Commutative Algebra II

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Contents

This course will cover a selection of basic topics in commutative algebra. I will be assuming knowledge of a first course in commutative algebra, as in the book of Atiyah-MacDonald [1]. I will also assume knowledge of Tor and Ext. Some of topics which will covered may include Cohen-Macaulay rings, Gorenstein rings, regular rings, Gröbner bases, the module of differentials, class groups, Hilbert functions, Grothendieck groups, projective modules, tight closure, and basic element theory. Eisenbuds book [3], the book of Bruns and Herzog [2], and Matsumuras book [4] are all good reference books for the course, but there is no book required for the course.

Chapter 1

Regular Local Rings

Through out these notes, a ring R is considered to be commutative. That is a set with two operations $+, \cdot$ such that under $+$, R is an abelian group with additive identity 0. Multiplication is associative with identity 1 (or 1_R), distributive: $a(b+c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$, and commutative: $ab = ba$.

Further, make note that there is no differentiation between the symbols $\subset$ and $\subseteq$. The symbol $\subsetneq$ will be used to represent a proper subset.

1 Definitions and Equivalences

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a d dimensional noetherian local ring. The following are equivalent:*

(1) $\operatorname{pdim}_R k < \infty$*;*

(2) $\operatorname{pdim}_R M < \infty$ *for all finitely generated R-modules M;*

(3) $\mathfrak{m}$ *is generated by a regular sequence;*

(4) $\mathfrak{m}$ *is generated by d elements.*

Definition. If R satisfies one of these (hence all) we say R is a *regular local ring*.

Definition. Let $x_1, x_2, \ldots, x_d \in R$ and M be an R-module. Say $x_1, x_2, \ldots, x_d$ is a *regular sequence on M* if

(1) x_1 is a non-zero divisor on M;

(2) For all $2 \leqslant i \leqslant d$, x_i is a non-zero divisor on $M/x_1, x_2, \ldots, x_{i-1}M$;

(3) $(x_1, x_2, \ldots, x_d)M \neq M$.

When $M = R$, we say $x_1, x_2, \ldots, x_d$ is a *regular sequence*.

Example 1. Let $R = \mathbb{Q}[x, y]/(xy)$. Note that $\mathfrak{p} = xR$ is a prime ideal since $R/\mathfrak{p}$ is a domain. If we let $M = R/\mathfrak{p}$ then we have that y is a non-zero divisor on M but not on R. Also, $M \neq My$, thus y is a regular sequence on M, but not on R.

1.1 Minimal Resolutions and Projective Dimension

Let $(R, \mathfrak{m}, k)$ be a local noetherian ring, M a finitely generated R-module and $b_0 = \dim_k(M/\mathfrak{m}M) < \infty$. By Nakayama's lemma (NAK), M is generated by b_0 elements, i.e. there exists $u_1, u_2, \ldots, u_{b_0}$ in M such that $M = Ru_1 + Ru_2 + \cdots + Ru_{b_0}$. To see this, choose a basis $\overline{u_1}, \overline{u_2}, \ldots, \overline{u_{b_0}}$ of $M/\mathfrak{m}M$ and lift back to $u_1, u_2, \ldots, u_{b_0}$ in M. by our choice,

$$M = u_1, u_2, \ldots, u_{b_0} + \mathfrak{m}M.$$

Thus NAK gives that $M = u_1, u_2, \ldots, u_{b_0}$.

We can then choose a free module $F_0 = R^{b_0}$ and a map taking each standard basis element e_i to a generator of M.

$$\begin{tikzcd}
 & & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
 & M_1 = \ker(\varphi_0) \arrow[ur] & e_i \arrow[r, mapsto] & u_i & \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

Since R is noetherian, M_1 is finitely generated (submodules of noetherian modules are again noetherian). So repeat the process to get

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[rr, "\varphi_2"] \arrow[dr] & & F_1 \arrow[rr, "\varphi_1"] \arrow[dr] & & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
 & & M_2 \arrow[ur] \arrow[dr] & & M_1 \arrow[ur] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where $M_i = \ker(\varphi_{i-1})$.

Proposition 2. *The (infinite) exact sequence*

$$F_. : \quad \ldots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

as constructed above, is called a minimal free resolution of M. Minimality is used to mean

$$\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}.$$

Proof that this holds. It is enough to show that $\varphi_1(F_1) \subset \mathfrak{m}F_0$ and then repeat. Note that $\varphi_1(F_1) = M_1$, so we are claiming $M_1 \subset \mathfrak{m}F_0$. Consider the short exact sequence

$$0 \longrightarrow M_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

and tensor with k. We get an exact sequence

$$M_1/\mathfrak{m}M_1 \longrightarrow F_0/\mathfrak{m}F_0 \longrightarrow M/\mathfrak{m}M \longrightarrow 0.$$

Notice that both $F_0/\mathfrak{m}F_0$ and $M/\mathfrak{m}M$ are isomorphic to k^{b_0}. Thus the surjection above becomes an isomorphism. This implies the image of the map from $M_1/\mathfrak{m}M_1$ to $F_0/\mathfrak{m}F_0$ is zero; hence $M_1 \subset \mathfrak{m}F_0$. □

Recall that we can compute $\operatorname{Tor}_i^R(M, N)$ by taking any free resolution of M, tensor with N, and then take the i^{th} homology. Therefore we can compute $\operatorname{Tor}_i^R(M, k)$ by tensoring $F_.$ with k and taking homology:

$$\begin{array}{lccccccccc} F_. \otimes k : & \dots \longrightarrow & F_i \otimes k & \xrightarrow{\overline{\varphi_i}} & \cdots \longrightarrow & F_1 \otimes k & \xrightarrow{\overline{\varphi_1}} & F_0 \otimes k \\ & & \| & & & \| & & \| \\ & & k^{b_i} & & & k^{b_1} & & k^{b_0} \end{array}$$

where b_i is the rank of F_i. Since $F_.$ is minimal, the image is in what we are moding out. In fact,

$$\varphi_i(F_i) \subset \mathfrak{m}F_{i-1} \iff \overline{\varphi_i} = 0.$$

Remark. To summarize the above discussion, we have the following equivalent statements about a free resolution

$$F_. : \quad \dots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

of an R-module M:

(1) $\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}$ for all i;

(2) $\overline{\varphi_i} = 0$ for all i;

(3) $\dim_k \operatorname{Tor}_i^R(M, k) = \operatorname{rank}(F_i) = b_i$ for all i.

In particular, the b_i are independent of the choices made to construct the minimal free resolution.

Definition. The *projective dimension* of an R-module M, denoted $\operatorname{pdim}_R(M)$, is the

$$\sup\{i \mid \operatorname{Tor}_i^R(M, k) \neq 0\}.$$

If $\operatorname{pdim}_R(M) = n < \infty$, then this means the minimal free resolution is finite:

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

Definition. The ranks of the F_i are called *betti numbers* of M.

Example 2. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $x \in \mathfrak{m}$. Then x is regular if and only if $\mathrm{pdim}_R(R/xR) = 1$. To see this just construct the minimal free resolution.

Example 3. Let $R = \mathbb{C}[x,y]/(x^3 - y^2)$ and consider the maximal ideal $\mathfrak{m}_0 = (x, y)$. We claim $R_{\mathfrak{m}_0}$ is not regular. This is because the minimal number of generators of $\mathfrak{m}_0$ is 2 but the dimension of R is 1. That $\mathfrak{m}_0$ requires 2 generators can be seen by noticing

$$\frac{\mathfrak{m}_0 R_{\mathfrak{m}_0}}{\mathfrak{m}_0^2 R_{\mathfrak{m}_0}} \simeq \frac{\mathfrak{m}_0}{\mathfrak{m}_0^2} \simeq \mathbb{C}^2$$

and applying NAK (see discussion on page 2).

However, any other maximal ideal has the form $\mathfrak{m} = (x - \alpha, y - \beta)$ where $\alpha^3 = \beta^2$ (by Nullstellensatz). It can be checked that $R_{\mathfrak{m}}$ is regular.

Definition. For a ring R, we define the *singular locus* to be

$$\mathrm{Sing}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_{\mathfrak{p}} \text{ is not regular}\}$$

and the *regular locus* as

$$\mathrm{Reg}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_{\mathfrak{p}} \text{ is regular}\}.$$

Remark. Later we will see a criteria for finitely generated algebras over a field of characteristic 0 to be regular local rings (the Jacobian criteria), but historically the first way of characterizing when an abstract ring is regular was part (4) of theorem 1 stated above.

After proving the theorem, we would like to prove stability under "generalization", i.e. if $\mathfrak{q} \subset \mathfrak{p}$ are elements of $\mathrm{Spec}(R)$ and $\mathfrak{p} \in \mathrm{Reg}(R)$, then $\mathfrak{q} \in \mathrm{Reg}(R)$.

1.2 Koszul Complex

Definition. A *complex* is a sequence of R-modules

$$C_. : \qquad \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

such that $d_{i-1} \circ d_i = 0$ for all i.

Definition. The n^{th} *homology module* of a complex $C_.$ is given by

$$H_n(C_.) = \frac{\ker(d_n)}{\mathrm{im}(d_{n+1})}$$

Remark. We can also think of complexes as a graded R-module

$$C_. = \bigoplus C_n$$

with an endomorphism $d : C_{.} \to C_{.}$ of degree -1 where $d^2 = 0$. With this in mind, we can tensor two complexes as follows: $(C_{.}, d) \otimes (C'_{.}, d')$ is a complex with n^{th} graded piece $\bigoplus_{i+j=n}(C_i \otimes C'_j)$ and an endomorphism

$$C_i \otimes C'_j \xrightarrow{\delta} (C_{i-1} \otimes C'_j) \oplus (C_i \otimes C'_{j-1})$$

where

$$x \otimes y \longmapsto (d(x) \otimes y) \oplus ((-1)^i x \otimes d'(y)).$$

The Koszul Complex For $x \in R$, define $K_{.}(x; R)$ as the complex

$$0 \longrightarrow R_1 \xrightarrow{\cdot x} R_0 \longrightarrow 0$$

where R_1 and R_0 are just R (the indices will keep track of the homological degree). For a complex $C_{.}$, consider $C_{.} \otimes K_{.}(x; R)$, which we denote as $C_{.}(x)$. From the above remark, we have

$$[C_{.}(x)]_n = (C_{n-1} \otimes R_1) \oplus (C_n \otimes R_0) = C_{n-1} \oplus C_n.$$

The map δ from $[C_{.}(x)]_n$ to $[C_{.}(x)]_{n-1}$ takes C_{n-1} to C_{n-2} via d_{n-1}, C_n to C_{n-1} via d_n, and C_{n-1} to C_{n-1} by multiplication by $(-1)^{n-1}x$. This gives rise to a short exact sequence of complexes

$$0 \longrightarrow C_{.} \longrightarrow C_{.}(x) \longrightarrow C_{.}(-1) \longrightarrow 0.$$

The n^{th} row of this sequence looks like

$$0 \longrightarrow C_n \xrightarrow{\alpha} C_n \oplus C_{n-1} \xrightarrow{\beta} C_{n-1} \longrightarrow 0$$

with the maps $\alpha : z \mapsto (z, 0)$ and $\beta : (*, u) \mapsto (-1)^{n-1}u$ which are compatible with the differentials. Then there exists a long exact sequence of homology

$$\cdots \longrightarrow H_n(C_{.}) \longrightarrow H_n(C_{.}(x)) \longrightarrow H_n(C_{.}(-1)) \xrightarrow{\delta} H_{n-1}(C_{.}) \longrightarrow \cdots$$

where δ is the connecting homomorphism. But note that $H_n(C_{.}(-1)) \simeq H_{n-1}(C_{.})$ so we have

$$\cdots \longrightarrow H_n(C_{.}) \longrightarrow H_n(C_{.}(x)) \longrightarrow H_{n-1}(C_{.}) \xrightarrow{\delta} H_{n-1}(C_{.}) \longrightarrow \cdots$$

which breaks up into short exact sequences of the form

$$0 \longrightarrow \frac{H_n(C_{.})}{xH_n(C_{.})} \longrightarrow H_n(C_{.} \otimes K_{.}(x; R)) \longrightarrow \operatorname{ann}_{H_{n-1}(C_{.})} x \longrightarrow 0 \quad (1.1)$$

Definition. Let $x_1, \ldots, x_n \in R$ and M an R-module. The *Koszul complex* is defined inductively as

$$K_.(x_1, \ldots x_n; M) := K_.(x_1, \ldots x_{n-1}; M) \otimes K_.(x_n, R)$$

where $K_.(x; M) := K_.(x; R) \otimes M$.

Theorem 3. *Let R be a ring, $x_1, x_2, \ldots, x_n \in R$ and M a finitely generated R-module.*

(1) $H_0(x_1, \ldots, x_n; M) \simeq M/(x_1, \ldots, x_n)M$*;*

(2) $H_n(x_1, \ldots, x_n; M) \simeq \operatorname{ann}_M(x_1, \ldots, x_n)$*;*

(3) If $x_1, \ldots, x_n$ is a regular sequence on M then $H_i(x_1, \ldots, x_n; M) = 0$ for all $i \geqslant 1$;

(4) If R is noetherian and $H_i(x_1, \ldots, x_n; M) = 0$ for all $i \geqslant 1$ and $x_1, \ldots, x_n \in$ $\operatorname{Jac}(R)$*, then $x_1, \ldots, x_n$ is a regular sequence on M.*

Proof. (1) : Induct on n. For $n = 1$, define $K_.(x_1; R)$ as the complex

$$0 \longrightarrow R \xrightarrow{\cdot x_1} R \longrightarrow 0.$$

So $H_0(x_1; M) \simeq M/x_1M$.

For $n > 1$, let $C_. = K_.(x_1, \ldots, x_{n-1}; M)$ and apply (1.1) with $n = 0$ and $x = x_n$ to get

$$\frac{H_0(C_.)}{x_n H_0(C_.)} \simeq H_0(C_. \otimes K_.(x_1)) = H_0(x_1, \ldots, x_n; M).$$

By induction $H_0(C_.) \simeq M/(x_1, \ldots, x_n)M$; completing the proof of (1).

(2) : This follows from the following complex of $K.(x_1, \ldots, x_n; M)$:

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix} \pm x_1 \\ \vdots \\ \pm x_n \end{pmatrix}} M^{\oplus n} \longrightarrow \cdots \longrightarrow M^{\oplus n} \xrightarrow{(x_1 \ldots x_n)} M \longrightarrow 0.$$

(3) : Induct on n. For $n = 1$, x_1 is regular on M if an only if the complex

$$0 \longrightarrow M \xrightarrow{\cdot x_1} R \longrightarrow 0.$$

is exact if and only if $H_1(x_1; M) = 0$. (Note that this is the base case of (4)).

