Hypergraph zeta functions and isospectral digraphs

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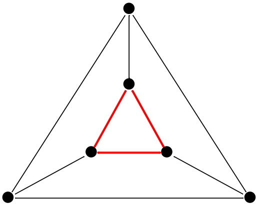
January 6, 2008

Can we be more creative with our paths?



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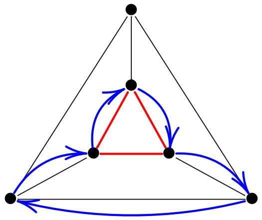

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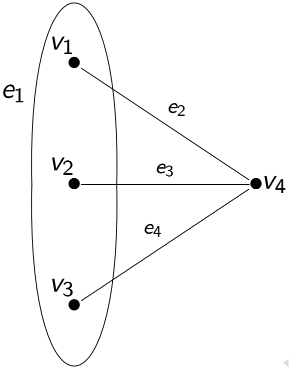
Hypergraphs

Definition

A hypergraph $\mathbb{H}$ is a set of *vertices* $V(\mathbb{H})$ and a set of *hyperedges* $E(\mathbb{H})$ such that each hyperedge is the nonempty union of elements of $V(\mathbb{H})$ and the union of all the hyperedges is $V(\mathbb{H})$. We call the cardinality of a hyperedge e the *order* of the hyperedge and denote it $|e|$.



Path definitions

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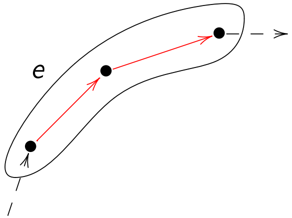
Definition

We say a path has hyperedge backtracking if we use a hyperedge twice in a row.



Generalized Ihara-Selberg zeta function

We keep the same ideas for tail-less, prime, and an equivalence relation. For a hypergraph $\mathbb{H}$, we define the generalized Ihara-Selberg zeta function for $u \in \mathbb{C}$ by

$$\zeta_{\mathbb{H}}(u) = \prod_{[c]} \left(1 - u^{|c|}\right)^{-1}.$$

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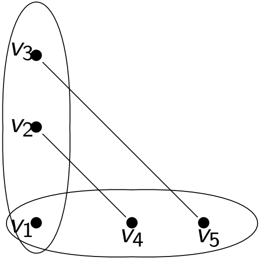
- This is still typically an infinite product.

Expressing the generalized zeta function as a determinant

To realize $\zeta_{\mathbb{H}}(u)$ as a determinant expression, we generalize a construction of Kotani and Sunada:



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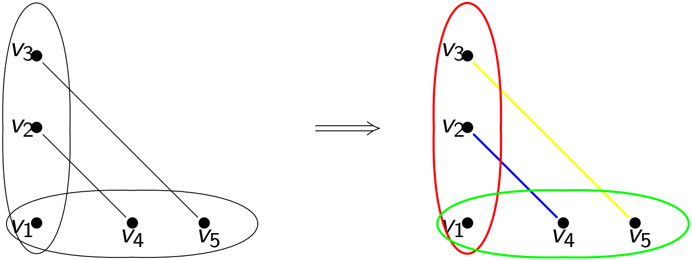

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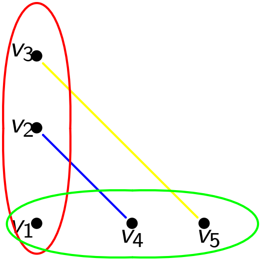
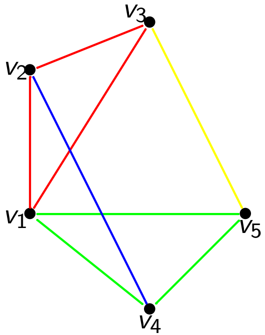


$\Longrightarrow$



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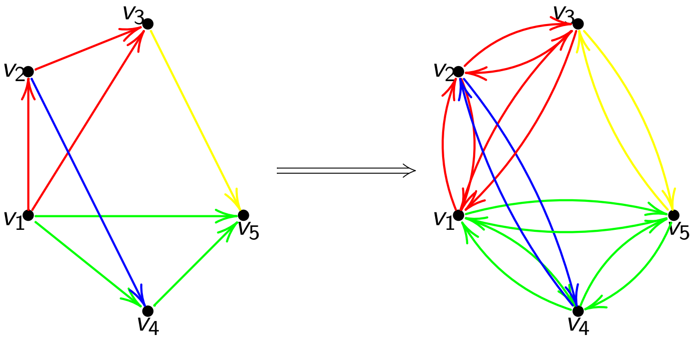
- We split each edge into two directed edges which point in opposite directions.



