Summation Notation

Consider again the example in which $A_k = 2^k$ represents the number of ancestors a person has in the kth generation back. What is the total number of ancestors for the past six generations? The answer is

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 126.$$

Joseph Louis Lagrange (1736–1813)

It is convenient to use a shorthand notation to write such sums. In 1772 the French mathematician Joseph Louis Lagrange introduced the capital Greek letter sigma, Σ, to denote the word *sum* (or *summation*), and defined the summation notation as follows:

Definition

If m and n are integers and $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from *k* equals *m* to *n* of *a*-sub-*k***, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Example 5.1.4 **Computing Summations**

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$. Compute the following:

a. $\sum_{k=1}^{5} a_k$ b. $\sum_{k=2}^{2} a_k$ c. $\sum_{k=1}^{2} a_{2k}$

Solution

a. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$ ■

Oftentimes, the terms of a summation are expressed using an explicit formula. For instance, it is common to see summations such as

$$\sum_{k=1}^{5} k^2 \quad \text{or} \quad \sum_{i=0}^{8} \frac{(-1)^i}{i+1}.$$

Example 5.1.5 **When the Terms of a Summation Are Given by a Formula**

Compute $\sum_{k=1}^{5} k^2$.

Solution

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$ ■

When the upper limit of a summation is a variable, an ellipsis is used to write the summation in expanded form.

Example 5.1.6 **Changing from Summation Notation to Expanded Form**

Write $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$ in expanded form:

Solution

$$\begin{aligned}\sum_{i=0}^{n} \frac{(-1)^i}{i+1} &= \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}\\ &= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}\\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}\end{aligned}$$ ■

Example 5.1.7 **Changing from Expanded Form to Summation Notation**

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}.$$

Solution The general term of this summation can be expressed as $\frac{i+1}{n+i}$ for each integer i from 0 to n. Hence

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{i=0}^{n} \frac{i+1}{n+i}.$$ ■

In Examples 5.1.6 and 5.1.7, the top index n of the summation is a *free variable* because it may be replaced by any integer greater than or equal to the bottom index, and each such replacement leads to a different summation. For any particular summation the top index acts like a constant. Thus when the top index also appears in the terms of the summation, as in Example 5.1.7, its value does not change from term to term. By contrast, the index variable in these examples is bound by the summation symbol. It must take every value from the bottom limit to the top limit in succession. The binding of an index variable in a summation is similar to the binding of a variable in a quantified statement or of a local variable in a computer program.

Writing a summation in expanded form helps relate it to our previous experience of working with sums. But for small values of n the expanded form may be misleading. For instance, consider trying to evaluate the following expression for $n = 1$:

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

It may be tempting to write that when $n = 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2$ equals

Caution!
Don't write this → $1^2 + 2^2 + 3^2 + \cdots + 1^2$. This is wrong!

The reason is that $1^2 + 2^2 + 3^2 + \cdots + n^2$ is simply a way of representing the sum of squares of consecutive integers starting with 1^2 and ending with n^2. Thus, when $n = 1$ the sum starts and ends with 1, and so it is just 1^2. If $n = 2$ the sum is $1^2 + 2^2$, and if $n = 3$ the sum is $1^2 + 2^2 + 3^2$.

Example 5.1.8

Evaluating $a_1 + a_2 + a_3 + \cdots + a_n$ for Small n

What is the value of $2^0 + 2^1 + 2^2 + \cdots + 2^n$ when $n = 0$, $n = 1$, and $n = 2$?

Solution When you evaluate a summation like $2^0 + 2^1 + 2^2 + \cdots + 2^n$ for small values of n, you can avoid a mistake by imagining it in summation notation. For instance,

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = \sum_{i=0}^{n} 2^i.$$

So when $n = 0$, $2^0 + 2^1 + 2^2 + \cdots + 2^n$ has the value $\displaystyle\sum_{i=0}^{0} 2^i = 2^0 = 1$.

When $n = 1$, $2^0 + 2^1 + 2^2 + \cdots + 2^n$ has the value $\displaystyle\sum_{i=0}^{1} 2^i = 2^0 + 2^1 = 1 + 2$.

When $n = 2$, $2^0 + 2^1 + 2^2 + \cdots + 2^n$ has the value $\displaystyle\sum_{i=0}^{2} 2^i = 2^0 + 2^1 + 2^2 = 1 + 2 + 4$. ■

A more mathematically precise definition of summation, called a *recursive definition*, is the following:* If m is any integer, then

$$\sum_{k=m}^{m} a_k = a_m \quad \text{and} \quad \sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n \quad \text{for every integer } n > m.$$

When solving problems, it is often useful to rewrite a summation using the recursive form of the definition, either by grouping summands using a single summation sign or by separating off the final term of a summation.

Example 5.1.9

Using a Single Summation Sign and Separating Off a Final Term

a. Write $\displaystyle\sum_{k=0}^{n} 2^k + 2^{n+1}$ as a single summation.

b. Rewrite $\displaystyle\sum_{i=1}^{n+1} \frac{1}{i^2}$ by separating off the final term.

Solution

a. $$\sum_{k=0}^{n} 2^k + 2^{k+1} = (2^0 + 2^1 + 2^2 + \cdots + 2^n) + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

b. $$\sum_{i=1}^{n+1} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$ ■

In certain sums each term is a difference of two quantities. When you write such sums in expanded form, you sometimes see that all the terms cancel except the first and the last.

*Other recursively defined sequences are discussed later in this section and, in greater detail, in Section 5.6.

Example 5.1.10 **A Telescoping Sum**

Some sums can be transformed so that successive cancellation of terms collapses the final result like a telescope. For instance, observe that for every integer $k \geq 1$,

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}.$$

Use this identity to find a simple expression for $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Solution

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}\end{aligned}$$

■

Product Notation

The notation for the product of a sequence of numbers is analogous to the notation for their sum. The Greek capital letter pi, Π, denotes a product. For example,

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5.$$

Definition

If m and n are integers and $m \leq n$, the symbol $\displaystyle\prod_{k=m}^{n} a_k$, read the **product from *k* equals *m* to *n* of *a*-sub-*k***, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

A recursive definition for the product notation is the following: If m is any integer, then

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for every integer } n > m.$$

Example 5.1.11 **Computing Products**

Compute the following products:

a. $\displaystyle\prod_{k=1}^{5} k$

b. $\displaystyle\prod_{k=1}^{1} \frac{k}{k+1}$

Solution

a. $\displaystyle\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

b. $\displaystyle\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

■