For $n > 1$, we shall use the same notation as in (1). By induction, $H_i(C_.) = 0$ for all $i \geqslant 1$. By (1.1), $h_i(C_. \otimes K(x_n)) = 0$ for all $i \geqslant 2$. If $i = 1$, we have by induction and the fact that x_n is a non-zero divisor on $M/(x_1,\ldots,x_{n-1})M = H_0(C_.)$ that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{H_0(C_.)}{x_nH_0(C_.)} & \longrightarrow & H_1(C_. \otimes K_.(x_n)) & \longrightarrow & \mathrm{ann}_{H_0(C_.)}(x_n) & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & 0 & & & & 0 & & \end{array}$$

Thus $H_1(C_. \otimes K_.(x_n)) = 0$, completing the proof of (3).

(4) : Induct on n. Notice that the $n = 1$ case is the same as the $n = 1$ case of part (3). Assume that $n > 1$ and sue the same notation as in (3). Note that by (1.1) we have

(i) $H_i(C_.)/x_nH_i(C_.) = 0$ for all $i \geqslant 1$;

(ii) $\mathrm{ann}_{H_0(C_.)}(x_n) = 0$.

Since R is noetherian, $H_i(C_.)$ is finitely generated as it is a subquotient of $M^{\oplus\binom{n}{i}}$. Since x_n is an element of $\mathrm{Jac}(R)$, NAK gives us that $H_i(C_.) = 0$. By induction, $x_1,\ldots,x_{n-1}$ is a regular sequence on M. Now (ii) shows $x_1,\ldots,x_n$ is a regular sequence on M. □

Corollary 4. *Assume $x_1,\ldots,x_n$ is a regular sequence on R. Then $K_.(x_1,\ldots,x_n;R)$ is a free resolution of $R/(x_1,\ldots,x_n)$.*

Proof. Apply statements (1) and (3) of theorem 3. □

Note. The above complex looks like

$$0 \longrightarrow R^{\binom{n}{n}} \longrightarrow R^{\binom{n}{n-1}} \longrightarrow R^{\binom{n}{n-2}} \longrightarrow \cdots \longrightarrow R^{\binom{n}{1}} \longrightarrow R \longrightarrow R/(x_1,\ldots,x_n)R \longrightarrow 0.$$

Remark (Base Change). If $\varphi : R \to S$ is an algebra homomorphism and $X_1,\ldots,x_n \in R$, then

$$K_.(x_1,\ldots,x_n;r) \otimes_R S \simeq K_.(\varphi(x_1),\ldots,\varphi(x_n);S).$$

Proof. If $n = 1$, apply $\_ \otimes_R S$ to

$$0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow 0$$

to get the sequence

$$0 \longrightarrow S \xrightarrow{\varphi(x_1)} S \longrightarrow 0.$$

The rest follows by induction. □

Note. What about $H_1(x, 0; R)$? By (1.1) we have

$$\begin{tikzcd} 0 \arrow[r] & \dfrac{H_1(x;R)}{0 \cdot H_i(x;R)} \arrow[r] \arrow[d, equal] & H_1(x,0;R) \arrow[r] & \operatorname{ann}_{H_0(x;R)}(0) \arrow[r] \arrow[d, equal] & 0. \\ & H_1(x;R) & & H_0(x;R) & \end{tikzcd}$$

In fact,

$$H_1(x, 0; R) \simeq H_1(x; R) \oplus H_0(x; R).$$

Remark (Koszul complexes as Tor). Let S be a ring, $x_1, \dots, x_n \in S$. Set $R = S[T_1, \dots, T_n]$. Then $T_1, \dots, T_n$ are a regular sequence on R. By the above corollary, $K_.(T_1, \dots, T_n; R)$ is a free resolution of $R/(T_1, \dots, T_n)$. Consider the map $\varphi : R \to S$ defined by $\varphi(T_i) = x_i$, $1 \leqslant i \leqslant n$ (this makes S an R-module). By the previous a base change we have

$$K_.(T_1, \dots, T_n; r) \otimes_R S \simeq K_.(x_1, \dots, x_n; S).$$

So,

$$H_i(K_0(T_1, \dots, T_n; r) \otimes_R S) = H_0(x_1, \dots, x_n; S).$$

But by definition of Tor, the above can also be written as

$$\operatorname{Tor}_i^R(R/(T_1, \dots, T_n), S).$$

In general we have (exercise)

$$\operatorname{ann}(\operatorname{Tor}_i^R(M, N)) \supset \operatorname{ann}(M) + \operatorname{ann}(N).$$

Therefore we have

$$\begin{aligned} \operatorname{ann}(H_i(x_1, \dots, x_n; S)) &= \operatorname{ann}(\operatorname{Tor}_i^R(R/(T_1, \dots, T_n), S)) \\ &\supset (T_1, \dots, T_n) + (T_1 - x_1, \dots, T_n - x_n) \\ &\supset (x_1, \dots, x_n). \end{aligned}$$

Definition. In a domain R, $x_1, \dots, x_m$ is called a *prime sequence* if the ideals $(x_1, \dots, x_i)$ are distinct prime ideals for all $i \geqslant 0$ ($i = 0$ means the zero ideal).

Lemma 5. *A prime sequence is a regular sequence.*

Proof. Straight forward application of the definition of regular sequence. □

We are now ready to prove theorem 1.

Proof of theorem 1. (2) $\Longrightarrow$ (1): Let $M = k$.

(3) $\Longrightarrow$ (1): Write $\mathfrak{m} = (x_1, \dots, x_t)$ where $x_1, \dots, x_t$ is a regular sequence. By the corollary on page 7, we know that $K_.(x_1, \dots, x_t; R)$ is a free resolution of $R/\mathfrak{m} = k$. Therefore the projective dimension of k is finite.

(1) $\Longrightarrow$ (2): We need to prove that given a finitely generated module M, that $\operatorname{Tor}_i^R(M,k) = 0$ for all $i >> 0$. But

$$\operatorname{Tor}_i^R(M,k) = \operatorname{Tor}_i^R(k,M).$$

Since $\operatorname{pdim}_R(k) < \infty$, $\operatorname{Tor}_i^R(k,M) = 0$ for all $i > \operatorname{pdim}_R(k)$. In particular,

$$\operatorname{pdim}_R(M) \leqslant \operatorname{pdim}_R(k).$$

(4) $\Longrightarrow$ (3): Let $\mathfrak{m} = (x_1, \dots, x_d)$ where d is the dimension of the ring R. By the above lemma, it is enough to show that $\mathfrak{m}$ is generated by a prime sequence. Induct on d. If $d = 0$, we have that $\mathfrak{m} = (0)$ and thus we are done since R is then a domain.

For $d > 0$, notice that $\mathfrak{m}$ is not minimal. So by prime avoidance,

$$\mathfrak{m} \nsubseteq \bigcup_{\mathfrak{p} \text{ min'l}} \mathfrak{p}.$$

Therefore there exists an element x_1' such that

$$x_1' = x_1 + r_2x_2 + \cdots + r_dx_d \notin \bigcup_{\mathfrak{p} \text{ min'l}} \mathfrak{p}.$$

Now $\mathfrak{m} = (x_1', x_2, \dots, x_n)$. So replace x_1 with x_1'.

Note that $\dim(R/x_1R) \geqslant d-1$ by the Krull principle ideal theorem. But $\mathfrak{m}/x_1R$ is generated by $d-1$ elements, so again by the principal ideal theorem, $\dim(R/x_1R) \leqslant d-1$. Hence $\dim(R/x_1R) = d-1$.

By induction, the images of $x_1, \dots, x_d$ in R/x_1R are a prime sequence. Therefore, the ideals (x_1), (x_1,x_2), $\dots$, $(x_1,x_2,\dots,x_d)$ are all prime ideals in R.

It is left to show that R is a domain. Since x_1 is prime (and the fact that x_1 avoids the minimal primes), there is a prime ideal $\mathfrak{p} \subsetneq (x_1)$. Let $y \in \mathfrak{p}$ and write $y = rx_1$. Since (x_1) avoids $\mathfrak{p}$, we have that $r \in \mathfrak{p}$. Thus $\mathfrak{p} = x_1\mathfrak{p}$ and NAK forces $\mathfrak{p} = (0)$. In other words, R is a domain.

(1) $\Longrightarrow$ (4): Consider the following

Claim. If $\operatorname{pdim}(k) < \infty$ then $\operatorname{pdim}(k) \leqslant \dim(R) = d$.

By the above claim and lemma 9 below, we have that $s = d$; finishing the proof of the theorem. To see this, notice that the claim implies $\operatorname{pdim}(k) \leqslant \dim(R) = d$. Hence, by the definition of projective dimension, we have that

$$\operatorname{Tor}_i^R(k,k) = 0, \text{ for all } i > d.$$

Now apply lemma 9 to get $s \leqslant d$, thus we have that $s = d$.

To prove the claim, recall that we already saw that for all M, finitely generated R-modules

$$\operatorname{pdim}_R(M) \leqslant \operatorname{pdim}_R(k).$$

Let $n = \operatorname{pdim}_R(k)$ and suppose that $n > d$. Choose a maximal regular sequence $y_1, \dots, y_t \in \mathfrak{m}$ ($t = 0$ is ok). Note that $t \leqslant d$ (this is left as an exercise). By choice we have that $\mathfrak{m}$ is associated to $R/(y_1, \dots, y_t)$. If note, then prime avoidance allows us to choose

$$y_{t+1} \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R/(y_1,\dots,y_t))} \mathfrak{p}$$

such that $(y_1, \dots, y_{t+1})$ is a regular sequence; a contradiction. Hence, k embeds into $R/(y_1, \dots, y_t)$. Consider the short exact sequence

$$0 \longrightarrow k \longrightarrow R/(y_1, \dots, y_t) \longrightarrow N \longrightarrow 0,$$

tensor with k and then look at Tor,

$$\begin{array}{ccccc} \mathrm{Tor}_{n+1}(N,k) & \longrightarrow & \mathrm{Tor}_n(k,k) & \longrightarrow & \mathrm{Tor}_n(R/(y_1,\dots,y_t),k) \\ \| & & \nparallel & & \| \\ 0 & & 0 & & 0 \end{array}$$

Note that $\mathrm{pdim}(N) \leqslant n$ gives the first vanishing and the third vanishing follows from the corollary on page 7; i.e. $K_.(y_1, \dots, y_t; R)$ is a free resolution of $R/(y_1, \dots, y_t)$ of length $t \leqslant d < n$. The fact that the middle Tor does not vanish is because $\mathrm{pdim}(k) = n$. Thus we have that $\mathrm{pdim}(k) \leqslant d$, proving the claim. $\square$

Corollary 6. *A regular local ring is a domain.*

Proof. This is a result of the proof (4) implies (3). $\square$

Lemma 7. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Let F, G be finitely generated free R-modules and $\varphi : F \to G$ be an R-module homomorphism. If $\overline{\varphi} : \overline{F} \to \overline{G}$ is one-to-one ($\overline{\cdot} = \cdot \otimes R/\mathfrak{m}$), then φ is split, i.e., there exists a map $\psi : G \to F$ such that $\psi \circ \varphi = 1_F$.*

Proof. Write $F = R^n$, $G = R^m$ and consider the standard basis elements $\{e_1, \dots, e_n\}$ of F. By assumption, $\overline{\varphi}(\overline{e_1}), \dots, \overline{\varphi}(\overline{e_n}) \in k^m$ are linearly independent in $\overline{G}$. Choose $\overline{f_{n+1}}, \dots, \overline{f_m}$ extending to a basis of $\overline{G}$. Then by NAK,

$$(\varphi(e_1), \dots, \varphi(e_n), f_{n+1}, \dots, f_m) = G.$$

Therefore, these form a basis (see discussion below). So for ψ, take the projection map onto the first n. $\square$

Discussion. Let $G = R^m$ and $(y_1, \dots, y_m) = G$ as an R-module. Assume that

$$\sum_{i=1}^{m} r_i y_i = 0.$$

We want to prove $r_i = 0$ for all i. By NAK, we know that $\overline{r_i} = 0$, i.e. $r_i \in \mathfrak{m}$. Consider the surjection

$$\begin{array}{c} G \xrightarrow{\varphi} G \\ e_i \longmapsto y_i \end{array}$$

By lemma 8 below, we have that φ is an isomorphism, thus $r_i = 0$.

Lemma 8. *Suppose that M is finitely generated R-module and that $\varphi : M \to M$ is surjective, then φ is an isomorphism.*

Proof. Let $S = R[t]$, a polynomial ring over R. Make M into an S-module by defining $f(t) \cdot m = f(\varphi)m$, i.e., $tm = \varphi(m)$. Now we have

$$M \xrightarrow{\cdot t} M \longrightarrow 0$$

as S-modules. So $M = tM$ and by NAK there exists a $g(t) \in S$ such that

$$(1 - tg)M = 0.$$

Suppose $tm = 0$ for some m in M. Then $(1 - tg)m = 0$ and thus $m = 0$. □

Lemma 9 (Serre)**.** *If $\mathfrak{m}$ is minimally generated by s elements (i.e. $\mathfrak{m} = x_1, \dots, x_s$ where $s = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$), then*

$$\dim_k(\mathrm{Tor}_i^R(k,k)) \geqslant \binom{s}{i}.$$

Proof. Consider $K_.(x_1, \dots, x_s; R)$ where $(x_1, \dots, x_s) = \mathfrak{m}$. Note that $K_i \simeq R^{\binom{s}{i}}$. But $K_.$, in general, is not exact. Let $F_.$ be the minimal free resolution of of $k = R/\mathfrak{m}$.

Key Claim. K_i is a direct summand of F_i.