Oriented line graph

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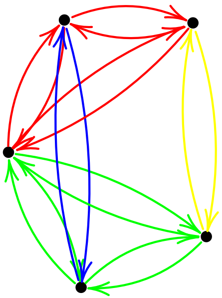

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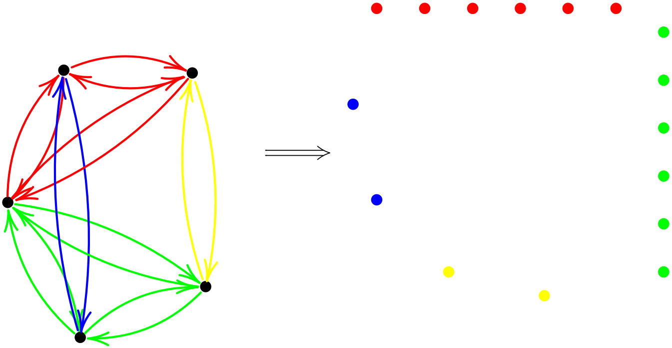

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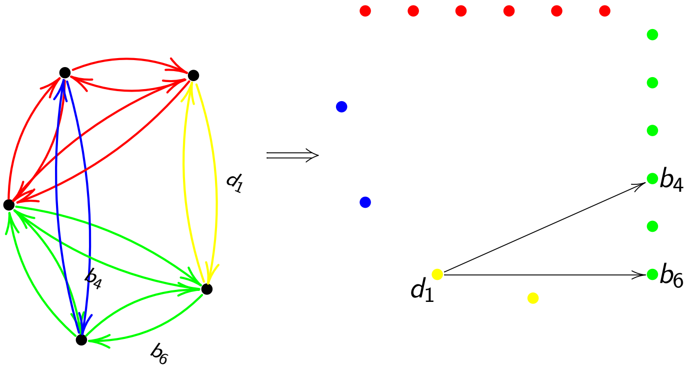

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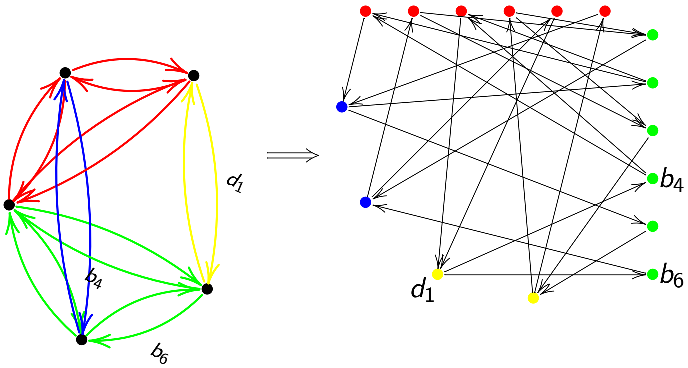

Perron-Frobenius operator

Definition

For a directed graph, the Perron–Frobenius operator T is a matrix given by setting the i, j-entry to 1 if there is an oriented edge with v_i as the start and v_j as the terminus, and setting it to be zero otherwise. This is an oriented version of the adjacency operator of a graph.

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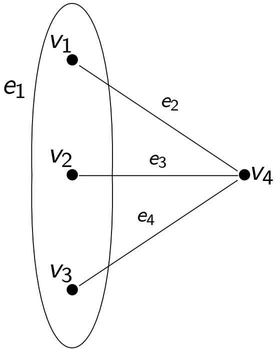
Then,

$$\zeta_{\mathbb{H}}(u) = \det\left(I - uT\right)^{-1}.$$

Associated bipartite graph

A hypervertex v is incident to a hyperedge e if $v \in e$. We can use this to define a bipartite graph associated to $\mathbb{H}$.



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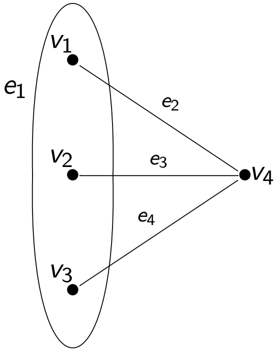

