The Ihara-Selberg Zeta Function

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- Now: a lot of people going in a lot of different directions.
**Definition**

A graph $X = (V, E)$ is

- a set $V$ of vertices
- and a set $E$ of unordered pairs of vertices, called edges.
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The Adjacency Matrix of a Graph

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Two vertices $u$ and $v$ are adjacent, written $u \sim v$, if $\{u, v\}$ is an edge.

We can use the adjacency relation to associate a matrix $A$ with a graph as follows: the rows and columns of $A$ are parametrized by the vertices. The $(v_i, v_j)$ entry of $A$ is 1 if $v_i$ is adjacent to $v_j$ and 0 otherwise.
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\[ A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \]
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\end{pmatrix}$$

**Question:** What does the $(v_i, v_j)$ entry of $A^k$ represent when $k$ is a positive integer?
The Spectrum

- The adjacency relation is a symmetric relation; ie, if $u$ is adjacent to $v$, then $v$ is adjacent to $u$. Hence, the adjacency matrix of a graph is a symmetric matrix.
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The spectrum of this graph is $\{\frac{1}{2} \left(1 + \sqrt{17}\right), 0, -1, \frac{1}{2} \left(1 - \sqrt{17}\right)\}$. 
Some Properties associated with the eigenvalues

Remarkably, the spectrum of a graph contains quite a bit of useful information regarding the graph. In general, it’s associated with more intangible properties like “expansion”, but there are some concrete properties encoded in them as well. We list a few very basic properties:
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- If the graph has no loops, the eigenvalues sum to zero.
- The degree of a vertex $u$, denoted $d(x)$, is the number of vertices to which it is adjacent. If $\Delta$ is the maximum degree of all of the vertices in $X$, then
  \[ \Delta \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|} \geq -\Delta. \]
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  \[ \Delta \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|} \geq -\Delta. \]
- In particular, if $X$ is $k$-regular, $\lambda_1 = k$.
- $X$ is bipartite if and only if $\lambda_1 = -\lambda_{|V|}$. 
The Second Eigenvalue

Assume $X$ is a $k$-regular graph; then, $\lambda_1 = k$. The key question is: how large, in absolute value, are the other eigenvalues?

Let’s first see why this question is important before we give an answer. To do this, we need to know what the $(v_i, v_j)$-entry of powers of the adjacency matrix represents. Let’s look at an example:
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$$ k = 1 $$

The $(1, 2)$-entry of $A^1$ is: 0
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The $(1, 2)$-entry of $A^3$ is: 1

\[ k = 3 \]

\[ 1 \quad 1 \quad 1 \quad 1 \quad 1 \]

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\[ v_1 \quad v_2 \]
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The $(1, 2)$-entry of $A^4$ is: 2
The Second Eigenvalue

The \((v_i, v_j)\) entry of \(A^k\) is the number of ways to go from \(v_i\) to \(v_j\) in \(k\) steps.

Thus, random walks, mixing problems, and data expansion can all be modeled by successive multiplication of the adjacency matrix by itself.
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As you raise \(A\) to higher and higher powers, you will be forced into the eigenspace corresponding to the largest eigenvalue in absolute value. The speed at which you reach that eigenspace is determined by how large the next eigenvalue is. If you want good mixing, expansion, etc., you are interested in having all of the other eigenvalues as small as possible.
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**Definition**

A \(k\)-regular graph is **Ramanujan** if

\[ |\lambda| \leq 2\sqrt{k - 1} \]

with the exception of \(\lambda_1 = k\).
The Riemann Zeta Function

Let \( s = \sigma + it \). The Riemann zeta function is defined by

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**Remark**

*There is a different way to express the zeta function which illustrates how it connects to the prime numbers. We can write $\zeta(s)$ as an Euler Product Expansion by

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$*
The Riemann hypothesis

After a lot of complex analysis, it’s possible to analytically continue $\zeta(s)$ to the entire complex plane, excepting a simple pole at $s = 1$. This means there is some other function $\hat{\zeta}(s)$ such that $\hat{\zeta}(s)$ agrees exactly with $\zeta(s)$ on $\sigma > 1$ and is analytic throughout the complex plane (minus the pole).
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Remark

Neat fact: when you analytically continue and get $\hat{\zeta}(s)$, you get functional equations that let you evaluate the function for certain values. One well known value is:

$$\hat{\zeta}(-1) = -\frac{1}{12} \neq \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$$
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Definition

The Riemann Hypothesis states that if $\hat{\zeta}(s) = 0$; then $s$ is a negative even integer or $\text{Re } s = \frac{1}{2}$. 
We’d like to define a zeta function on a graph that mimics many of the properties of the Riemann zeta function.

- Our strategy is to use the Euler product expansion for the definition.
A Graph Zeta Function

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Throughout, we will let $X$ be a finite, connected graph such that the degree of every vertex is at least 2.
What is a prime cycle?

**Definition**

A prime cycle is a closed path with no backtracking or tails and is not the $m$-multiple of some other closed path. We impose an equivalence relation on cycles by identifying cycles that differ by cyclic permutation.
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A *prime cycle* is a closed path with no backtracking or tails and is not the $m$-multiple of some other closed path. We impose an equivalence relation on cycles by identifying cycles that differ by cyclic permutation.
We define the **Ihara-Selberg Zeta Function** for a finite graph $X$ by

$$Z_X(u) = \prod_{[c]} \left( 1 - u^{l(c)} \right)^{-1},$$

with $u \in \mathbb{C}$. Here, the product runs over all prime cycles, and $l(c)$ is the length of the cycle $c$. 