If this is true, then

$$\dim_k \mathrm{Tor}_i^R(k,k) = \mathrm{rank}(F_i) \geqslant \mathrm{rank}(K_i) = \binom{s}{i}.$$

To prove the key claim, induct on i. By lemma 7, it is enough to show there exists $\varphi_i : K_i \to F_i$ such that $\overline{\varphi_i} : \overline{K}_i \to \overline{F}_i$ is one-to-one. Consider

$$\begin{tikzcd}
\cdots \arrow[r] & F_s \arrow[r] & \cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & R \arrow[r] & k \arrow[r] & 0 \\
0 \arrow[r] & K_s \arrow[r] \arrow[u, "\varphi_s"] & \cdots \arrow[r] & K_2 \arrow[r] \arrow[u, "\varphi_2"] & K_1 \arrow[r] \arrow[u, "\varphi_1"] & R \arrow[r] \arrow[u, "id"] & k \arrow[r] \arrow[u, "id"] & 0
\end{tikzcd}$$

by the comparison theorem. Assume this is true for $i-1$, i.e., there exists ψ_i such that

$$\begin{tikzcd}
F_i \arrow[r, "\delta_i"] & F_{i-1} \arrow[d, shift left, "\psi_{i-1}"] \\
K_i \arrow[u, "\varphi_i"] \arrow[r, "d_i"] & K_{i-1} \arrow[u, shift left, "\varphi_{i-1}"]
\end{tikzcd}$$

and $\psi_{i-1} \circ \varphi_{i-1} = 1_{K_i}$. Suppose $z \in K_i$ and that $\varphi_i(z) \in \mathfrak{m}F_i$ (i.e. $\overline{\varphi}_i(\overline{z}) = 0$). By minimality of $F_.$, $\delta_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. This implies that $\delta_i\varphi_i(z) \in \mathfrak{m}^2F_{i-1}$. Therefore, $\varphi_{i-1}d_i(z)$ is in $\mathfrak{m}^2F_{i-1}$ as well. This forces $\psi_{i-1}\varphi_{i-1}d_i(z) \in \mathfrak{m}^2K_{i-1}$ and thus we have that $d_i(z) \in \mathfrak{m}^2K_{i-1}$.

Finally, we are done if we show

Claim. If $d_i(z) \in \mathfrak{m}^2 K_{i-1}$, then $z \in \mathfrak{m}K_i$ (hence $\overline{z} = 0$ and $\overline{\varphi}_i$ is injective).

Recall K_i has a basis e_J, $J \in [S]$, $|J| = i$.

$$d_i(e_J) = \sum_{j \in J} \pm x_j e_{J \setminus \{j\}}$$

where

$$z = \sum_{\substack{J \subset [s] \\ |J| = i}} z_J e_J$$

and

$$\begin{aligned} d_i(z) &= \sum_{j \in J} \sum_{\substack{J \subset [S] \\ |J| = i}} \pm z_J x_j e_{J \setminus \{j\}} \\ &= \sum_{\substack{L \subset [S] \\ |L| = i-1}} \left(\sum_{j \notin L} \pm z_{L \cup \{j\}} x_j \right) e_L. \end{aligned}$$

Since $d_i(z) \in \mathfrak{m}^2 K_{i-1}$, we have that

$$\sum_{j \notin L} \pm z_{L \cup \{j\}} x_j \in \mathfrak{m}^2.$$

But $\overline{x}_1, \ldots, \overline{x}_s$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$ and hence $z_{L \cup \{j\}} \in \mathfrak{m}$, forcing $z \in \mathfrak{m}K_i$. □

Conjecture 1 (Buchsbaum-Eisenbud, Horrocks)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, M a finitely generated module over R with finite length (i.e. M is artinian) and finite projective dimension. Then*

$$\dim_k \operatorname{Tor}_i^R(M, k) \geqslant \binom{d}{i}$$

where $d = \dim(R)$.

Remark. Huneke asked if finite projective dimension is needed and Serre showed the conjecture is true if $M = k$.

2 Corollaries of a Regular Local Ring

In this section we develop several corollaries of theorem 1. We start off with some definitions that are needed to state the corollaries.

Definition. For a ring R and an ideal $I \subset R$, the *codimension* of I in R is

$$\operatorname{codim}(I) = \dim(R) - \dim(R/I).$$

Definition. By $\mu(M)$, we mean the minimal number of generators of a module M.

Definition. Let $(R, \mathfrak{m}, k)$ and (S, η, l) be local rings. A ring homomorphism $(R, \mathfrak{m}, k) \to (S, \eta, l)$ is a *local homomorphism* if $\mathfrak{m}S \subset \eta$.

Definition. A ring homomorphism from R into S is a *flat homomorphism* if S is a flat R-module.

Note. If we have flat local ring homomorphism $(R, \mathfrak{m}, k) \to (S, \eta, l)$, then it is not necessary that $k = l$. Consider the natural map from the rationals to the reals; this is flat.

Definition. Let R be a noetherian ring. We say R is *regular* if it is locally, i.e., $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$. (Equivalently, $R_\mathfrak{m}$ is a regular local ring for all maximal ideals $\mathfrak{m}$ of R.)

Note. A regular ring is not necessarily a domain. For example, consider $\mathbb{Q} \times \mathbb{Q}$.

The following are corollaries of theorem 1.

Corollary 10. *Unless otherwise stated, let $(R, \mathfrak{m}, k)$ be a regular local ring and $I \subset \mathfrak{m}$*

(1) Let x be a non-zero element of $\mathfrak{m}$. Then R/x is a regular local ring if and only if $x \notin \mathfrak{m}^2$.

(2) (For Jacobian Criterion) R/I is a regular local ring if and only if

$$\dim_k\left(\frac{I + \mathfrak{m}^2}{\mathfrak{m}^2}\right) = \operatorname{codim}(I).$$

(3) For all prime ideals $\mathfrak{q}$ in R, $R_\mathfrak{q}$ is a regular local ring.

(4) If $\widehat{R}$ is the completion with respect to $\mathfrak{m}$, then R is a regular local ring if and only if $\widehat{R}$ is a regular local ring.

(5) Let (S, η, l) be a local ring and $(R, \mathfrak{m}, k) \to (S, \eta, l)$ a local flat homomorphism. If $S/\mathfrak{m}S$ is a regular local ring, then S is a regular local ring.

(6) If R is a regular ring, not necessarily local, then $R[x_1, \ldots, x_n]$ is regular.

(7) If R is a regular local ring, then $R[\![x_1, \ldots, x_n]\!]$ is a regular local ring.

(8) Let $(R, \mathfrak{m}, k)$ have dimension d and $y_1, \ldots, y_t \in \mathfrak{m}$ be a maximal regular sequence. Then $t = d$.

Proof of (1). Since R is a regular local ring, we know that R is a domain. Hence

$$\dim(R/x) = \dim(R) - 1.$$

Therefore by theorem 1 part (4), R/x is regular if and only if

$$\mu(\mathfrak{m}/x) = \dim(R) - 1,$$

But by NAK we have

$$\mu(\mathfrak{m}/x) = \begin{cases} \mu(\mathfrak{m}) & \text{if } x \in \mathfrak{m}^2 \\ \mu(\mathfrak{m}) & \text{if } x \notin \mathfrak{m}^2 \end{cases}$$

where $\mu(\mathfrak{m}) = \dim(R)$. □

Note. The proof of corollary (2) is left as an exercise.

Proof of (3). By part (2) of theorem 1 we have that

$$\operatorname{pdim}_R(R/\mathfrak{q}) < \infty.$$

Since localization is flat, we also have that

$$\operatorname{pdim}_{R_\mathfrak{q}}(R/\mathfrak{q})_\mathfrak{q} < \infty.$$

But $(R/\mathfrak{q})_\mathfrak{q}$ is the residue field of $R_\mathfrak{q}$. Thus by part (1) of theorem 1, $R_\mathfrak{q}$ is a regular local ring. □

Proof of (4). By part (1) of theorem 1, the projective dimension of k is finite. Since $\widehat{R}$ is flat over R, and

$$k \otimes_R \widehat{R} = \widehat{R}/\mathfrak{m}\widehat{R} = R/\mathfrak{m} = k,$$

we see that $\operatorname{pdim}_{\widehat{R}} k < \infty$ and therefore $\widehat{R}$ is regular.

Conversely, fix a minimal free resolution $F_.$ of k over R:

$$F_. : \cdots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow k \longrightarrow 0.$$

Now apply $\widehat{R} \otimes_R \cdot$ to the above resolution to get the exact sequence

$$\cdots \longrightarrow \widehat{F}_i \xrightarrow{\widehat{\varphi}_i} \widehat{F}_{i-1} \longrightarrow \cdots \longrightarrow \widehat{F}_1 \longrightarrow \widehat{R} \longrightarrow \widehat{k} \longrightarrow 0.$$

Moreover, $\widehat{F}_i$ are free $\widehat{R}$-modules and $\widehat{\varphi}_i(\widehat{F}_i) \subseteq \widehat{m}\widehat{F}_{i-1}$, so it is also minimal. □

Proof of (5). By part (3) of theorem 1, we know there exists $(x_1, \ldots, x_d) = \mathfrak{m}$ where $x_1, \ldots, x_d$ is a regular sequence in R. We also know there exists $y_1, \ldots, y_n \in S$ such that

(i) $(\mathfrak{m}S, y_1, \ldots, y_n) = \eta$;

(ii) $\overline{y}_1, \ldots, \overline{y}_n$ are a regular sequence in $\overline{S} = S/\mathfrak{m}S$.

Note that
$$\eta = (\varphi(x_1), \dots, \varphi(x_d)), y_1, \dots, y_n).$$
So it is enough to show these elements are a regular sequence in S; further, we only need to show
$$H_i(\varphi(x_1), \dots, \varphi(x_d); S) = 0$$
for all $i \geqslant 1$. By a change of base, remark (1.2), we have
$$H_i(\varphi(x_1), \dots, \varphi(x_d); S) = H_i(x_1, \dots, x_d; R) \otimes_R S$$
since S is flat over R and that $H_i(x_1, \dots, x_d; R) = 0$ for all $i \geqslant 1$. □

Proof of (6). By induction, it is enough to show that $R[x]$ is regular. Let $Q \in \operatorname{Spec}(R[x])$, and $Q \cap R = \mathfrak{q}$. We need to prove that $R[x]_Q$ is a regular local ring. By first localizing at $R \setminus \mathfrak{q}$, without losing any generality, R is local and $Q \cap R = \mathfrak{m}$ is maximal in R. We now have
$$(R, \mathfrak{m}) \longrightarrow (R[x]_Q, \eta).$$
We know R is a RLR by assumption. By part (5) of the above corollary, it is enough to show the map is flat and that $R[x]_Q/\mathfrak{m}R[x]_Q$ is regular.

It is flat since $R \to R[x]$ is free and $R[x] \to R[x]_Q$ is flat (composition of flat is flat). Now $(R[x]/\mathfrak{m}R[x])_Q$ is a localization of $k[x]$, a PID. For $Q \in \operatorname{Spec}(k[x])$ we either have $Q = 0$ or $Q = (f)$. If $Q = 0$, then $k[x]_Q = k(x)$ is a field. If $Q = (f)$, then by part (4), $R[x]_Q$ is regular (DVR). □

Note. A one dimensional regular domain is a Dedekind domain.

Note. The proof of part (7) is left as an exercise.

Proof of (6). We know that $\operatorname{pdim}_R(k) = d$ and thus $\operatorname{pdim}_R(M) \leqslant d$ for all finitely generated R-modules M. By Assumption, we also have
$$0 \longrightarrow k \longrightarrow \frac{R}{(y_1, \dots, y_t)} \longrightarrow N \longrightarrow 0.$$
Now, $\operatorname{pdim}_R(R/(y_1, \dots, y_t)) = t$ as $y_1, \dots, y_t$ is a regular sequence, hence $t \leqslant d$ and
$$\operatorname{Tor}_{d+1}(k, N) \longrightarrow \operatorname{Tor}_d(k, k) \longrightarrow \operatorname{Tor}_d(k, \frac{R}{(y_1, \dots, y_t)})$$
The first term is zero since $\operatorname{pdim}_R(N) \leqslant d$. Likewise, the second term is non-zero as $\operatorname{pdim}_R(k) = d$. Thus we have
$$t = \operatorname{pdim}_R(\frac{R}{y_1, \dots, y_t}) \geqslant d,$$
forcing $t = d$. □

Example 4. The ring $k[x]/(x^2)$ is not regular.

Example 5. Below is a ring that is a domain, but not regular:

$$k[t^2, t^3] \simeq \frac{k[x,y]}{(x^2 - y^3)}.$$

3 The Jacobian Criterion

The purpose of this section is to develop the Jacobian criterion stated below. The proof will be broken into two parts, the complete case and the general case.

Definition. A field k is *perfect* if either the characteristic is zero, or $k = k^p$ when the characteristic is p.

Theorem 11. *[Jacobian Criterion] Let k be a perfect field, $S = k[x_1, \ldots, x_n]$, and $\mathfrak{p} \subseteq S$ a prime ideal of height h. Write*

$$\mathfrak{p} = (F_1, \ldots, F_m); \quad J = (\frac{\partial F_i}{\partial x_j}); \quad R = S/\mathfrak{p}.$$

Let $\mathfrak{q} \in \operatorname{Spec}(R)$ and write

$$L = Q.F.(R/\mathfrak{q}) = k(\xi_1, \ldots, \xi_n),$$

where $\xi_i = x_i + \mathfrak{q}$. Then

(1) $\operatorname{rank}(J(\xi_1, \ldots, \xi_n)) \leqslant h$*;*

(2) $\operatorname{rank}(J(\xi_1, \ldots, \xi_n)) = h$ *if and only if $R_{\mathfrak{q}}$ is RLR.*

3.1 The Complete Case

By corollary 10, we know that if R is a regular local ring, then $R[\![x_1, \ldots, x_n]\!]$ is also a regular local ring.

Proposition 12. *Suppose $(R, \mathfrak{m}, k)$ is a d-dimensional, complete, regular local ring containing a field. Then*

$$R \simeq k[\![T_1, \ldots, T_d]\!].$$

Note. The ring $\widehat{\mathbb{Z}}_{\mathfrak{p}}$ is not isomorphic to the power series ring; it does not contain a field.

Proof. By Cohen's structure theorem [3, theorem 7.7][1], R contains a copy of k. I.e.

$$\begin{tikzcd} k \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & R \arrow[r] & R/\mathfrak{m}. \end{tikzcd}$$

[1] This was discussed in the first semester, but the notes are not finished. When the first semester notes are complete, this reference will point there.

We also know $\mathfrak{m} = (x_1, \dots, x_d)$. Therefore, consider the map

$$\begin{aligned} k[\![T_1, \dots, T_d]\!] &\xrightarrow{\varphi} R \\ k &\longmapsto k \\ T_i &\longmapsto x_i \end{aligned}$$

Note, if

$$f = \sum \alpha_{\underline{v}} T^{\underline{v}} \in k[\![T_1, \dots, T_d]\!],$$

then

$$\varphi(f) = \sum \alpha_{\underline{v}} x^{\underline{v}}$$

has meaning in R since R is complete.

Claim. The map φ is an isomorphism.