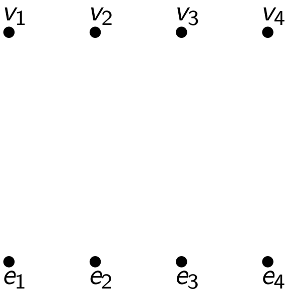

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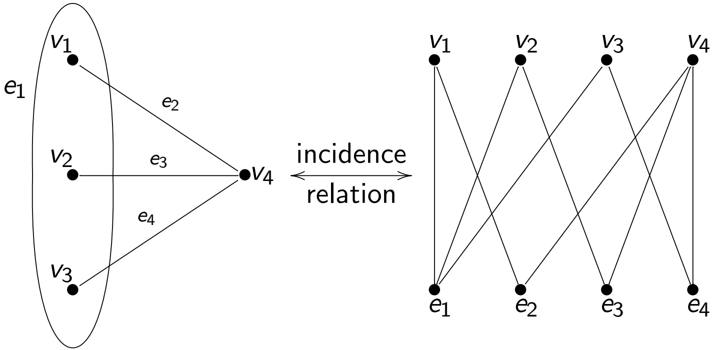
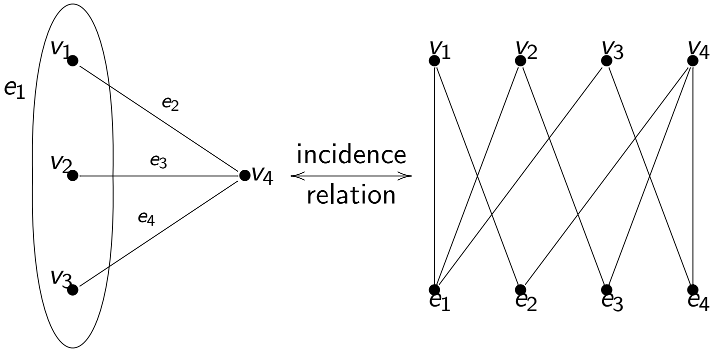
- (v, e) is an edge if v is incident to e.



Bipartite graph and the hypergraph zeta function

The associated bipartite graph is our second structure which we can study to realize the hypergraph zeta function.



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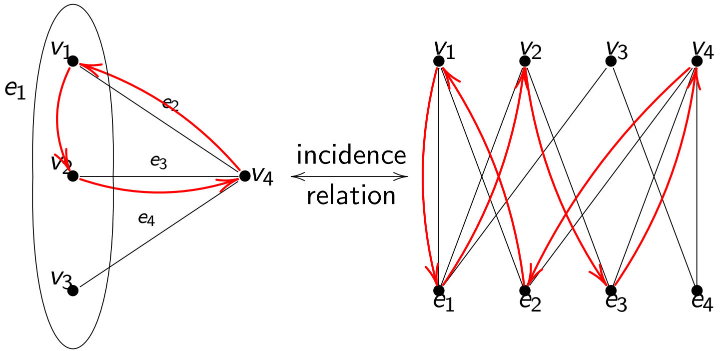
- Let's look at what happens to a prime cycle in $\mathbb{H}$ when we change to the associated bipartite graph B.



A cycle of length 3 has become a cycle of length 6 in the bipartite graph!

Hashimoto's determinant expressions

Remark

There is a 1-to-1 correspondence between prime cycles of length ℓ in $\mathbb{H}$ and prime cycles of length 2ℓ in B.

This gives us a different expression for the generalized zeta function:

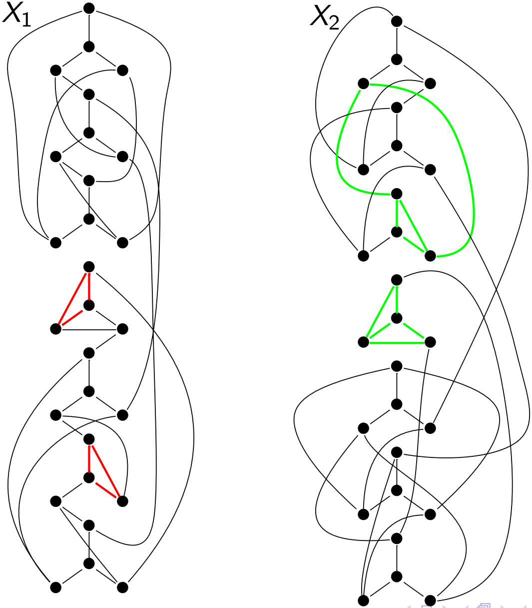
$$\zeta_{\mathbb{H}}(u) = Z_B(\sqrt{u}).$$

Properties of ζ

These expressions lead us to some interesting properies of the generalized zeta function:

- $\zeta_{\mathbb{H}}(u)$ is a rational function.
- There exists hypergraphs with $\zeta_{\mathbb{H}}(u)$ such that no graph has $Z(u) = \zeta_{\mathbb{H}}(u)$.
- There are lots of functional equations.
- There is a meaningful Riemann hypothesis for regular hypergraphs.

