When the value of the variable lies close to the centre of the interval, the *Newton-Gauss algorithm* is more suitable:

$$f(x_0+ct)=f(x_0)+t\Delta f(x_0)+\frac{t(t-1)}{2!}\Delta^2 f(x_0-c)+\frac{(t+1)t(t-1)}{3!}\Delta^3 f(x_0-c)$$
$$+\frac{(t+1)t(t-1)(t-2)}{4!}\Delta^4 f(x_0-2c)+\frac{(t+2)(t+1)t(t-1)(t-2)}{5!}\Delta^5 f(x_0-2c)+\ldots.$$

The formula can be rewritten in terms of finite central differences, with a notation introduced by Sheppard in 1899 [27]:

$$\delta f(x_0) = f(x_0 + c/2) - f(x_0 - c/2)$$

where c is an interval increment, assumed constant.

In the same way as with the Gregory-Newton formula, we can also write the Newton-Gauss algorithm both in terms of forward differences and backward differences. A combination of the two leads to the *Newton-Stirling formula*:

$$f(x_0+ct)=f(x_0)+t\left[\frac{\Delta f(x_0)+\Delta f(x_0-c)}{2}\right]+\frac{t^2}{2!}\Delta^2 f(x_0-c)$$
$$+\frac{t(t^2-1^2)}{3!}\left[\frac{\Delta^3 f(x_0-c)+\Delta^3 f(x_0-2c)}{2}\right]$$
$$+\frac{t^2(t^2-1^2)}{4!}\Delta^4 f(x_0-2c)$$
$$+\frac{t(t^2-1^2)(t^2-2^2)}{5!}\left[\frac{\Delta^5 f(x_0-2c)+\Delta^5 f(x_0-3c)}{2}\right]+\ldots$$

This formula, which can be found in Newton's *Methodus Differentialis* ([22], Prop. iii, Case 1) was studied by Stirling in 1730 ([29], Prop. xx). It is useful when the value to be found by interpolation corresponds to a point $x = x_0 + ct$ very close to x_0. There are also a number of other variations of the Newton formulas, such as the Newton-Bessel formula (Newton [22], Prop. iii, Case ii) and the Laplace-Everett formula (Laplace [15], p. 15; Everett [7], p. 648).

10.5 The Lagrange Interpolation Polynomial

In one of his lectures at the Ecole Normale in Year III of the Revolution (1795), Lagrange referred to the question, treated previously by Newton, of using a 'parabolic curve' to interpolate a curve, that is interpolating a function by means of a polynomial function. Newton applied his result to the trajectory of a comet, whereas Lagrange was motivated by a practical problem of surveying:

From a point whose position is unknown, are observed three objects whose relative distances are known, and the three angles formed by the visual rays from the eye of the observer to these three have been determined. We wish to find the position of the observer with respect to these same objects. ([14], p. 280)

Lagrange proposed a solution by the use of trial and error; this gives rise to a curve of the errors through a finite number of points, the points corresponding to the trials; finally, to solve the problem, the curve needs to be approximated by a polynomial. This Lagrange interpolation polynomial, is none other than Newton's polynomial, but it is expressed in a different way, and this difference is interesting.

J.-L. Lagrange

From Leçons élémentaires de mathématiques donnés à l'Ecole Normale en 1795, *Journal de l'École Polytechnique*, VII^e et VIII^e cahiers, t. II (1812). *Oeuvres*, t. VII, Paris: Gauthier-Villars, 1877 (pp. 284–286).

Newton was the first to consider this Problem; this is the solution he gives for it:

Let P, Q, R, S, … be the values of the ordinates y which correspond to the values $p, q, r, s, \ldots$ of the abscissas x; we shall have the following equations

$$P = a + bp + cp^2 + dp^3 + \ldots,$$
$$Q = a + bq + cq^2 + dq^3 + \ldots,$$
$$R = a + br + cr^2 + dr^3 + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

the number of these equations being equal to those of the undetermined coefficients $a, b, c, \ldots$. Subtracting these equations one from the other, the remainders will be divisible by $q - p$, $r - q, \ldots$, and we shall have, after dividing,

$$\frac{Q-P}{q-p}=b+c(q+p)+d(q^2+qp+p^2)+\ldots,$$
$$\frac{R-Q}{r-q}=b+c(r+q)+d(r^2+rq+q^2)+\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Let $$\frac{Q-P}{q-p}=Q_1,\ \frac{R-Q}{r-q}=R_1,\ \frac{S-R}{s-r}=S_1,\ldots;$$

then we shall find in like manner, by subtraction and division,

$$\frac{R_1-Q_1}{r-p}=c+d(r+q+p)+\ldots,$$
$$\frac{S_1-R_1}{s-q}=c+d(s+r+q)+\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Similarly, let $$\frac{R_1-Q_1}{r-p}=R_2,\ \frac{S_1-R_1}{s-q}=S_2,\ldots;$$

and we shall find $$\frac{S_2-R_2}{s-r}=d+\ldots,$$
$$\ldots\ldots\ldots\ldots,$$

and so on.

... the coefficients $a, b, c, \ldots$, starting with the last and substituting them in the general equation

$$y = a + bx + cx^2 + dx^3 + \ldots,$$

then, after cancellations, the following formula is derived, which it is easy to continue as far as is wished

$$y = P + Q_1(x-p) + R_2(x-p)(x-q) + S_3(x-p)(x-q)(x-r) + \ldots$$

But this solution can be reduced to much greater simplicity by the following consideration.