To show that φ is onto, let $r \in R$ and define $\alpha_0 \in k$ such that

$$\alpha_0 \equiv r \pmod{\mathfrak{m}}.$$

Then define $r_1 \equiv r - \alpha_0$. Thus r_1 is an element of $\mathfrak{m}$ and we can write

$$r_1 = \sum_{j=1}^{d} s_{1j} x_j$$

for $s_{1j} \in \mathfrak{m}$. Choose $\alpha_{1j} \in k$ such that

$$s_{1j} \equiv \alpha_{1j} \pmod{\mathfrak{m}}.$$

Now define

$$r_2 = r_1 - \sum \alpha_{1j} x_j \in \mathfrak{m}^2.$$

Therefore,

$$r_2 = \sum_{|\underline{v}|=2} s_{1\underline{v}} x^{\underline{v}}.$$

Choose $\alpha_{1\underline{v}} \in k$ such that $s_{\underline{v}} - \alpha_{\underline{v}} \in \mathfrak{m}$. Now repeat.

If we define

$$f = \alpha_0 + \alpha_1 T_1 + \cdots + \alpha_d T_d + \sum_{|\underline{v}| \geqslant 2} \alpha_{\underline{v}} T^{\underline{v}} \in k[\![T_1, \dots, T_d]\!],$$

then $\varphi(f) = r$ and thus φ is onto.

Now assume φ is not injective, that is, $\ker(\varphi) \neq 0$. So

$$R \simeq \frac{k[\![t_1, \dots, t_d]\!]}{\ker(\varphi)}$$

has dimension less than d; a contradiction. $\square$

Example 6. Below are a couple of examples of proposition (12):

(1)
$$k\widehat{[T_1,\ldots,T_d]}_{(T_1,\ldots,T_d)} \simeq k[\![T_1,\ldots,T_d]\!];$$

(2)
$$\left(\widehat{\frac{k[x,y]}{(x^2+y^2-1)}}\right)_{\mathfrak{m}} \simeq k[\![T]\!].$$

Introduction and Remarks Let $f \in \mathbb{C}[x_1,\ldots,x_n]$ be non-zero and irreducible. Then every maximal ideal $\mathfrak{m} \in \mathfrak{m} - \operatorname{Spec}(R)$, $R = \mathbb{C}[x_1,\ldots,x_n]/(f)$, is of the form $(x_1-\alpha_1,\ldots,x_n-\alpha_n)$ such that $f(\alpha_1,\ldots,\alpha_n)=0$ (Nullstellensatz). Which $\mathfrak{m}$ are such that $R_{\mathfrak{m}}$ is RLR?

Note. We have that $\dim(R) = n-1$. Therefore, $R_{\mathfrak{m}}$ is RLR if and only if $\dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2) = n-1$.

Let $M = (x_1-\alpha_1,\ldots,x_n-\alpha_n)$ be a maximal ideal in $\mathbb{C}[x_1,\ldots,x_n]$ such that $M/(f) = \mathfrak{m}$. There exists a short exact sequence of vector spaces

$$0 \longrightarrow \frac{(f,M^2)}{M^2} \longrightarrow \frac{M}{M^2} \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow 0.$$

Therefore $R_{\mathfrak{m}}$ is RLR iff $\dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2) = n-1$ iff $f \notin M^2$. By the taylor expansion,

$$\begin{aligned} f &= f(\alpha_1,\ldots,\alpha_n) + \sum \frac{\partial f}{\partial x_i}(\underline{\alpha})(x_i-\alpha_i) + M^2 \\ &= \sum \frac{\partial f}{\partial x_i}(\underline{\alpha})(x_i-\alpha_i) + M^2. \end{aligned}$$

So the image of f in M/M^2, with basis $\langle x_i-\alpha_i\rangle$, is just $\left(\frac{\partial f}{\partial x_i}(\underline{\alpha})\right)_{1\times n}$. Therefore,

$$\begin{aligned} R_{\mathfrak{m}} \text{ is RLR } &\iff M \not\supseteq \left(\frac{\partial f}{\partial x_i}\right) \\ &\iff \text{there exists } i \text{ such that } \frac{\partial f}{\partial x_i}(\underline{\alpha}) \neq 0. \end{aligned}$$

Example 7. Find the singular locus

$$\operatorname{Sing}\left(\frac{k[x,y,z,u]}{(xy-z^2)}\right),$$

where k is algebraically closed and $\operatorname{Char}(k) \neq 2$. That is, find all prime ideals $\mathfrak{p}$ such that $R_{\mathfrak{p}}$ is not a RLR. Let $f = xy - z^2$. Let I be the ideal of partials of f;

$$I = (y, x, -2z, 0) = (x,y,z)$$

Therefore, the singular locus is $V(I) = \{(x,y,z,u-\alpha)$.

Lemma 13. *Theorem 11 holds if $\mathfrak{q}$ is maximal.*

Proof. In this case, $R/\mathfrak{q} = L$. By nullstellensatz, L is algebraic over k. Choose G_i inductively as follows:

$$G_i = \text{ minimal polynomial of } \xi_i \text{ over } k(\xi_1, \ldots, \xi_{i-1}).$$

Lifting back to S gives

$$\begin{aligned} G_1(x_1) &= \text{ irreducible polynomial of } \xi_1 \text{ over } k \\ G_2(x_1, x_2) &= \text{ irreducible polynomial of } \xi_2 \text{ over } k(\xi_1) \\ &\vdots \\ G_n(x_1, \ldots, x_n) &= \text{ irreducible polynomial of } \xi_n \text{ over } k(\xi_1, \ldots, \xi_{n-1}). \end{aligned}$$

It is left to the reader to show that $\mathfrak{q} = (G_1, \ldots, G_n)$. Noctice the Jacobian matrix for $\mathfrak{q}$ is

$$\begin{aligned} K &= \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \cdots & \frac{\partial G_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial G_n}{\partial x_1} & \frac{\partial G_n}{\partial x_2} & \cdots & \frac{\partial G_n}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & & 0 \\ & \ddots & \\ * & & \frac{\partial G_n}{\partial x_n} \end{pmatrix}. \end{aligned}$$

Therefore, the determinant of $K(\xi_1, \ldots, \xi_n)$ is

$$\prod_{i=1}^{n} \frac{\partial G_i}{\partial x_i}(\xi_1, \ldots, \xi_n).$$

By separability this determinant is not zero (derivative of minimal polynomials are not evaluated to zero.) since $\mathfrak{p} \subseteq \mathfrak{q}$, we can write

$$F_i = \sum_{l=1}^{n} r_{il} G_l.$$

Therefore

$$\frac{\partial F_i}{\partial x_j} = \sum_{l=1}^{n} \frac{\partial r_{il}}{\partial x_j} G_l + \sum_{l=1}^{n} r_{il} \frac{\partial G_l}{\partial x_j}$$

for $j = 1, 2, \ldots, n$ and

$$\frac{\partial F_i}{x_j}(\xi_1, \ldots, \xi_n) = \sum_{l=1}^{n} r_{il}(\xi_1, \ldots, \xi_n) \frac{\partial G_l}{\partial x_j}(\xi_1, \ldots, \xi_n)$$

for $j = 1, 2, \ldots, n$. Therefore

$$J(\xi_1, \ldots, \xi_n) = (r_{il}(\xi_1, \ldots, \xi_n))_{m\times n} K(\xi_1, \ldots, \xi_n).$$

By above, $K(\xi_1, \ldots, \xi_n)$ is invertible, so

$$\text{rank } J(\xi_1, \ldots, \xi_n) = \text{rank}(r_{il}(\xi_1, \ldots, \xi_n)).$$

Notice that $\dim(R_{\mathfrak{q}}) = n - h$. Thus we have the $R_{\mathfrak{q}}$ is RLR if and only if

$$\dim_L \frac{\mathfrak{q}}{\mathfrak{p} + \mathfrak{q}^2} = n - h.$$

(We always have $\geqslant$ by Krull's principal ideal theorem.) We have a short exact sequence of vector spaces over L

$$0 \longrightarrow \frac{\mathfrak{p} + \mathfrak{q}^2}{\mathfrak{q}^2} \longrightarrow \frac{\mathfrak{q}}{\mathfrak{q}^2} \longrightarrow \frac{\mathfrak{q}}{\mathfrak{p} + \mathfrak{q}^2} \longrightarrow 0,$$

where $\mathfrak{q}/\mathfrak{q}^2$ is n-dimensional over L. Therefore,

$$\dim_L \frac{\mathfrak{p} + \mathfrak{q}^2}{\mathfrak{q}^2} \leqslant h$$

with equality if and only if $R_{\mathfrak{q}}$ is a RLR. Note that

$$(G_1(\xi_1, \ldots, \xi_n), \ldots, G_n(\xi_1, \ldots, \xi_n))$$

are a basis of $\mathfrak{q}/\mathfrak{q}^2$. Thus

$$\begin{aligned}\dim_L \frac{\mathfrak{p} + \mathfrak{q}^2}{\mathfrak{q}^2} &= \dim_L \frac{(F_1, \ldots, F_m) + \mathfrak{q}^2}{\mathfrak{q}^2} \\ &= \text{ row rank of } (r_{il}(\xi_1, \ldots, \xi_n)) \\ &= \text{rank } J(\xi_1, \ldots, \xi_n).\end{aligned}$$

Here the $r_{il}(\xi_1, \ldots, \xi_n)$ are coordenents of $F_1(\xi_1, \ldots, \xi_n), ldots, F_m(\xi_1, \ldots, \xi_n)$ in terms of $G_1(\xi_1, \ldots, \xi_n), \ldots, G_n(\xi_1, \ldots, \xi_n)$. □

Conjecture 2 (Baduendi-Rothschild). *Let $S = \mathbb{C}[\![x_1, \ldots, x_d]\!]$ and $F_1, \ldots, F_d$ be an system of parameters in S. Let $R = \mathbb{C}[\![F_1, \ldots, F_d]\!]$ be a subring of S and $\mathfrak{p}$ a prime ideal in R. If $S/\sqrt{\mathfrak{p}S}$ is regular, then $R/\mathfrak{p}$ is also regular.*

4 Exercises

(1) Prove that for a ring R and a non-zero divisor $x \in R$ that $H_1(x, 0; R) \simeq H_1(x; R) \oplus H_0(x; R)$.

(2) Let R be a ring and M, N be R-modules. Then

$$\operatorname{ann}(\operatorname{Tor}_i^R(M, N)) \supset \operatorname{ann}(M) + \operatorname{ann}(N).$$

(3) Let R be a regular local ring and $I \subset R$. Then R/I is a regular local ring if and only if

$$\dim_k(\frac{I + \mathfrak{m}^2}{\mathfrak{m}^2}) = \operatorname{codim}(I).$$

(4) Let R be a Noetherian local domain. Then $R \simeq R_1 \times \cdots \times R_t$ where R_i are domains.

(5) If R is a RLR, then $R[\![x_1, \ldots, x_n]\!]$. (Hint: use $R \to R[\![x_1, \ldots, x_n]\!]$ is a flat extension.)

(6) Find the singular locus of $\mathbb{C}[x, y, z, u]/(x^2 + y^2 + z^2)$.

Chapter 2

Depth, Cohen-Macaulay Modules, and Projective Dimension

Definition. Let I be an ideal in a Noetherian ring R and M a finitely generated R module such that $IM \neq M$. The *depth* of M in I, denoted $\operatorname{depth}_I(M)$, is the length of the longest regular sequence on M contained in I. As a convention, if R is local we write $\operatorname{depth}(M) := \operatorname{depth}_{\mathfrak{m}}(M)$.

Proposition 14. *Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finitely generated R-module such that $IM \neq M$. Suppose $y_1, \ldots y_t \in I$ is a maximal regular sequence on M. Then $t = \min\{i | \operatorname{Ext}^i_R(R/I, M) \neq 0\}$. In particular, t is independent of the regular sequence chosen.*

Proof. Suppose first that $t = 0$, then we need to show

$$\operatorname{Ext}^0_R(R/I, M)(= \hom_R(R/I, M)) \neq 0$$

Since there is no regular sequence on M contained in I, no element of I is a NZD on M, i.e.

$$I \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$$

By prime avoidance, $I \subseteq \mathfrak{p} \in \operatorname{Ass}(M)$ for one of these $\mathfrak{p}$. Then $R/\mathfrak{p} \hookrightarrow M$ (by definition of $\operatorname{Ass}(M)$), so we have

$$R/I \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$$

so that $\hom_R(R/I, M) \neq 0$ (the composition of the maps above is there). On the other hand, if $\hom_R(R/I, M) \neq 0$, then $\exists\ 0 \neq z \in M$ s.t. $Iz = 0$. Thus $\nexists$ a NZD on M in I, so $t = 0$.
We proceed by induction on t. Consider the short exact sequence

$$0 \to M \overset{*y_1}{\to} M \to \overline{M} \to 0$$

where $\overline{M} := M/y_1M$. Note $y_2, \ldots, y_t$ is a maximal regular sequence on $\overline{M}$, so by induction $t-1 = \min\{i | \mathrm{Ext}^i_R(R/I, \overline{M}) \neq 0\}$. Apply $\hom_R(R/I, \_)$ to the sequence to obtain

$$\cdots \to \mathrm{Ext}^{j-1}_R(R/I, \overline{M}) \to \mathrm{Ext}^j_R(R/I, M) \overset{*y_1}{\to} \mathrm{Ext}^j_R(R/I, M) \to \mathrm{Ext}^j_R(R/I, \overline{M}) \to \cdots$$

When $j = t$, $\mathrm{Ext}^{j-1}_R(R/I, \overline{M}) \neq 0$ and when $j < t-1$, $\mathrm{Ext}^{j=1}_R(R/I, \overline{M}) = \mathrm{Ext}^j_R(R/I, \overline{M}) = 0$. Hence

$$\mathrm{Ext}^j_R(R/I, M) \overset{*y_1}{\to} \mathrm{Ext}^j_R(R/I, M)$$

is an isomorphism, but since $y_1 \in \mathrm{ann}(R/I) \subseteq \mathrm{ann}(\mathrm{Ext}^j_R(R/I, M))$, we must have $\mathrm{Ext}^j_R(R/I, M) = 0$.
Finally for $j = t-1$, we have

$$0 \to \mathrm{Ext}^{t-1}_R(R/I, M) \overset{*y_1}{\to} \mathrm{Ext}^{t-1}_R(R/I, M) \to \mathrm{Ext}^{t-1}_R(R/I, \overline{M}) \to \mathrm{Ext}^t_R(R/I, M) \overset{*y_1}{\to} \cdots$$

Note that the first and last maps (multiplication by y_1) are the 0 map, so $\mathrm{Ext}^{t-1}_R(R/I, M) = 0$, which implies $\mathrm{Ext}^t_R(R/I, M) \simeq \mathrm{Ext}^{t-1}_R(R/I, \overline{M}) \neq 0$. □

Remark 1. If y is a NZD on M and $y \in I$,

$$\mathrm{depth}_I(M/yM) = \mathrm{depth}_I(M) - 1$$

Remark 2. If $t = \mathrm{depth}_I(M)$, then $\mathrm{Ext}^t_R(R/I, M) \simeq \hom_R(R/(y_1, \ldots, y_t), M)$ where $y_1, \ldots, y_t$ is a maximal regular M-sequence in I and S.

