, as we have seen here, admit arbitrary constants into their expression.