Since y has to take the values P, Q, R, …, when x becomes $p, q, r, \ldots$, it is easy to see that the expression for y will be of the form

$$y = AP + BQ + CR + DS + \ldots,$$

where the quantities A, B, C, … must be expressible in terms of x in such a way that in putting $x = p$ we would have

$$A = 1, \quad B = 0, \quad C = 0, \ \ldots,$$

and likewise, on putting $x = q$, we would have

$$A = 0, \quad B = 1, \quad C = 0, \quad D = 0, \ \ldots,$$

and on putting $x = r$, we have similarly,

$$A = 0, \quad B = 0, \quad C = 1, \quad D = 0, \ \ldots, \text{ etc.},$$

from which it is easy to conclude that the values of A, B, C, … must be of the form

$$A = \frac{(x-q)(x-r)(x-s)\cdots}{(p-q)(p-r)(p-s)\cdots},$$

$$B = \frac{(x-p)(x-r)(x-s)\cdots}{(q-p)(q-r)(q-s)\cdots},$$

$$C = \frac{(x-p)(x-q)(x-s)\cdots}{(r-p)(r-q)(r-s)\cdots},$$

....,

in taking as many factors, in the numerators and in the denominators, as there are given points of the curve, less one.

This is certainly the same polynomial since the *Lagrange polynomial* and Newton's polynomial take the same values for a number of values of the variable, equal to the degree of the polynomial plus one. Lagrange's ploynomial has the advantage of being expressed much more simply since its coefficients are given by a rule that is global and non-iterative. Newton's polynomial, on the other hand, turns out to be better suited to the case, which happens in practice, where, after having interpolated the function using n values, we wish to take account of a further value: here we simply have to add to the initial polynomial of degree $n - 1$ the term $\Delta^{n+1}y_0(x - x_0)\ldots(x - x_n)$, the coefficient $\Delta^{n+1}y_0$ being obtained by extending the table of differences by one line to the right. By contrast, with the Lagrange polynomial all the terms have to be changed.

However, if the points x_k are equidistant, $x_k = x_0 + ck$ ($k = 0, \ldots, n$), then P_n, the Lagrange polynomial of degree n, of the function f is given by the following formula where the $g_k(t)$ are independent of the point x_0 and of the increment step c and can so be tabulated:

$$P_n(x_0 + ct) = \sum_{0 \le k \le n} f(x_k) g_k(t) \text{ where } g_k(t) = \prod_{\substack{0 \le i \le n \\ i \ne k}} \frac{(t-i)}{(k-i)}$$

In 1821 Cauchy [4] proposed the use of rational functions, being more general than polynomials. Noting that there is only one rational function u, having numerator of degree p and denominator of degree q, taking the determined values u_i for $p + q + 1$ values x_i of the variable, he gives a general formula as a solution to the question. For example, for $p = q = 1$, he obtains:

$$u = \frac{u_0 u_1 \dfrac{x - x_2}{(x_0 - x_2)(x_1 - x_2)} + u_0 u_2 \dfrac{x - x_1}{(x_0 - x_1)(x_2 - x_1)} + u_1 u_2 \dfrac{x - x_0}{(x_1 - x_0)(x_2 - x_0)}}{u_0 \dfrac{x_0 - x}{(x_0 - x_1)(x_0 - x_2)} + u_1 \dfrac{x_1 - x}{(x_1 - x_0)(x_1 - x_2)} + u_2 \dfrac{x_2 - x}{(x_2 - x_0)(x_2 - x_1)}}$$

It is interesting to note that in the text by Lagrange cited above, he wrote: "… it is clear that, for whatever proposed curve, the parabolic curve that is thus drawn will always be closer to it as the number of given points becomes greater, and their distance apart lessens." While such an assertion accords with intuition, it would be well perhaps to look at this a little more closely (see the Runge phenomenon, Section 13.3). There is also the problem of trying to determine limits to the error arising from using interpolation. Cauchy addressed this problem, and we shall look at how he dealt with it in Section 13.4.

Gaspard de Prony

It has to be said that Lagrange was himself more interested in the algebraic aspect of interpolation than in putting the theory to practical use. In contrast, one of his colleagues, Gaspard de Prony, *professeur* at the Ecole Polytechnique from 1794, held that mathematical theories ought to lead to practical calculations. To put this into practice, de Prony organised the production of tables of great accuracy by adopting the principle of division of labour in order "to manufacture logarithms as one manufactures pins". A first group of mathematicians chose the formulas (an adaptation of the calculation of finite differences); a second group of calculators prepared instructions and the lay-out of the tables (in such a way that only additions and subtractions remained to be done); finally the calculations themselves were carried out by a third group of at least 60 to 80 persons (of which many were hairdressers reduced to unemployment by the Revolution). The calculations, completed in 1801, produced tables of logarithms from 1 to 200 000 with 14 places of decimals (as well as tables of sines). But the vast size of the tables made their publication far too expensive. In the event they were only printed partially, and even that a century later. It is possible that the 'industrial' approach adopted by Prony had an influence on Babbage and his project to carry out calculations accurately and repetitively by machine (see Grattan-Guinness [10]).