Remark 3. Why do we require $IM \neq M$? If $IM = M$, $1 + i \in \mathrm{ann}(M)$ for some i, so $(1+i)\mathrm{Ext}^j_R(\_, M) = 0$ and clearly $I\mathrm{Ext}^j_R(R/I, \_) = 0$, so we have $\mathrm{Ext}^j_R(R/I, M) = 0$ for all j. Thus depth doesn't make sense in this context.

Lemma 15 (Depth Lemma part 1)**.** *Suppose R is Noetherian, $I \subseteq R$ an ideal, and N, M, K finitely generated R-modules such that $IN \neq N$, $IM \neq M$, and $IK \neq K$. If*

$$0 \to N \to M \to K \to 0$$

is a short exact sequence, then

$$\mathrm{depth}_I(K) \geq \min\{\mathrm{depth}_I(N), \mathrm{depth}_I(M)\} - 1$$

Proof. Set $a = \min\{\mathrm{depth}_I(N), \mathrm{depth}_I(M)\}$. By proposition 14, $\mathrm{Ext}^j_R(R/I, M) = \mathrm{Ext}^j_R(R/I, N) = 0$ for all $j < a$ and one of $\mathrm{Ext}^a_R(R/I, N)$ or $\mathrm{Ext}^a_R(R/I, M)$ is nonzero. Apply $\hom_R(R/I, \_)$ to the short exact sequence to obtain

$$\cdots \to \mathrm{Ext}^j_R(R/I, M) \to \mathrm{Ext}^j_R(R/I, K) \to \mathrm{Ext}^{j+1}_R(R/I, N) \to \cdots$$

so for $j \leq a-2$, both ends are 0 which forces $\mathrm{Ext}^j_R(R/I, K) = 0$. Thus $\mathrm{depth}_I(K) \geq a - 1$. □

Theorem 16 (Auslander Buchsbaum Formula)**.** *Let* $(R, \mathfrak{m}, k$ *be a Noetherian local ring,* $0 \neq M$ *a finitely generated* R*-module such that* $\mathrm{pdim}_R(M) < \infty$*. Then*

$$\mathrm{depth}(M) + \mathrm{pdim}_R(M) = \mathrm{depth}(R)$$

Proof. If $\mathrm{depth}(R) = 0$, we need to show $\mathrm{pdim}_R(M) = \mathrm{depth}(M) = 0$. We claim that M is free. Since $\mathrm{depth}(R) = 0$, $\mathfrak{m} \in \mathrm{Ass}(R)$ and $R/\mathfrak{m} \hookrightarrow R$ so $\exists$ nonzero $z \in R$ such that $\mathfrak{m}z = 0$. If M is not free, we can write a minimal free resolution of M:

$$0 \to F_n \overset{\varphi_n}{\to} \cdots \to F_1 \overset{\varphi_1}{\to} F_0 \to M \to 0$$

where $n > 0$. Recall by minimality, $\varphi_n(F_n) \subseteq \mathfrak{m}F_{n-1}$ so that $\varphi_n((z, 0, \ldots, 0)) = 0$ because $\varphi_n((z, 0, \ldots, 0)) = z\varphi((1, 0, \ldots, 0)) \in z\mathfrak{m} = 0$, so φ_n is not 1-1, a contradiction. Thus M is free, so $\mathrm{pdim}_R(M) = 0$ and $\mathrm{depth}(M) = \mathrm{depth}(R^n) = \mathrm{depth}(R) = 0$.
Next assume $\mathrm{depth}(M) = 0$. Set $t = \mathrm{depth}(R)$ and choose a maximal regular sequence $y_1, \ldots, y_t \in \mathfrak{m}$. Note $\mathrm{pdim}(R/(y_1, \ldots, y_t)) = t$. Let $\mathrm{pdim}_R(M) = n$, we want to show that $n = t$. Consider $\mathrm{Tor}_j^R(R/(y_1, \ldots, y_t), M)$. It's enough to show this is nonzero for $j = t$ and for $j = n$. For $j = t$, consider (the end of) a free resolution of $R/(y_1, \ldots, y_t)$

$$0 \to R \overset{[\pm y_i]}{\to} R^t \to \cdots$$

and tensor with M we have

$$0 \to M \overset{[\pm y_i]}{\to} M^t \to \cdots$$

then $\mathrm{Tor}_t^R(R/(y_1, \ldots, y_t), M)$ is the kernel of the map defined by $[\pm y_i]$, which is nonzero because $\mathrm{depth}(M) = 0$ and the y_i's are in $\mathfrak{m}$. By a similar argument, $\mathrm{Tor}_n^R(R/(y_1, \ldots, y_t), M) \neq 0$ (by computing the other way).
Finally assume $\mathrm{depth}(M)$ and $\mathrm{depth}(R)$ are greater than 0 and proceed by induction. By prime avoidance, $\exists$ $x \in \mathfrak{m}$ which is a NZD on R and M. Then $\mathrm{Tor}_1^R(R/(x), M) = 0$ for all $i \geq 1$ (because the projective dimension of $R/(x)$ is 1 by the koszul complex). Now if

$$0 \to F_n \to \cdots \to F_1 \to M \to 0$$

is a free resolution of M and we tensor with $R/(x)$, we have

$$0 \to \overline{F_n} \to \cdots \to \overline{F_0} \to \overline{M} \to 0$$

is exact, so $\mathrm{pdim}_R(M) = \mathrm{pdim}_{\overline{R}}(\overline{M})$. By induction hypothesis, $\mathrm{depth}(\overline{M}) + \mathrm{pdim}_{\overline{R}}(\overline{M}) = \mathrm{depth}(\overline{R})$, so that

$$\mathrm{depth}(M) - 1 + \mathrm{pdim}(M) = \mathrm{depth}(R) - 1$$

which proves the formula. $\square$

Definition. Let $(R, \mathfrak{m}, k)$ be a local, noetherian ring and M a finitely generated R-module. We say M is *Cohen-Macaulay* (or *C-M*) if $\operatorname{depth}(M) = \dim(M)$.

Theorem 17. *Let $(R, \mathfrak{m}, k)$ be a local, noetherian ring and M, N finitely generated R-modules. Then $\operatorname{Ext}^i_R(M, N) = 0$ for all $i < \operatorname{depth}(N) - \dim(M)$.*

Proof. First of all note that there is a prime filtration of M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_{i+1}/M_i \simeq R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec}(R)$. By induction we can show:
Claim: If $\operatorname{Ext}^j_R(R/p_i, N) = 0$ for all i, then $\operatorname{Ext}^j_R(M, N) = 0$.
This is done by breaking the filtration into short exact sequences of the form

$$0 \to M_{n-1} \to M \to R/\mathfrak{p}_{n-1} \to 0$$

and applying $\hom_R(\_, N)$ to obtain

$$\cdots \to \operatorname{Ext}^j_R(R/\mathfrak{p}_{n-1}, N) \to \operatorname{Ext}^j_R(M, N) \to \operatorname{Ext}^j_R(M_{n-1}, N) \to \cdots$$

The first and last Ext's are 0 by assumption and induction hypothesis, so $\operatorname{Ext}^j_R(M, N) = 0$. Finally note that $\dim(R/p_i) \leq \dim(M)\ \forall\ i$, so we can assume without loss of generality that $M = R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$.

Proceed by induction on $\dim(M)$. If $\dim(M) = 0$, we must have $\mathfrak{p} = \mathfrak{m}$, the maximal ideal of R, so that $M = k$ and we know $\operatorname{Ext}^i_R(k, N) = 0$ for all $i < \operatorname{depth}(N)$.
If $\dim(M) > 0$, let $M = R/\mathfrak{p}$ and choose $x \in \mathfrak{m} \setminus \mathfrak{p}$. Consider the short exact sequence

$$0 \to M \stackrel{*x}{\to} M \to \overline{M} \to 0$$

where $\overline{M} = R/(\mathfrak{p}, x)$. This induces the long exact sequence of Ext :

$$\cdots \to \operatorname{Ext}^i_R(\overline{M}, N) \to \operatorname{Ext}^i_R(M, N) \stackrel{*x}{\to} \operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^{i+1}_R(\overline{M}, N) \to \cdots$$

By induction, $\operatorname{Ext}^i_R(\overline{M}, N) = 0$ for all $i < \operatorname{depth}(N) - \dim(M) + 1$. Now if $i < \operatorname{depth}(N) - \dim(M)$ (so that $i + 1 < \operatorname{depth}(N) - \dim(M) + 1$), we have $\operatorname{Ext}^i_R(\overline{M}, N)\operatorname{Ext}^{i+1}(\overline{M}, N) = 0$, and thus $\operatorname{Ext}^i_R(M, N) = 0$ by NAK. □

Corollary 18. *Let $(R, \mathfrak{m}, k)$ be a local, Noetherian ring and M a finitely generated R-module. If $\mathfrak{p} \in \operatorname{Ass}(M)$, then*

$$\operatorname{depth}(M) \leq \dim(R/\mathfrak{p})$$

Proof. Apply theorem 17 to $\operatorname{Ext}^i_R(R/\mathfrak{p}, M)$. This is 0 for $i < \operatorname{depth}(M) = \dim(R/\mathfrak{p})$. But note that $\hom_R(R/\mathfrak{p}, M) \neq 0$ because $R/\mathfrak{p} \hookrightarrow M$, so $\operatorname{depth}(M) - \dim(R/\mathfrak{p}) \leq 0$. □

Corollary 19. *Let R and M be as above, and assume M is Cohen-Macaulay. Then $\forall\ \mathfrak{p} \in \operatorname{Ass}(M)$,*

$$\dim(R/\mathfrak{p}) = \dim(M)$$

Proof. By the first corollary,

$$\operatorname{depth}(M) \leq \dim(R/\mathfrak{p}) \leq \dim(M)$$

and since M is C-M, $\operatorname{depth}(M) = \dim(M)$ so equality holds. □

Example 1. The ring $R = k[x, y, z]/(xy, xz)$ cannot be Cohen-Macaulay since $\operatorname{Ass}(R) = \{(x), (y, z)\}$ and $\dim(R/(x)) = 2 \neq 1 = \dim(R/(y, z))$ (the corollary doesn't hold so R is not C-M).

Example 2. The ring $R = k[x, y, u, v]/(x, y) \cap (u, v)$ could be Cohen-Macaulay since $\operatorname{Ass}(R) = \{(x, y)R, (u, v)R\}$ and $\dim(R/(x, y)R) = \dim(R/(u, v)R) = 2$, but in fact it is not Cohen-Macaulay (so the converse of corollary 19 does not hold.

Example 3. Over a ring R of dimension 0, any finitely generated R-module is Cohen-Macaulay (because $\dim(M) = \operatorname{depth}(M) = 0$).

Example 4. If R is a dimension 1 local domain, R is Cohen-Macaulay (because any nonzero element is a NZD).

Example 5. There exist 2-dimensional domains which are not Cohen-Macaulay.

Theorem 20. *If $(R, \mathfrak{m}, k)$ is a local Cohen-Macaulay ring such that $\mathbb{Q} \subseteq R$ and G is a finite group of automorphisms of R, then the fixed ring $S = R^G$ is Cohen-Macaulay.*

Proof. Exercise □

Theorem 21. *Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring. Then for all ideals $I \subseteq R$,*

(1) $\operatorname{ht}(I) = \operatorname{depth}_I(R)$

(2) $\operatorname{ht}(I) + \dim(R/I) = \dim(R)$

(3) If $x_1, \ldots, x_i \in \mathfrak{m}$ and $h((x_1, \ldots, x_i)) = i$, then $x_1, \ldots, x_i$ is a regular sequence on R.

Proof of (3). Extend $x_1, \ldots, x_i$ to a full system of parameters as follows: if $i = \dim(R)$, it already is an s.o.p. so we're done. Otherwise choose x_{i+1} through x_d ($d = \dim(R)$) inductively so that x_j is not in any minimal prime of $(x_1, \ldots, x_{j-1})$ (possible by prime avoidance). Then $j \leq \operatorname{ht}(x_1, \ldots, x_j)$ by construction and $\operatorname{ht}(x_1, \ldots, x_j) \leq j$ by Krull height theorem, so $\operatorname{ht}(x_1, \ldots, x_j) = j$. *Claim:* $x_1, \ldots, x_d$ is a regular sequence on R: First assume by way of contradiction that x_1 is a zero divisor on R. Then $x_1 \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(R)$. Note $\dim(R/\mathfrak{p}) = \dim(R) = d$. But $\overline{x_2}, \ldots, \overline{x_d}$ are an s.o.p. in $R/\mathfrak{p}$, which contradicts Krull's height theorem. Thus x_1 is a nonzero divisor. Once we note that $\operatorname{depth}(R/x_1R) = \operatorname{depth}(R) - 1 = \dim(R) - 1 = \dim(R/x_1R)$ so that R/x_1R is Cohen-Macaulay, we can proceed by induction and see that $x_1, \ldots, x_d$ is a regular sequence on R. Then clearly the truncated sequence $x_1, \ldots, x_i$ is a regular sequence on R. □

Proof of (1). Note for any ring R and ideal $I \subseteq R$, $\operatorname{depth}_I(R) \leq \operatorname{ht}(I)$ because if $x_1, \ldots, x_s$ is a regular sequence, then $(x_1, \ldots, \mathbf{x}_s)$ has height s. On the other hand, if I is an ideal with height s, we can choose $x_1, \ldots, x_s \in I$ such that $\operatorname{ht}(x_1, \ldots, x_s) = s$ by using prime avoidance. By (3), such a $x_1, \ldots, x_s$ is a regular sequence so $\operatorname{depth}_I(R) \geq s = \operatorname{ht}(I)$. □

Proof of (2). Note $\operatorname{ht}(I) = \min\{\operatorname{ht}(\mathfrak{p}) : I \subseteq \mathfrak{p}, \mathfrak{p} \text{ prime}\}$ and $\dim(R/I) = \max\{\dim(R/\mathfrak{p}) : I \subseteq \mathfrak{p}, \mathfrak{p} \text{ prime}\}$. So without loss of generality we may assume I is a prime ideal $\mathfrak{p}$. Set $s = \operatorname{ht}(\mathfrak{p})$ and choose $(x_1, \ldots, x_s) \subseteq \mathfrak{p}$ such that $\operatorname{ht}(x_1, \ldots, x_s) = s$. By (3), $x_1, \ldots, x_s$ is a regular sequence $R/(x_1, \ldots, x_s)$ is C-M of dimension $\dim(R) - s$. Finally, $\mathfrak{p}/(x_1, \ldots, x_s) \in \operatorname{Ass}(R/(x_1, \ldots, x_s)$ since it is minimal, so $\dim(R/\mathfrak{p}) = \dim(R/(x_1, \ldots, x_s)$. Thus $\operatorname{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}) + \dim(R) - \operatorname{ht}(\mathfrak{p}) = \dim(R)$. □

We pause to give a brief overview of what we know about Cohen-Macaulay rings:

- $(R, \mathfrak{m}, k)$ is Cohen-Maculay if and only if there exists a system of parameters which forms a regular sequence on R (or equivalently, all systems of parameters form regular sequences on R).
- If $(R, \mathfrak{m}, k)$ is Cohen-Macaulay, then it is *unmixed*, i.e. $\dim(R/\mathfrak{p}) = \dim(R)$ for all $\mathfrak{p} \in \operatorname{Ass}(R)$.
- R is Cohen-Macaulay if and only if R/Rx is Cohen-Macaulay for all nonunit, non zero divisors $x \in R$.
- The main examples of C-M rings are (all Noetherian local) RLR's, 0-dimensional rings, 1-dimensional domains, and complete intersections (RLR modulo a regular sequence)

Definition. A ring R is *catenary* if for all primes $\mathfrak{p} \subseteq \mathfrak{q}$, all maximal chains of primes between $\mathfrak{p}$ and $\mathfrak{q}$ have the same length.

