CSC 344 – Algorithms and Complexity

Lecture #1 – Review of Mathematical Induction

Proof by Mathematical Induction

• Many results in mathematics are claimed true for every positive integer.
• Any of these results could be checked for a specific value of n (e.g., 1, 2, 3, ..) but it would be impossible to check every possible case. For example, let $S_n$ represent the statement that the sum of the first n positive integers is
Proof by Mathematical Induction, (continued)

• Let \( S_n \) represent the statement that the sum of the first \( n \) positive integers is \( n(n+1)/2 \)

\[
S_n : \quad 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}
\]

If \( n = 1 \), then \( S_1 \) is 

\[
1 = \frac{1(1 + 1)}{2}, \quad \text{which is true.}
\]

If \( n = 2 \), then \( S_2 \) is

\[
1 + 2 = \frac{2(2 + 1)}{2}, \quad \text{which is true.}
\]

If \( n = 3 \), then \( S_3 \) is

\[
1 + 2 + 3 = \frac{3(3 + 1)}{2}, \quad \text{which is true.}
\]

If \( n = 4 \), then \( S_4 \) is

\[
1 + 2 + 3 + 4 = \frac{4(4 + 1)}{2}, \quad \text{which is true.}
\]
Proof by Mathematical Induction, (continued)

- Continuing in this way for any amount of time would still not prove that $S_n$ is true for every positive integer value of $n$.
- To prove that such statements are true for every positive integer value of $n$, the principle shown on the following slide is often used.

Principle of Mathematical Induction

- Let $S_n$ be a statement concerning the positive integer $n$. Suppose that
  - 1. $S_1$ is true;
  - 2. for any positive integer $k$, $k \leq n$, if $S_k$ is true, then $S_{k+1}$ is also true.
- Then $S_n$ is true for every positive integer value of $n.$
Principle of Mathematical Induction (continued)

• By assumption (1), the statement is true when \( n = 1 \).

• By assumption (2), the fact that the statement is true for \( n = 1 \) implies that it is true for \( n = 1 + 1 = 2 \).

• Using (2) again, the statement is thus true for \( 2 + 1 = 3 \), for \( 3 + 1 = 4 \), for \( 4 + 1 = 5 \), etc.

• Continuing in this way shows that the statement must be true for every positive integer.

How Does Mathematical Induction Work?
How To Prove by Mathematical Induction

• Step 1
  – Prove that the statement is true for $n = 1$.

• Step 2
  – Show that, for any positive integer $k$, $k \leq n$, if $S_k$ is true, then $S_{k + 1}$ is also true.

Example 1 - Proving An Equality Statement

Let $S_n$ represent the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.$$ 

Prove that $S_n$ is true for every positive integer $n$.

Solution

**Step 1** Show that the statement is true when $n = 1$. If $n = 1$, $S_1$ becomes

$$1 = \frac{1(1+1)}{2},$$

which is true.
Example 1 - Proving an Equality Statement

**Step 2** Show that $S_k$ implies $S_{k + 1}$, where $S_k$ is the statement

$$1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2},$$

and $S_{k + 1}$ is the statement

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2}.$$

Example 1 - Proving an Equality Statement

**Step 2** Start with $S_k$ and assume it is a true statement.

$$1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2},$$

Add $k + 1$ to both sides of this equation to obtain $S_{k + 1}$.

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)$$
Example 1 - Proving An Equality Statement

Step 2

\[ 1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \]

\[ = (k + 1)\left(\frac{k}{2} + 1\right) \quad \text{Factor out } k + 1. \]

\[ = (k + 1)\left(\frac{k + 2}{2}\right) \quad \text{Add inside the parentheses.} \]

\[ = \frac{(k + 1)[(k + 1) + 1]}{2} \quad \text{Multiply; } k + 2 = (k + 1) + 1. \]

Example 1 - Proving An Equality Statement

- This final result is the statement for \( n = k + 1 \); it has been shown that if \( S_k \) is true, then \( S_{k+1} \) is also true.

- The two steps required for a proof by mathematical induction have been completed, so the statement \( S_n \) is true for every positive integer value of \( n \).
Prove By Mathematical Induction

• Please note that the left side of the statement $S_n$ always includes all the terms up to the nth term, as well as the nth term.