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Some Properties:

- Generally the product is infinite.
- This actually turns out to be a rational function.
- There are nice, explicit factorizations.
To show that the zeta function is a rational function, we will realize it as a determinant expression. To define the appropriate operators, we must first change our framework. A graph is actually a pretty hard model to really work with here.
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Consider an edge which is part of some graph:

A prime cycle could use this edge by going from left to right, or... by going from right to left.
Oriented Line Graph Construction

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- We replace each edge with two oriented edges to model the two ways we could use the edge.
- Now we construct a new graph $L$ via

$$V_L = E(X_o),$$
$$E_L^o = \{(e_i, e_j) \in E(X_o) \times E(X_o); \bar{e}_i \neq e_j, t(e_i) = o(e_j)\}.$$
The oriented line graph has several important properties:

- It is strongly connected.
- It exactly mimics the prime cycle structure of the original graph.
- The zeta function of a strongly connected, oriented graph is easy to factor!
Perron-Frobenius Operator

Definition

For a strongly connected, oriented graph the **Perron-Frobenius operator** $T$ is a matrix given by setting the $i,j$-entry to 1 if there is an oriented edge with $v_i$ as the start and $v_j$ as the terminus, and setting it to be zero otherwise. This is an oriented version of the adjacency operator of a graph.
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Then,

$$Z_X(u) = \det (I - uT)^{-1}.$$
Finishing our example
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\[
T = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
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\end{pmatrix}
\]
Still finishing our example...

Now that we have the $T$ operator, we can work out the zeta function explicitly:

$$Z_X(u) = \frac{1}{1 - 4u^3 - 2u^4 + 4u^6 + 4u^7 + u^8 - 4u^{10}}.$$
Bass’ Factorization

So far, we’ve realized that the zeta function is a rational function and have the means to compute it. There is a much prettier factorization due to Hyman Bass which is easier to compute and gives us a better tool for theoretical calculations.
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**Theorem (Bass, 1992)**

Let $X$ be a finite, connected graph as before. Then

$$Z_X(u) = (1 - u^2)^\chi \det(I - uA + u^2Q)^{-1},$$

where $\chi = |V| - |E|$ is the Euler Number of $X$, $A$ is the adjacency matrix, and $Q$ is a diagonal matrix with entries $d(v_i) - 1$.

**Remark**

As a corollary to Bass’ theorem, when $X$ is $k$-regular, we get functional equations which relate the value at $u$ to the value at $\frac{1}{(k-1)u}$.
Let’s take a closer look at Bass' factorization, particularly when our graph is \((q + 1)\)-regular, and look for poles:

\[
Z_X(u)^{-1} = (1 - u^2)^{-\chi} \times \det[l - Au + qlu^2].
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- All of the complex poles lie on the circle in \(\mathbb{C}\) given by \(|r| = \frac{1}{\sqrt{q}}\).
- All other poles are in a specified interval of the real line.
Let’s take a closer look at the polynomial $f(u) = qu^2 - \lambda u + 1$. The discriminant is

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The Complex Poles

Let’s take a closer look at the polynomial $f(u) = qu^2 - \lambda u + 1$. The discriminant is

$$\Delta = \lambda^2 - 4q.\$$

We get a complex pole whenever the discriminant is negative; ie, whenever

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This relation looks very familiar!
Graph Riemann Hypothesis

We could state a Riemann Hypothesis for this zeta function as follows:

**Definition**

Let $X$ be a $(q + 1)$-regular graph. Then $Z_X(u)$ is said to satisfy the Riemann Hypothesis if the only real poles are a simple pole at $u = \frac{1}{q}$ and poles with absolute value 1.

Equivalently, let $u = q^{-s}$. Then $Z_X(q^{-s})$ satisfies the Riemann Hypothesis if whenever $Z_X(q^{-s}) = 0$ and $\text{Re } s \in (0, 1)$, we have $\text{Re } s = \frac{1}{2}$. 

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**Theorem**

A $(q + 1)$-regular graph satisfies the Riemann Hypothesis if and only if $X$ is a Ramanujan graph.
As a graph invariant

Most of the previous discussion has been concerned with actual properties of the function. Perhaps a more important question is: what in the world does this function tell us about the graph?
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Most of the previous discussion has been concerned with actual properties of the function. Perhaps a more important question is: what in the world does this function tell us about the graph?

There are two directions this question can take:

- Someone hands me the zeta function, what physical properties (number of triangles, colourability, ...) can I attribute to the graph?
- Someone gives me two graphs which have the same (or different) zeta function. What conclusions can I draw about the graphs?
Distinguishing graphs

We will look very briefly at the second question. What can I saw about graphs which have the same zeta function?
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**Theorem (Mellein, 2001)**

Suppose $X$ and $Y$ are both $k$-regular graphs. Then $Z_X(u) = Z_Y(u)$ if and only if $\text{Spec}(X) = \text{Spec}(Y)$.

**Remark**

If we remove the regularity condition, all bets are off. There are plenty of examples of graphs with the same adjacency matrix spectrum or the same laplacian spectrum that have different zeta functions. There aren’t currently any known examples of graphs with the same zeta function but different operator spectrums, though.
The previous theorem suggests that knowing all about the cycle structure of a graph is only enough to get you spectral information when the graph is regular. We need some way to tweak the zeta function to rely more on the actual structure and less on the spectrum.
Beating the theorem...

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Can we take prime cycles as before but throw away any prime cycle that uses two red edges in a row? When we form the product, what happens? This has the potential to be quite interesting and to broaden the theory.
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Distinguishing Cospectral Graphs

\[ X_1 \]

\[ X_2 \]