Theorem 1.17 implies any ring of the form R/I, where R is C-M, is catenary.

Theorem 22. *Let $(R, \mathfrak{m}, k)$ be a local, Noetherian ring, $\mathfrak{p}$ a prime ideal of R, and M a finitely generated R-module. If M is C-M and $M_\mathfrak{p} \neq 0$, then $M_\mathfrak{p}$ is C-M over $R_\mathfrak{p}$.*

Proof. First note that $\dim(M_\mathfrak{p}) \geq \operatorname{depth}(M_\mathfrak{p}) \geq \operatorname{depth}_\mathfrak{p}(M)$ because a maximal regular sequence on M in $\mathfrak{p}$ is still a regular sequence on $M_\mathfrak{p}$. Thus it's enough to show that $\operatorname{depth}_\mathfrak{p}(M) = \dim(M_\mathfrak{p})$. Induct on $\operatorname{depth}_\mathfrak{p}(M)$: If $\operatorname{depth}_\mathfrak{p}(M) = 0$, then $\mathfrak{p} \subseteq \mathfrak{q} \in \operatorname{Ass}(M)$ because $\mathfrak{p}$ consists of zero divisors of M. But every associated prime of M is minmal (M C-M implies M is unmixed), so $\mathfrak{p}$ is minimal, and thus $\dim(M_\mathfrak{p}) = 0$. If $\operatorname{depth}_\mathfrak{p}(M) > 0$, choose $x \in \mathfrak{p}$ a NZD on M. By induction, $\operatorname{depth}_\mathfrak{p}(M/xM) = \dim((M/xM)_\mathfrak{p})$, which is simply $\operatorname{depth}_\mathfrak{p}(M) - 1 = \dim(M_\mathfrak{p}) - 1$ which proves the result. □

Theorem 23. *Suppose $(A, \mathfrak{m}_A) \subseteq (R, \mathfrak{m}_R)$ is a finite extension of Noetherian local rings (i.e. R is a finitely generated A-module), and assume A is a RLR. Then R is C-M if and only if R is free over A.*

Proof. Let $d = \dim(A)$. Note that since A is a RLR, $\operatorname{pdim}_A(R) < \infty$. Thus by the Auslander-Buchsbaum formula, $\operatorname{depth}(R) + \operatorname{pdim}(R) = \operatorname{depth}(A) = d$. But $\operatorname{depth}(R) = d \iff R$ is C-M $\iff \operatorname{pdim}_A(R) = 0 \iff R$ is free over A. □

Theorem 24. *Suppose $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is a flat map of Noetherian local rings. Then*

(1) $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S)$

(2) $\operatorname{depth}(S) = \operatorname{depth}(R) + \operatorname{depth}(S/\mathfrak{m}S)$

(3) S is C-M if and only if R is C-M and $S/\mathfrak{m}S$ is C-M.

To prove this we need the following two lemmas:

Lemma 25. *Let R and S be as above and $I \subseteq R$ an ideal. Then $R/I \to S/IS$ is also flat.*

Proof. Let $0 \to M\ toN \to K \to 0$ be a s.e.s. of R/I modules. Since $R \to S$ is flat and M, N, K are also R-modules, the sequence

$$0 \to M \otimes_R S \to N \otimes_R S \to K \otimes_R S \to 0$$

is exact. But $M \simeq M \otimes_R R/I$ and $N \simeq N \otimes_R R/I$, so $M \otimes_R S \simeq (M \otimes_R R/I) \otimes_{R/I} S/IS \simeq M \otimes_{R/I} S/IS$, so the above short exact sequence shows that S/IS is flat over R/I. □

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Lemma 26. *Let R and S be as above and $x \in \mathfrak{n}$ a NZD on $S/\mathfrak{m}S$. Then*

(a) x is a NZD on S/IS for all $I \subseteq R$.

(b) The induced map $R \to S/xS$ is flat.

Proof of (a). By lemma 25, $R/I \to R/IS$ is flat if $I \subseteq R$, and

$$(S/IS)/((M/I)(S/IS)) \simeq S/\mathfrak{m}S,$$

so WLOG I = 0 in (a). We claim x is a NZD on $S/\mathfrak{m}^n S$ for all $n \geq 1$ and proceed by induction on n. The $n = 1$ case is the hypothesis of the lemma, so assume $n > 1$. Consider the s.e.s.

$$0 \to \mathfrak{m}^{n-1}S/\mathfrak{m}^n S \to S/\mathfrak{m}^n S \to S/\mathfrak{m}^{n-1}S \to 0$$

Note that $\mathfrak{m}^{n-1}/\mathfrak{m}^n \simeq (R/\mathfrak{m})^l$ where l is the minimum number of generators of $\mathfrak{m}^{n-1}$, so $\mathfrak{m}^{n-1}S/\mathfrak{m}^n S \simeq (S/\mathfrak{m}S)^l$. Since x is a NZD on the first and last modules of the s.e.s. (by assumption and induction), it is a NZD on the middle module, i.e. $S/\mathfrak{m}^n S$. Thus x is a NZD on $S/\cap_{n\geq 1} \mathfrak{m}^n S$, and since $\mathfrak{m}S \subseteq \mathfrak{n}$, Krull's intersection theorem implies that $\bigcap \mathfrak{m}^n S = 0$, so x is a NZD on S. □

Proof of (b). In general, to prove $R \to T$ is flat, it suffices to prove $\mathrm{Tor}_1^R(M,T) = 0$ for all finitely generated R-modules M. This is because if $0 \to N \to K \to M \to 0$ is a s.e.s. of R-modules and we apply $\otimes_R T$, we get $\mathrm{Tor}_1^R(M,T) \to N \otimes_R T \to K \otimes_R T \to M \otimes_R T \to 0$, and $\mathrm{Tor}_1^R(M,T) = 0$ implies this sequence is exact, i.e. that T is flat. So to prove (b), it is enough to show $\mathrm{Tor}_1^R(M, S/xS) = 0$ for all finitely generated R-modules M. Given such an M, take a prime filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

where $M_{i+1}/M_i \simeq R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. If $\mathrm{Tor}_1^R(R/\mathfrak{p}_i, S/xS) = 0$ for all i, then $\mathrm{Tor}_1^R(M, S/xS) = 0$ (simple induction on the length of the filtration proves this). Finally, let $\mathfrak{p} \in \mathrm{Spec}(R)$, consider

$$0 \to S \xrightarrow{*x} S \to S/xS \to 0$$

and apply $\otimes_R R/\mathfrak{p}$. We have

$$\cdots \to \mathrm{Tor}_1^R(S/xS, R/\mathfrak{p}) \to \mathrm{Tor}_1^R(S/xS, R/\mathfrak{p}) \to S/\mathfrak{p}S \xrightarrow{*x} S/\mathfrak{p}S \to \cdots$$

. Then we can see $\mathrm{Tor}_1^R(S/xS, R/\mathfrak{p}) = 0$ because S is flat over R, so S/xS is flat over R. □

Proof of (1). Induct on $\dim(R)$: If $\dim(R) = 0$, the nilradical of R is $\mathfrak{m}$, so $\mathfrak{m}^n = 0$ for some n. Then $(\mathfrak{m}S)^n = 0$ for some n, so $\mathfrak{m}S \subseteq \mathrm{nilrad}(S)$ which implies $\dim(S/\mathfrak{m}S) = \dim(S)$. If $\dim(R) > 0$, note that we can pass to $R/N \to S/NS$ where $N = \mathrm{nilrad}(R)$. This doesn't change the dimension of S, R, or $S/\mathfrak{m}S$ and the map is still flat by the lemma, so we can assume WLOG that R is reduced (i.e. $\mathrm{nilrad}(R) = 0$). Choose $x \in \mathfrak{m}$ $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$. Then $\dim(R/xR) = \dim(R) - 1$ and x is a NZD since the associated primes are minimal for a reduced ring. By the lemma $R/xR \to S/xS$ is still flat and by induction, $\dim(S/xS) = \dim(R/xR) + \dim(S/\mathfrak{m}S) = \dim(R) - 1 + \dim(S/\mathfrak{m}S)$. In order to show $\dim(S/xS) = \dim(S) - 1$, it's enough to show x is not in a minimal prime of S. This is true because $0 \to R \xrightarrow{*x} R$ is exact and S is flat, so $0 \to S \xrightarrow{*x} S$ is exact. Thus x is a NZD on S and therefore not in any minimal prime of S. □

Proof of (2). If $\mathrm{depth}(S/\mathfrak{m}S) > 0$, choose $x \in \mathfrak{n}$ a NZD on $S/\mathfrak{m}S$ and pass to $R \to S/xS := \overline{S}$ (by lemma 26 this is still flat). Now $\mathrm{depth}(\overline{S}/\mathfrak{m}\overline{S}) = \mathrm{depth}(S/\mathfrak{m}S) - 1$ and $\mathrm{depth}(\overline{S}) = \mathrm{depth}(S) - 1$ and by induction we have $\mathrm{depth}(S) = \mathrm{depth}(R) + \mathrm{depth}(S/\mathfrak{m}S)$. If $\mathrm{depth}(S/\mathfrak{m}S) = 0$ and $\mathrm{depth}(R) > 0$, choose $y \in \mathfrak{m}$ a NZD on R, then y is also a NZD on S (flatness). Passing to $R/yR \to S/yS$, by induction $\mathrm{depth}(R/yR) = \mathrm{depth}(S/yS)$, so $\mathrm{depth}(R) - 1 = \mathrm{depth}(S) - 1$ as we wanted. The final case is $\mathrm{depth}(R) = \mathrm{depth}(S/\mathfrak{m}S) = 0$. We want to show that $\mathrm{depth}(S) = 0$. We have $0 \to R/\mathfrak{m} \to R$ is exact because m is an associated prime of R, and tensoring with S yields $0 \to S/\mathfrak{m}S \to S$ is exact (S is flat). But $0 \to S/\mathfrak{n} \to S/\mathfrak{m}S$ is exact because $\mathrm{depth}(S/\mathfrak{m}S) = 0$, so $0 \to S/\mathfrak{n} \to S$ is exact which implies $\mathrm{depth}(S) = 0$. □

Proof of (3). Follows directly from (1) and (2). □

Corollary 27. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. R is C-M if and only if $\widehat{R}$ is C-M.*

Proof. $R \to \widehat{R}$ is flat and $\widehat{R}/\mathfrak{m}\widehat{R} = \widehat{R}/\widehat{\mathfrak{m}} = R/\mathfrak{m}$ has depth 0 (it's a field), so by theorem 24, R is C-M if and only if $\widehat{R}$ is C-M. □

Definition. A Noetherian ring R is *Cohen-Macaulay* if and only if for all $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_{\mathfrak{p}}$ is C-M (equivalently, we just need $R_{\mathfrak{m}}$ C-M for all maximal ideals $\mathfrak{m}$ of R.

Corollary 28. *If R is C-M, then the polynomial ring $R[x_1, \ldots, x_n]$ is C-M.*

Proof. Clearly it's enough to show $R[x]$ is C-M. Let $Q \in \operatorname{Spec}(R[x])$ and $Q \cap R = q$. Then $R_q \to (R[x])_Q$ is flat. By theorem 24, $R[x]_Q$ is C-M if and only if R_q and $(R[x]/qR[x])_Q$ are C-M. R_q is by assumption, and $(R[x]/qR[x])_Q$ is a localization of $k(q)[x]$ where $k(q) = R_q/qR_q$, which is C-M (it's a 1-dimensional domain). □

Definition. A *Determinental ring* is defined as follows: Let R be C-M and $n \leq m$. Adjoin $n * m$ variables x_{ij} to R to obtain $S := R[x_{ij}$. Let $I = I_t(X)$, the $t \times t$ minors of $X = [x_{ij}]$. Then S/I is a determinental ring and is C-M.

Theorem 29. *Any 2-dimensional Noetherian local integrally closed domain is C-M*

Proof. Note $\operatorname{depth}(R) \geq 1$ because any nonzero $x \in \mathfrak{m}$ is a NZD. Begin a regular sequence with some nonzero $x \in \mathfrak{m}$. If $\operatorname{depth}(R) = 1$, then this is the longest regular sequence on R and thus $\mathfrak{m} \in \operatorname{Ass}(R/xR)$. Then $R/\mathfrak{m} \hookrightarrow R/xR$. Consider the image of 1 under this injection, call it y, and consider $\alpha = y/x \in R_{(0)}$. Note that $\mathfrak{m}\alpha \subseteq R$. If $\mathfrak{m}\alpha \subseteq \mathfrak{m}$, then by the determinant trick, α is integral over R, so $\alpha \in R$ (integrally closed) and thus $\exists\ r \in R$ such that $y/x = r$, i.e. $y = xr$, which contradicts y being the image of 1 under the injection. Thus $\mathfrak{m}\alpha \nsubseteq \mathfrak{m}$, so $1 \in \mathfrak{m}\alpha$, and thus $\exists\ r \in \mathfrak{m}$ such that $1 = r(\alpha)$, so $x = ry$. Then $\mathfrak{m} = (x : y) = (ry : y) = (r)$, so $\operatorname{ht}(\mathfrak{m}) = 1$, a contradiction because $\dim(R) = 2$. Thus $\mathfrak{m} \notin \operatorname{Ass}(R/xR)$, and $\operatorname{depth}(R) = 2$. □

1 Some Characterizations of C-M Rings

Theorem 30. *Let R be a RLR and I an ideal of R. Then R/I is C-M if and only if $\operatorname{ht}(I) = \operatorname{pdim}_R(R/I)$.*

Remark 4. For a Noetherian local ring R and a finitely generated R module M, the depth of M as an R-module is the same as the depth of M as an $R/\operatorname{ann}(M)$-module.