Example 2 - Proving An Inequality Statement

Prove: If $x$ is a real number between 0 and 1, then for every positive integer $n$,

$$0 < x^n < 1.$$

Solution

Step 1 Here $S_1$ is the statement

if $0 < x < 1$, then $0 < x^1 < 1$,

which is true.
Example 2 - Proving An Inequality Statement

**Step 2** Here $S_k$ is the statement

if $0 < x < 1$, then $0 < x^k < 1$.

To show that this implies that $S_{k+1}$ is true, multiply all three parts of $0 < x^k < 1$ by $x$ to get

$$x \times 0 < x \times x^k < x \times 1$$

Example 2 - Proving An Inequality Statement

**Step 2** (Here the fact that $0 < x$ is used.) Simplify to obtain

$$0 < x^{k+1} < x.$$  

Since $x < 1$,

$$0 < x^{k+1} < x < 1$$

and thus

$$0 < x^{k+1} < 1.$$  

This work shows that if $S_k$ is true, then $S_{k+1}$ is true. Since both steps for a proof by mathematical induction have been completed, the given statement is true for every positive integer $n$.  

Generalized Principle of Mathematical Induction

• Some statements $S_n$ are not true for the first few values of $n$, but are true for all values of $n$ that are greater than or equal to some fixed integer $j$.

• The following slightly generalized form of the principle of mathematical induction takes care of these cases.

Generalized Principle of Mathematical Induction

Let $S_n$ be a statement concerning the positive integer $n$. Let $j$ be a fixed positive integer. Suppose that

– **Step 1** $S_j$ is true;

– **Step 2** for any positive integer $k$, $k \geq j$, $S_k$ implies $S_{k+1}$.

– Then $S_n$ is true for all positive integers $n$, where $n \geq j$. 
Example 3 - Using The Generalized Principle

Let \( S_n \) represent the statement \( 2^n > 2n + 1 \). Show that \( S_n \) is true for all values of \( n \) such that \( n \geq 3 \).

**Solution**

**Step 1** Show that \( S_n \) is true for \( n = 3 \). If \( n = 3 \), then \( S_3 \) is

\[ 2^3 > 2 \times 3 + 1 \]

or

\[ 8 > 7. \]

Thus, \( S_3 \) is true.

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Example 3 - Using The Generalized Principle

Let \( S_n \) represent the statement \( 2^n > 2n + 1 \). Show that \( S_n \) is true for all values of \( n \) such that \( n \geq 3 \).

**Solution**

**Step 2** Now show that \( S_k \) implies \( S_{k+1} \), where \( k \geq 3 \), and where

\[ S_k \text{ is } 2^k > 2k + 1, \]

and

\[ S_{k+1} \text{ is } 2^{k+1} > 2(k + 1) + 1. \]
Example 3 - Using The Generalized Principle

**Step 2**
Multiply both sides of $2^k > 2k + 1$ by 2, obtaining

$$2 \times 2^k > 2(2k + 1)$$

$$2^{k+1} > 4k + 2.$$  

Rewrite $4k + 2$ as $2k + 2 + 2k = 2(k + 1) + 2k$.

$$2^{k+1} > 2(k + 1) + 2k \quad (1)$$

Since $k$ is a positive integer greater than 3,

$$2k > 1. \quad (2)$$


Example 3 - Using The Generalized Principle

**Step 2**
Adding $2(k + 1)$ to both sides of inequality (2) gives

$$2(k + 1) + 2k > 2(k + 1) + 1. \quad (3)$$

From inequalities (1) and (3),

$$2^{k+1} > 2(k + 1) + 2k > 2(k + 1) + 1,$$

or

$$2^{k+1} > 2(k + 1) + 1, \quad \text{as required.}$$
Example 3 - Using The Generalized Principle

**Step 2**

Thus, $S_k$ implies $S_{k+1}$, and this, together with the fact that $S_3$ is true, shows that $S_n$ is true for every positive integer value of $n$ greater than or equal to 3.

Example 4 - Sum of Odd Integers

- Proposition: $1 + 3 + \ldots + (2n-1) = n^2$
  for all integers $n \geq 1$.
- Proof (by induction):
  1. Basis step:
      The statement is true for $n=1$: $1=1^2$.
  2. Inductive step:
      Assume the statement is true for some $k \geq 1$
      (inductive hypothesis)
      show that it is true for $k+1$.  