Distinguishing cospectral graphs



Structural interplay: isospectral digraph construction

- Given a hypergraph, we have relied upon two key structures so far: the oriented line graph $L^o\mathbb{H}$ and the associated bipartite graph $B_{\mathbb{H}}$.

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- Given a bipartite graph, there are two ways to form a hypergraph, depending upon which set you choose to represent hypervertices and which you choose to represent hyperedges. The other hypergraph which comes from $B_{\mathbb{H}}$ is the dual hypergraph of $\mathbb{H}$ denoted $\mathbb{H}^*$.

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- Given a bipartite graph, there are two ways to form a hypergraph, depending upon which set you choose to represent hypervertices and which you choose to represent hyperedges. The other hypergraph which comes from $B_{\mathbb{H}}$ is the dual hypergraph of $\mathbb{H}$ denoted $\mathbb{H}^*$.
- We will be interested in studying the oriented line graphs which arise from $\mathbb{H}$ and $\mathbb{H}^*$. With appropriate conditions on our initial hypergraph $\mathbb{H}$, we will see that $L^o\mathbb{H}$ and $L^o\mathbb{H}^*$ have the same T spectra and are not isomorphic.

Structural interplay: fitting the structures together

$$\begin{tikzcd}[column sep=small]
 & & B \arrow[dl] \arrow[dr] & & \\
 & \mathbb{H} \arrow[dl] & & \mathbb{H}^* \arrow[dr] & \\
L^o\mathbb{H} & & & & L^o\mathbb{H}^*
\end{tikzcd}$$

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$$

Conditions for isospectrality

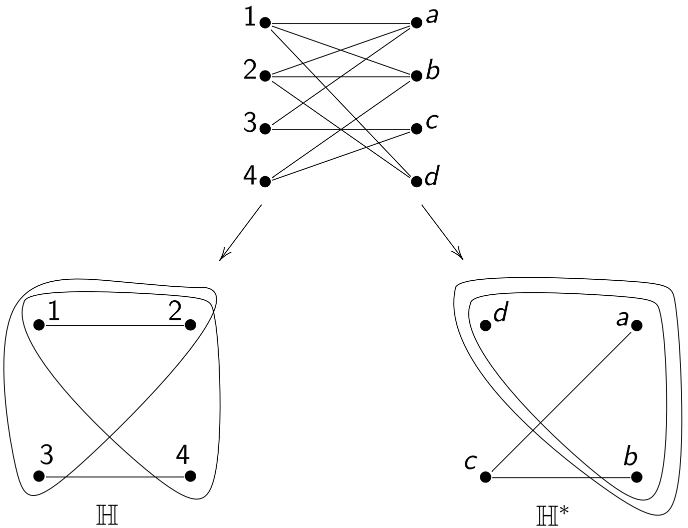
Theorem (B, S)

Let $\mathbb{H}$ be a connected hypergraph which is not just a cycle and for which every hypervertex is in at least 2 hyperedges. Then the non-zero part of the spectra of $T(L^o\mathbb{H})$ and $T(L^o\mathbb{H}^*)$ are identical. In particular, if

$$\sum_{e \in E(\mathbb{H})} [|e|(|e|-1)] = \sum_{v \in V(\mathbb{H})} [i(v)(i(v)-1)],$$

then $L^o\mathbb{H}$ and $L^o\mathbb{H}^*$ are isospectral (with respect to T).

Cospectral digraphs



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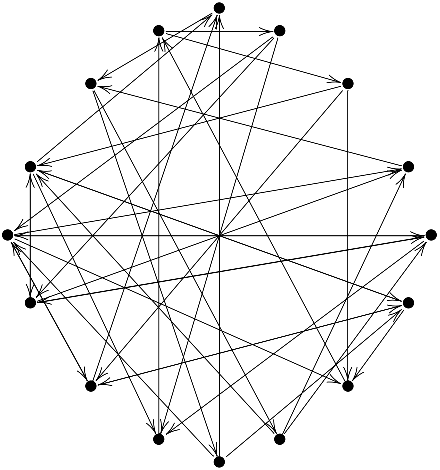

Conditions for non-isomorphism

Theorem (B,S)

Suppose $\mathbb{H}$ *is a hypergraph where every vertex is in at least 3 hyperedges. If* $L^o\mathbb{H} \cong L^o\mathbb{H}^*$, *then* $\mathbb{H} \cong \mathbb{H}^*$.

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- We're pretty sure we can remove the condition on each vertex being in at least 3 hyperedges.
- This theorem combined with the previous theorem give us an easy recipe for constructing isospectral digraphs which are not isomorphic.