Applying this remark to the theorem with $M = R/I$, we can consider $\operatorname{depth}(R/I)$ over R or over R/I.

Proof. Using the Auslander-Buchsbaum formula, we have

$$\mathrm{depth}(R/I) + \mathrm{pdim}(R/I) = \mathrm{depth}(R) = \dim(R)(\text{b/c } R \text{ is C-M})$$

Thus

$$\mathrm{pdim}(R/I) = \dim(R) - \mathrm{depth}(R/I) \geq \dim(R) - \dim(R/I) = \mathrm{ht}(I)(\text{ b/c } R \text{ is catenary})$$

with equality holding if and only if $\dim(R/I) = \mathrm{depth}(R/I)$, i.e. if and only if R/I is C-M. □

Theorem 31 (Unmixedness Theorem)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then R is C-M if and only if the following condition holds:*

$$\textit{if } x_1, \ldots, x_i \in \mathfrak{m} \textit{ and } \mathrm{ht}(x_1, \ldots, x_i) = i, \textit{ then } (x_1, \ldots, x_i) \textit{ is unmixed.}$$

Here unmixed means that for every $\mathfrak{p} \in \mathrm{Ass}(R/(x_1, \ldots, x_i))$, $\dim(R/\mathfrak{p}) = \dim(R/(x_1, \ldots, x_i))$.

Chapter 3

Gorenstein Rings

There are many equivalent definitions for a Gorenstein ring. We give now the following one.

Definition. A Noetherian local ring (R, m, k) is *Gorenstein* if and only if

(1) R is Cohen-Macaulay.

(2) There exists a system of parameters (shortly sop), say $x_1, \ldots, x_d$, such that the generated ideal $(x_1, \ldots, x_d)$ is irreducible, i.e.

$$I \cap J \neq (x_1, \ldots, x_d) \qquad \text{if } I \neq (x_1, \ldots, x_d) \text{ and } J \neq (x_1, \ldots, x_d).$$

1 Criteria for irreducibility

Let (R, m, k) be a Noetherian local ring as above.

Remark 1. We need to understand the 0-dimensional case, since $(x_1, \ldots, x_d)$ is irreducible in R if and only if (0) is irreducible in the 0-dimensional ring $R/(x_1, \ldots, x_d)$.

Remark 2. Any Regular Local Ring (RLR in the notation of Chapter 1) is Gorenstein.

Proof. Suppose R is a d-dimensional RLR, then $m = (x_1, \ldots, x_d)$ a regular sequence (a sop in fact). R is C-M since it is regular and

$$R/(x_1, \ldots, x_d) \simeq R/m \simeq k,$$

where k is the residue field. Finally (0) is of course irreducible in a field. □

Remark 3. Not every 0-dimensional ring is Gorenstein, e.g. $R = K[x, y]/(x, y)^2$ is such that $(0) = xR \cap yR$, hence it is not irreducible.

Remark 4. There exist 0-dimensional rings which are Gorenstein but not RLR, e.g. $R = K[x]/(x^2)$ has three ideals $(0) \subseteq (x) \subseteq R$ and clearly (0) can not be obtained as intersection of two non-zero ideals in R.

Remark 5. It is natural to ask if in (2), of the definition Gorenstein, the condition is independent of the chosen sop. The answer is yes (see corollary 42), but we need some results before proving it.

Remark 6. Remark 5 implies that any RLR modulo any sop is Gorenstein.

Example 8. Consider the ring

$$\left(\frac{k[x_1,\ldots,x_d]}{(x_1^n,\ldots,x_d^n)}\right)_{(x_1,\ldots,x_d)}$$

with $n \geq 1$. This is Gorenstein since $k[x_1,\ldots,x_d]_{(x_1,\ldots,x_d)}$ is a RLR and $(x_1^n,\ldots,x_d^n)_{(x_1,\ldots,x_d)}$ is a sop.

Definition. Given a neotherian local ring (R,m,k) we define the *Socle* of R to be

$$\mathrm{Soc}(R) = \{x \in R : xm = 0\}$$

Clearly it is a k-vector space since it is annihilated by m.

Definition. Let $N \subseteq M$ be modules over a ring R. Then M is said to be *essential* over N if every non-zero submodule $K \subseteq M$ has non-zero intersection with N, i.e.

$$K \subseteq M \Rightarrow K \cap N \neq (0).$$

Lemma 32. *Let (R,m,k) be a 0-dimensional local Noetherian ring. Then R is essential over* $\mathrm{Soc}(R)$.

Proof. Let $I \subseteq R$ be an ideal and choose $n \geq 0$ maximal such that $m^n I \neq 0$. Note that such a maximum exists since R is 0-dimensional and hence $m^N = 0$ for all $N >> 0$. Also $m^n I \subseteq (0:m) \cap I$ by choice. Therefore $\mathrm{Soc}(R) \cap I \neq (0)$ and R is essential over $\mathrm{Soc}(R)$. □

Proposition 33. *Let (R,m,k) be a 0-dimensional Noetherian local ring. Then:*

$$(0) \textit{ is irreducible (i.e. } R \textit{ is Gorenstein) if and only if } \dim_k \mathrm{Soc}(R) = 1$$

Proof. Assume $\dim_k \mathrm{Soc}(R) = 1$. Let $x \in \mathrm{Soc}(R)$ be a basis and suppose $(0) = I \cap J$. Since R is essential over $\mathrm{Soc}(R)$ by Lemma 32, $I \cap \mathrm{Soc}(R) \neq (0)$ and $J \cap \mathrm{Soc}(R) \neq (0)$. But $\mathrm{Soc}(R)$ is a 1-dimensional vector space, so $x \in I$ and $x \in J$, and this is a contradiction since $x \neq 0$.

Conversely assume (0) is irreducible and suppose $\dim_k \mathrm{Soc}(R) \geq 2$. Choose $x, y \in \mathrm{Soc}(R)$ linearly independent. The linear independence and the fact that x and y annihilate the maximal ideal imply $(x) \cap (y) = (0)$, which is a contradiction since we assumed (0) was irreducible. □

Theorem 34. *Let (R,m,k) be a d-dimensional RLR and $I \subseteq R$ an ideal such that $\sqrt{I} = m$, so that $\dim R/I = 0$. Then*

$$R/I \textit{ is Gorenstein if and only if } \dim_k \mathrm{Tor}_d^R(k, R/I) = 1.$$

Remark. By Auslander-Buchsbaum formula pdim $R/I = \dim R - \text{depth } R/I = d$ since depth $R/I \leq \dim R/I = 0$. Hence a minimal free resolution of R/I looks like

$$0 \longrightarrow F_d \xrightarrow{\varphi_d} F_{d-1} \longrightarrow \ldots\ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} R/I \longrightarrow 0.$$

Tensoring with k and taking homologies we get

$$\mathrm{Tor}_d^R(k, R/I) \simeq F_d \otimes k \simeq k^{rk(F_d)}.$$

Proof of Theorem 34. R is a d-dimensional RLR, so the maximal ideal is generated by d elements which form a regular sequence in R, write $m = (x_1, \ldots, x_d)$. The Koszul complex gives now a free minimal resolution of $k = R/m$:

$$0 \longrightarrow R \xrightarrow{[\pm x_1 \cdots \pm x_d]} R^d \longrightarrow \ldots\ldots \longrightarrow R/m \longrightarrow 0.$$

Tensoring with R/I and taking the d-th homology:

$$\mathrm{Tor}_d^R(k, R/I) = \ker(R/I \xrightarrow{[\pm x_1 \cdots \pm x_d]} (R/I)^d) = \mathrm{Soc}(R),$$

so R/I is Gorenstein iff $\dim_k \mathrm{Soc}(R) = \dim_k \mathrm{Tor}(k, R/I) = 1$. □

Corollary 35. *If (R, m, k) is a RLR and $x_1, \ldots, x_d$ is a sop, then $R/(x_1, \ldots, x_d)$ is Gorenstein.*

Proof. R is a RLR, so $x_1, \ldots, x_d$ form a regular sequence. Hence a minimal free resolution of $R/(x_1, \ldots, x_d)$ is given by the Koszul complex

$$0 \longrightarrow R \longrightarrow R^d \longrightarrow \ldots\ldots \longrightarrow R/m \longrightarrow 0.$$

which has last Betti number equal to one. □

Definition. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, S be a RLR, and $x_1, \ldots, x_t$ be a regular sequence in S. The ring R is said to be a *complete intersection* if the completion with respect the the maximal ideal, $\widehat{R}$, is isomorphic to $S/(x_1, \ldots, x_t)$.

Corollary 36. *Let (R, m, k) be a RLR and $x_1, \ldots, x_i$ be a regular sequence. Then $R/(x_1, \ldots, x_i)$ is Gorenstein, i.e. complete intersections are Gorenstein.*

Proof. Extend $x_1, \ldots, x_i$ to a complete sop and use Corollary 35. □

Remark. There is the following hierarchy:

RLR $\Rightarrow$ complete intersections $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay

and in general the arrows are not reversible.

Example 9. Consider $R = k[x, y, z]/I$ where

$$I = (x^2 - y^2, x^2 - z^2, xy, xz, yz).$$

$\mathrm{Soc}(R) = x^2 R$ and so R is Gorenstein. However it is not a complete intersection since the minimal number of generators of I is five.

Lemma 37. *Let (R, m, k) be a RLR and let M be a finitely generated torsion R-module. Consider a free resolution of M (not necessarily minimal):*

$$0 \longrightarrow R^{b_n} \xrightarrow{\varphi_n} R^{b_{n-1}} \longrightarrow \dots\dots \longrightarrow R^{b_1} \xrightarrow{\varphi_1} R^{b_0} \xrightarrow{\varphi_0} M \longrightarrow 0.$$

Then

$$\sum_{i=0}^{n} (-1)^i b_i = 0.$$

Proof. R is a RLR and hence a domain. Let $Q = R_{(0)}$ be its fraction field, then $M \otimes_R Q = 0$ by the assumption M torsion module. Also Q is flat over R, so:

$$0 \longrightarrow Q^{b_n} \xrightarrow{\varphi_n} Q^{b_{n-1}} \longrightarrow \dots \longrightarrow Q^{b_1} \xrightarrow{\varphi_1} Q^{b_0} \xrightarrow{\varphi_0} 0.$$

is exact. But these are Q-vector spaces, and for vector spaces the result is well known. $\square$

Theorem 38. *Let (R, m, k) be a 2-dimensional RLR and let $I \subseteq R$ be an ideal such that $\sqrt{I} = m$. Then*

$$R/I \text{ is Gorenstein} \iff R/I \text{ is a complete intersection} \iff I = (f, g).$$

Proof. Corollary 36 shows that every complete intersection R/I is Gorenstein, no matter what is the dimension of R. Conversely note that by Auslander-Buchsbaum formula $\mathrm{pdim} R/I = \dim R - \mathrm{depth} R/I = 2$ since $\mathrm{depth} R/I \leq \dim R/I = 0$. Take a free minimal resolution:

$$0 \longrightarrow R^{b_2} \longrightarrow R^{b_1} \longrightarrow R \longrightarrow R/I \longrightarrow 0 \qquad \text{with } b_1 = \mu(I).$$

Now R/I is Gorenstein $\iff$ $b_2 = 1$ $\iff$ $\mu(I) = b_1 = 2$ $\iff$ $I = (f, g)$ a complete intersection. $\square$

Recall that, given two modules M, N over any ring R and given $F.$ a free resolution of M and $G.$ a free resolution of N, then

$$\mathrm{Tor}_i^R(M, N) = H_i(F. \otimes G.) \quad \text{for all } i \geq 0.$$

In particular if $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, then $F. \otimes G.$ is a free resolution of $H_0(F. \otimes G.) \equiv M \otimes_R N$ by right exactness. Finally, if (R, m, k) is local and $F.$, $G.$ are minimal, then so is $F. \otimes G.$ (because of the way the maps are defined).

Lemma 39. *Let (R, m, k) be a d-dimensional local Cohen-Macaulay ring and $I \subseteq R$ be an ideal of height $\mathrm{ht} I = h$. Then there exists a regular sequence $x_1, \ldots, x_{d-h}$ such that the images $\overline{x_1}, \ldots, \overline{x_{d-h}}$ form a sop in R/I. Conversely, given a sop $\overline{y_1}, \ldots, \overline{y_{d-h}}$ in R/I, then there exist $y_1, \ldots, y_{d-h} \in R$ such that they form a regular sequence in R.*

Remark. Note that $\dim R/I = \dim R - \mathrm{ht} I = d - h$ since R is Cohen-Macaulay.

Remark. Since R is Cohen-Macaulay, to say that a sequence $x_i, \ldots, x_i$ is regular it is enough to show that $\mathrm{ht}(x_1, \ldots, x_i) = i$.

Proof of Lemma 39. It is enough to prove the second statement, since the first one follows by the second one.

Let $\overline{y_1}, \ldots, \overline{y_{d-h}}$ be a sop and lift it to any $z_1, \ldots, z_{d-h}$ (i.e. $\overline{z_i} = \overline{y_i}$ for all $1 \leq i \leq d-h$). By induction we claim that we can choose $y_1, \ldots, y_{d-h} \in R$ such that $\mathrm{ht}(y_1, \ldots, y_i) = i$ for all $1 \leq i \leq d-h$.

- $i = 1$: We need to find $y_1 \in R$ of height one, i.e. $y_1 \notin P$, for all $P \in \mathrm{Min}(R)$, and we can choose between any $y_1 \in (z_1) + I$. But

$$(z_1) + I \not\subseteq \bigcup_{P \in \mathrm{Min}(R)} P$$

so by prime avoidance there exists $t \in I$ such that $z_1 + t \notin \bigcup_{P \in \mathrm{Min}(R)} P$. Set $y_1 = z_1 + t$.

- $1 < i < d-h$: Suppose we have chosen $y_1, \ldots, y_i \in R$ such that $\mathrm{ht}(y_1, \ldots, y_i) = i$ and $\overline{y_j} = \overline{z_j}$ for all $1 \leq j \leq i$. We need to choose $y_{i+1} \in z_{i+1} + I$ such that $y_{i+1} \notin \bigcup_{P \in \mathrm{Min}(y_1, \ldots, y_i)} P$. To use prime avoidance we need

$$(z_{i+1}) + I \not\subseteq \bigcup_{P \in \mathrm{Min}(y_1, \ldots, y_i)} P.$$

This is true unless there exists $Q \in \mathrm{Min}(y_1, \ldots, y_i)$ such that $(z_{i+1}) + I \subseteq Q$.