Example 4 - Sum of Odd Integers (continued)

The statement is true for $k$:

$$1+3+\ldots+(2k-1) = k^2 \quad (1)$$

We need to show it for $k+1$:

$$1+3+\ldots+(2(k+1)-1) = (k+1)^2 \quad (2)$$

Showing (2):

$$1+3+\ldots+(2(k+1)-1) = 1+3+\ldots+(2k+1)$$

$$= 1+3+\ldots+(2k-1)+(2k+1)$$

$$= k^2+(2k+1)$$

$$= (k+1)^2$$

We proved the basis and inductive steps, so we conclude that the given statement true.

Example 5 - The Geometric Series

- Any sum of the form: $1 + r + r^2 + r^3 + \ldots + r^n$ is called a **Geometric Series**.
- Thus, $1 + 2 + 4 + 8 + 16 + \ldots + 2^n$ is a geometric series.
- To find the sum of this series, consider:

  $$S = 1 + r + r^2 + r^3 + \ldots + r^n.$$ 

  So

  $$-rS = -r - r^2 - r^3 - \ldots - r^{n+1}$$

  and

  $$(1-r)S = 1 - r^{n+1}$$

- Therefore, $1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$
Proof of the Geometric Series

• Prove: $1 + r + r^2 + ... + r^n = \left[ r^{(n+1)} - 1 \right] / (r - 1)$
• Proof: (by Induction)
• Basis: Show true for $n = 0$:
  \[
  \begin{align*}
  \text{LHS} &= 1 \\
  \text{RHS} &= \frac{r^{(0+1)} - 1}{r - 1} = \frac{r - 1}{r - 1} = 1
  \end{align*}
  \]
• Therefore LHS = RHS

Proof of the Geometric Series (continued)

• Induction:
  Assume $1 + r + r^2 + ... + r^k = \frac{r^{k+1} - 1}{r - 1}$
• Show:
  $1 + r + r^2 + ... + r^k + r^{k+1} = \frac{r^{k+2} - 1}{r - 1}$
• Now:
  $1 + r + r^2 + ... + r^k + r^{k+1}
  = \frac{r^{k+1} - 1 + r^{k+1}}{r - 1}$
Proof of the Geometric Series (continued)

\[
1 + r + r^2 + \ldots + r^k + r^{k+1} = \frac{r^{k+1} - 1}{r-1} + r^{k+1} \\
= \frac{r^{k+1} - 1 + (r-1)r^{k+1}}{r-1} \\
= \frac{r^{k+1} - 1 + r^{k+1} - r^{k+1}}{r-1} \\
= \frac{r^{k+2} - 1}{r-1}
\]

QED

Divisibility Property

- Proposition: For any integer \( n \geq 1 \),
  \[ 7^n - 2^n \text{ is divisible by 5.} \]  \((P(n))\)
- Proof (by induction):
  1. Basis:
     The statement is true for \( n = 1 \):
     \[ 7^1 - 2^1 = 7 - 2 = 5 \text{ is divisible by 5.} \]
Divisibility Property (continued)

• We are given that

\( P(k) : \)

\[ 7^k - 2^k \text{ is divisible by } 5. \quad (1) \]

Then

\[ 7^k - 2^k = 5a \quad \text{for some } a \in \mathbb{Z}. \]

(by definition) \quad (2)

Divisibility Property (continued)

• We need to show:

\( P(k+1): \)

\[ 7^{k+1} - 2^{k+1} \text{ is divisible by } 5. \]

\[ 7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = 5 \cdot 7^k + 2 \cdot 2^k \]

\[ = 5 \cdot 7^k + 2 \cdot (7^k - 2^k) = 5 \cdot 7^k + 2 \cdot 5a \]

(by (2))

\[ = 5 \cdot (7^k + 2a) \text{ which is divisible by } 5. \]

(by def.)

Thus, \( P(n) \) is true by induction.
Two Proofs to Try

\[ \sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4} \]

\[ \sum_{i=0}^{n} (2i + 1)^2 = \frac{(n + 1)(2n + 1)(2n + 3)}{3} \]