Consider an algebraic equation that has to be solved:

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + K = 0.$$

Suppose that an integer approximation p to the root is known, such that the ro lies in the interval $[p, p+1[$. If we change the unknown using the substitutio $x = p + 1/y$ $(0 < 1/y < 1)$, we obtain a new equation in y:

$$A'y^m + B'y^{m-1} + C'y^{m-2} + \ldots + K' = 0.$$

for which we are assured that it has at least one root greater than 1. If we let q be th integer part of this root, the substitution $y = q + 1/z$ leads to a new equation in which has at least one root greater than 1. Let r be its integer part ... and so on. Th continued fraction

$$p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{\ldots}}}$$

will provide the value of a root of the initial equation, and its successive convergent provide alternately upper and lower approximations for the root. Of course, if th root is rational, the process will end, and provide the root, after a finite number of it erations. For irrational roots, the process also gives us an estimation of the error o approximation at each stage, something that the Newton-Raphson method canno do. The text by Lagrange that follows needs no further commentary.

J.-L. Lagrange

Sur la résolution des équations numériques,
Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin, vol. XXIII (1769)
Œuvres, vol. II, Paris: Gauthier-Villars, 1868, pp. 539–578.

§III – A new method for approximating the roots of numerical equations

18. Consider the equation

(*a*) $$Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + K = 0$$

and suppose that we have already found, by the preceding method or otherwise, the integer value approximating to one of its roots, being real and positive: let this first value be p, such that we have

$$x > p \quad \text{and} \quad x < p + 1;$$

we make

$$x = p + \frac{1}{y},$$

and, substituting this value in the proposed equation, in the place of x, we shall have, after multiplying the whole equation by y^m and arranging the terms in order of powers of y, an equation of the form

(*b*) $$A'y^m + B'y^{m-1} + C'y^{m-2} + \ldots + K' = 0.$$

Now, since, by hypothesis, $\frac{1}{y} > 0$ and < 1, we have $y > 0$; therefore the equation (*b*) must necessarily have one real root greater than unity.

We therefore, by the methods of §1, look for the integer value approximating this root, and, since this root has to be positive, we only need to consider positive [values of] y (4).

Having found the integer value that approximates to y, which I shall call q, we now make

$$y = q + \frac{1}{z},$$

and, substituting this value of y in equation (*b*), we shall have a third equation of the form

(*c*) $$A''z^m + B''z^{m-1} + C''z^{m-2} + \ldots + K'' = 0,$$

which must necessarily have at least one real root greater than unity, for which we can find an approximate integer value in the same way.

This approximation to z being called r, we now have

$$z = r + \frac{1}{u},$$

and on substituting we shall have an equation in u which will have at least one real root greater than unity, and so on.

Continuing in the same manner, we shall approach closer and closer to the value of the required root; but, should it happen that some one of the numbers $p, q, \ldots$ should be an exact root, then $x = p$ or $y = q, \ldots$, and the operation will terminate; therefore in this case, we shall find for x a commensurable value.

In all other cases the value of the root must necessarily be incommensurable, and can only be approximated, as close as is wished.

[...]

§IV. – The application of the preceding methods to some examples.

25. I shall take as my first example the equation that Newton solved by his method, namely

$$x^3 - 2x - 5 = 0$$

[...]

I shall now, following the method of §III, put $x = 2 + \frac{1}{y}$; on substituting and arranging the terms as powers of y, we have the equation

$$y^3 - 10y^2 - 6y - 1 = 0,$$

in which I have changed the signs in order to make the first term positive.

This equation will therefore necessarily have a single root greater than unity (19), so that, in order to find an approximate value we need only substitute the numbers 0, 1, 2, 3, ..., until we find two consecutive substitutions that give results of contrary sign.

In order not to carry out many unnecessary substitutions, I note that on putting $y = 0$ I get a negative result, and in putting $y = 10$ the result is still negative; I start therefore with the number 10, and I make successively $y = 10, 11, \ldots$. I find straight away the results $-61, 54, \ldots$; from which I conclude that the approximate value for y is 10; therefore $q = 10$.

I therefore make $y = 10 + \frac{1}{z}$, and I have the equation

$$61z^3 - 94z^2 - 20z - 1 = 0,$$

and successively letting $z = 1, 2, \ldots$, I have the results $-54, 71, \ldots$; therefore $r = 1$.

Again, I let $z = 1 + \frac{1}{u}$, and I shall now have the equation

$$54u^3 + 25u^2 - 89u - 61 = 0,$$

and, letting $u = 1, 2, \ldots$, I shall have the results $-71, 293, \ldots$; therefore $s = 1$ and so on.

By continuing in this way, we find the numbers

$$2,\ 10,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 1,\ 12,\ \ldots,$$

so that the required root can be expressed by this continued fraction

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

from which we can obtain the fractions (23)

$$\frac{2}{1},\ \frac{21}{10},\ \frac{23}{11},\ \frac{44}{21},\ \frac{111}{53},\ \frac{155}{74},\ \frac{576}{275},\ \frac{731}{349},\ \frac{1307}{624},\ \frac{16415}{7837},\ \ldots$$

which are alternately smaller and greater than the value of x.

The last fraction $\frac{16415}{7837}$ is greater than the required root; but the error will be less than $\frac{1}{(7837)^2}$, (23, 2°), that is less than 0.000 000 016 3; therefore, if the fraction $\frac{16415}{7837}$ is reduced to decimals, it will be exact up to the seventh decimal; now, in carrying out the division, we obtain 2.094 551 486 5...; therefore, the required root will be between the numbers 2.094 551 49 and 2.094 551 47.

Newton, by his method, found the fraction 2.094 551 47 (*see his Method of infinite series*), from which we can see that that method gives, in this case, an extremely exact result; but we would be wrong to assume we would always have such accuracy.

Horner like Transformations of Polynomial Equations

7.9 The Ruffini-Budan Schema

At the beginning of the 19th century ingenious techniques for the transformation of polynomials were developed, apparently independently, by three mathematicians: Ruffini (1804) [18], Budan (1807 and 1813) [4] and Horner (1819) [7]. These techniques, combined with the results on locating the roots of a polynomial, allowed the decimals of the required root to be determined by successive approximations, often digit by digit, with considerable saving in operations.

Traces of this method of evaluating a polynomial using this approach, are already to be found in a text by Newton (see Section 5.1), however, a systematic method for finding, not only the constant term, but also all the other coefficients of a transformed polynomial $Q(x) = P(x + u)$ did not properly appear before the beginning of the 19th century. Mathematicians in the 18th century had not, in effect, succeeded in finding a practical and simple way of transforming one equation into a second equation, the roots of which were to be simply a constant added to each of the roots of the first equation.

It would appear that Paolo Ruffini [18], at least in Europe, was the first to formulate an algorithm for the transformation of the coefficients of an equation. He was followed, a little later, by François Budan [4]. Paolo Ruffini, a medical doctor and professor of mathematics at Medina, is principally known for his work on the impossibility of solving algebraically equations of degree 5. Budan, who was a doctor of medicine and professor of mathematics at Nantes, worked on the solution of polynomials.

What follows are extracts from the work of both authors. To make the reading of Ruffini's extract easier, we make some remarks about the notation used by Ruffini. The equation under consideration is

$$(A) \qquad Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + Sx^2 + Tx + V = 0.$$

The following quantities are introduced:

$$P' = A,\ P'' = P'p + B = Ap + B,\ P''' = P''p + C = Ap^2 + Bp + C, \ldots$$
$$P^{(m+1)} = P^{(m)}p + V = Ap^m + Bp^{m-1} + \ldots + Tp + V.$$

The notation $P^{(k)}$ does not therefore correspond with the derivative.
Ruffini also introduces:

$$Q^{(k)} = \frac{dP^{(k+1)}}{dp}, \quad R^{(k)} = \frac{dQ^{(k+1)}}{2dp}, \quad S^{(k)} = \frac{dR^{(k+1)}}{3dp}, \quad \text{etc.}$$

and shows that:

$$Q^{(k)} = Q^{(k-1)}p + P^{(k)},\ R^{(k)} = R^{(k-1)}p + P^{(k)},\ S^{(k)} = S^{(k-1)}p + P^{(k)}, \text{etc.}$$

The first formula in the text is equivalent to the Taylor series for a polynomial Z of degree m:

$$Z(p+y) = Z(p) + Z'(p)y + \frac{Z''(p)}{2}y^2 + \frac{Z'''(p)}{2.3}y^3 + \ldots + \frac{Z^{(m)}(p)}{m!}y^m$$

where the derivative $Z^{(k)}$ is given as $\frac{d^k Z}{dx^k}$.