Since $\mathrm{ht}(y_1, \ldots, y_i) = i$ and R is Cohen-Macaulay, $y_1, \ldots, y_i$ are a regular sequence, therefore all $Q \in \mathrm{Min}(y_1, \ldots, y_i)$ has height i. Also $\dim R/Q = d - i$ again because R is Cohen-Macaulay. Since

$$J := I + (y_1, \ldots, y_i, z_{i+1}) \subseteq Q$$

we have that R/Q is a homomorphic image of R/J. But

$$d - i = \dim R/Q \leq \dim R/J = \dim \frac{R}{I + (\overline{y_1}, \ldots, \overline{y_{i+1}})} = d - h - (i+1) < d - i$$

which is of course a contradiction. Therefore we can apply prime avoidance and conclude as for the case $i = 0$.

□

Theorem 40. *Let (R, m, k) be a d-dimensional RLR and let $I \subseteq R$ be an ideal of height $\mathrm{ht} I = h$. Then R/I is Gorenstein if and only if*

(1) R/I is Cohen-Macaulay.

(2) $\dim_k \mathrm{Tor}_h^R(R/I, k) = 1$.

Proof. First of all notice that since a Gorenstein ring is Cohen-Macaulay we have R/I is Cohen-Macaulay in both the assumptions. We proved that R/I is Cohen-Macaulay if and only if $\mathrm{pdim} R/I = \mathrm{ht} I = h$. Let $F_.$ be a free minimal resolution of R/I and let $\overline{x_1}, \ldots, \overline{x_{d-h}}$ be any sop of R/I. By Lemma 39 we can lift them to a regular sequence $x_1, \ldots, x_{d-h} \in R$. Let $K_. := K_.(x_1, \ldots, x_{d-h}; R)$ be the Koszul complex, which is a minimal free resolution of $R/(x_1, \ldots, x_{d-h}$ since they form a regular sequence. Tensoring $K_.$ with R/I we can calculate now

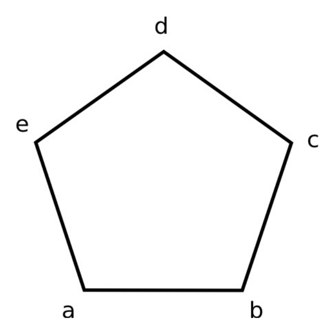
$$\mathrm{Tor}_i^R(R/I, R/(x_1, \ldots, x_{d-h})) \simeq H_i(x_1, \ldots, x_{d-h}; R/I) \simeq H_i(\overline{x_1}, \ldots, \overline{x_{d-h}}; R/I).$$

Since R/I is Cohen-Macaulay and $\overline{x_1}, \ldots, \overline{x_{d-h}}$ form a sop, they are a regular sequence in R/I and so the Koszul complex is exact, i.e. $\mathrm{Tor}_i^R(R/I, R/(x_1, \ldots, x_{d-h})) = 0$ for all $i \geq 1$. Then $F_. \otimes K_.$ is a free minimal resolution of the 0-dimensional ring $R/I \otimes R/(x_1, \ldots, x_{d-h}) = R/I + (x_1, \ldots, x_{d-h}) := S$ and by Theorem 34 S is Gorenstein if and only if the last Betti number of $F_. \otimes K_.$ is one. But the last element of the tensor product is just

$$F_h \otimes K_{d-h} = F_h \otimes R \simeq F_h$$

since $h = \mathrm{ht} I = \mathrm{pdim} R/I$ and the Koszul complex has always R in the last position. So R/I is Gorenstein if and only if $\mathrm{rank}(F_h) = 1$, that is $\dim_k \mathrm{Tor}_h^R(R/I, k) = 1$. □

Example 10. Consider a 5-cycle:



Let I be the graph ideal $I = (ab, bc, cd, de, ae) \subseteq S = k[a, b, c, d, e]$. This ideal has height $\mathrm{ht} I = 3$ and a free minimal resolution is given by

$$0 \longrightarrow S \longrightarrow S^5 \longrightarrow S^5 \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Then S/I is Gorenstein by Theorem 40, since the last shift (corresponding to $\mathrm{ht}I = 3$) is one. Notice that S/I is another example of Gorenstein ring which is not a complete intersection, since $\mu(I) = 5$.

Fact. Any complete local ring is the homomorphic image of a RLR. We have already prove this result only in the case in which the ring contains a field.

Corollary 41. *Let (R, m, k) be a local ring. Then*

$$R \text{ is Gorenstein} \iff \widehat{R} \text{ is Gorenstein}$$

Proof. Assume R is Gorenstein. By definition R is Cohen-Macaulay and there exists a sop $x_1, \ldots, x_d$ such that $(x_1, \ldots, x_d)$ is irreducible, if and only if (0) is irreducible in $R/(x_1, \ldots, x_d)$. We already know that $\widehat{R}$ is Cohen-Macaulay if R is Cohen-Macaulay and

$$\frac{\widehat{R}}{(x_1, \ldots, x_d)\widehat{R}} \simeq \frac{R}{(x_1, \ldots, x_d)}$$

since $(x_1, \ldots, x_d)$ is m-primary (every Cauchy sequence converges in a 0-dimensional ring). Hence (0) is irreducible in $\widehat{R}/(x_1, \ldots, x_d)\widehat{R}$ and $\widehat{R}$ is Gorenstein.

Conversely assume $\widehat{R}$ is Gorenstein, then $\widehat{R}$ is Cohen-Macaulay and this implies that R is Cohen-Macaulay. Since $\widehat{R}$ is the homomorphic image of a RLR, we have proved that for all sop $y_1, \ldots, y_d$ we have that (0) is irreducible in $\widehat{R}/(y_1, \ldots, y_d)\widehat{R}$. Since it is possible to choose any sop, just choose a sop $y_1, \ldots, y_d$ in R and then use again the isomorphism

$$\frac{\widehat{R}}{(x_1, \ldots, x_d)\widehat{R}} \simeq \frac{R}{(x_1, \ldots, x_d)}.$$

□

Corollary 42. *Let (R, m, k) be a Cohen-Macaulay local ring. The following facts are equivalent:*

(1) There exists a sop generating an irreducible ideal.

(2) All sop generate irreducible ideals.

(3) R is Gorenstein.

Remark. A theorem by Rees states that if all sop generate irreducible ideals, then R is automatically Cohen-Macaulay, and hence Gorenstein. So (2) $\iff$ (3) without the assumption R Cohen-Macaulay.

Remark. (1) is not equivalent to (2) and (3) if R is not Cohen-Macaulay.

2 Injective modules over Noetherian rings

Definition. Le t R be a Noetherian ring. A R-module E is *injective* if whenever there exists a diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "i"'] & N \end{tikzcd}$$

there exists a map $g : N \to E$ which makes the diagram commute:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "i"'] & N \arrow[ul, dotted, "g"'] \end{tikzcd}$$

Remark. A R-module E is injective if and only if for all N, M R-modules the map

$$\operatorname{Hom}_R(N, E) \to \operatorname{Hom}_R(M, E)$$

is surjective, if and only if the functor $\operatorname{Hom}_R(\cdot, E)$ is right exact. Since $\operatorname{Hom}_R(\cdot, E)$ is left exact this is equivalent to say that this functor is exact.

Proposition 43. *Let $R \to S$ be an algebra homomorphism and let E be an injective R-module. Notice that $\operatorname{Hom}_R(S, E)$ is a S-module with the following multiplication:*

$$S \times \operatorname{Hom}_R(S, E) \to \operatorname{Hom}_R(S, E)$$
$$(s, f(s')) \mapsto f(ss') \quad \textit{for all } s' \in S$$

Then $\operatorname{Hom}_R(S, E)$ is an injective S-module.

Proof. We have the following isomorphisms:

$$\operatorname{Hom}_S(\cdot, \operatorname{Hom}_R(S, E)) \simeq \operatorname{Hom}_R(\cdot \otimes_S S, E) \simeq \operatorname{Hom}_R(\cdot, E).$$

By assumption E is injective, so $\operatorname{Hom}_R(\cdot, E)$ is exact. Therefore $\operatorname{Hom}_S(\cdot, \operatorname{Hom}_R(S, E))$ is exact and this is equivalent to say that $\operatorname{Hom}_R(S, E)$ is in injective. □

Remark. A particular case of Proposition 43 is $S = R/I$. If E is an injective R-module, then $\operatorname{Hom}_R(R/I, E) \simeq ann_E I \subseteq E$ is an injective S-module.

Theorem 44 (Baer's Criterion)**.** *Let R be a ring and let E be an R-module. Then E is injective if and only if for all $I \subseteq R$ ideal and for all diagrams*

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r] & R \end{tikzcd} \tag{3.1}$$

there exists a map $g : R \to E$ which makes it commute:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r] & R \arrow[ul, dotted, "g"'] \end{tikzcd}$$

Proof. If E is injective than clearly there exists a map which makes (3.1) commute. Conversely assume that for all ideals $I \subseteq R$ and diagrams (3.1) there exists $g : R \to E$ which makes it commute. Suppose we have

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "i"'] & N \end{tikzcd}$$

and consider

$$\Lambda := \{(K, f_K) : M \subseteq K \subseteq N, f_K : K \to E, f_K|_M = f\}.$$

Partially order this set by

$$(K, f_K) \leq (L, f_L) \iff K \subseteq L \;\text{ and }\; f_L|_K = f_K.$$

Use Zorn's Lemma to get $(K, f_K) \in \Lambda$ maximal. Suppose $K \subsetneq N$ and choose $x \in N \smallsetminus K$, $x \neq 0$, and let $I = K :_R x \subseteq R$ be an ideal in R. Consider:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & K \arrow[u, "f_K"] \arrow[r] & K + Rx \end{tikzcd}$$

and we want to define $h : K + Rx \to E$ which extends f_K, getting a contradiction since (K, f_K) is maximal. Note that for all $i \in I$ we have $ix \in K$ (by definition of I), hence if such an h exists it has necessarily to be

$$ih(x) = h(ix) = f_K(ix).$$

Consider

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "f_K(\cdot x)"] \arrow[r] & R \arrow[ul, dotted, "\varphi"'] \end{tikzcd}$$

A map $\varphi : R \to E$ as above exists by assumption. Therefore, for all $i \in I$:

$$f_K(ix) = \varphi(i \cdot 1) = i\varphi(1).$$

Define $h(x) := \varphi(1)$, so that

$$\begin{aligned} h : K + Rx &\to E \\ k &\mapsto f_K(k) \;\text{ for all } k \in K \\ x &\mapsto \varphi(1) \end{aligned}$$

We have to show that h is well defined. Suppose $k + rx = k' + r'x$, then $k - k' = (r' - r)x$, then $r' - r \in I$ and necessarily

$$f_K(k - k') = f_K((r' - r)x) = (r' - r)\varphi(1)$$

that is $h(k+rx) = f_K(k)+r\varphi(1) = f_K(k')+r'\varphi(1) = h(k'+r'x)$. So $(K+Rx, h)$ properly extends (K, f_K), which is maximal, and this is a contradiction. Hence $K = N$ and so E is injective. $\square$

2.1 Divisible modules

Definition. A module M over a ring R is said to be *divisible* if for all $x \in R$, x NZD in R, and for all $u \in M$, there exists $v \in M$ (not necessarily unique) such that $u = xv$.

Examples. (1) All modules over a field k are divisible, since all x NDZ in k, i.e. all $x \neq 0$, are units.

(2) $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module. More generally if R is a domain its quotient field $Q(R)$ is divisible.

(3) If M is divisible and $N \subseteq M$, then M/N is divisible.

(4) Direct sums and direct products of divisible modules are divisible.

(5) Any injective module E is divisible.

Proof. Let $x \in R$ be a NZD and $u \in E$. Consider

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & R \arrow[u, "f"] \arrow[r, "\cdot x"] & R \end{tikzcd}$$

where $f(1) = u$. Then there exists $g : R \to E$ such that

$$u = f(1) = g(1 \cdot x) = xg(1).$$

Just set $v := g(1)$. □

(6) If R is a PID, then an R-module E is injective if and only if it is divisible.

Proof. By (5) if E is injective, then it is divisible. Assume now E is divisible. Notice that for proving (5) it was enough to consider the following diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & (x) \arrow[u, "f"] \arrow[r, "i"] & R \end{tikzcd}$$

with $f(x) = u$ and i the inclusion. So E divisible means that for all diagrams with principal ideals, there exists a map $g : R \to E$ that makes it commute. But R is a PID, so all its ideals are principal, therefore by Baer's Criterion E is injective. □

(7) Let $N \subseteq M$ be R-modules and assume N and M/N are divisible. Then M is divisible.

Proof. Let $x \in R$ be a NZD and let $u \in M$. Since M/N is divisible $\overline{u} = x\overline{v}$ for some $\overline{v} \in M/N$, that is $u - xv \in N$. But also N is divisible, so $u - xv = xw$ for some $w \in N$. Hence $u = x(v + w)$. □

Proposition 45. *Let R be a ring and let M be an R-module. Then there exists an injective module $I \supseteq M$.*

Proof. First assume $R = \mathbb{Z}$. Every $\mathbb{Z}$-module M is such that:

$$M \simeq \frac{\bigoplus \mathbb{Z}}{H} \hookrightarrow \frac{\bigoplus \mathbb{Q}}{H} := I_{\mathbb{Z}}.$$

By Example (2) $\mathbb{Q}$ is divisible, so $I_{\mathbb{Z}}$ is a divisible $\mathbb{Z}$-module by (4) and (3). Finally $I_{\mathbb{Z}}$ is injective by (6) since $\mathbb{Z}$ is a PID. Coming back to the general case there is a canonical ring map $\mathbb{Z} \to R$ which sends 1 to 1. So by Proposition 43 $I := \mathrm{Hom}_{\mathbb{Z}}(R, I_{\mathbb{Z}})$ is an injective R-module. M is an R-module, and so a $\mathbb{Z}$-module, so we have an injective map as above

$$0 \longrightarrow M \longrightarrow I_{\mathbb{Z}}$$

which, applying $\mathrm{Hom}_{\mathbb{Z}}(R, \cdot)$ (left-exact) becomes

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I_{\mathbb{Z}}) = I.$$

Then $M \simeq \mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M)$ and hence

$$0 \longrightarrow M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I_{\mathbb{Z}}) = I.$$

□

Remark. If M is divisible over R, then M is not necessarily divisible over $\mathbb{Z}$. For example $\mathbb{Z}_p$ is a field, so it is divisible and therefore injective as a $\mathbb{Z}_p$-module ($\mathbb{Z}_p$ is a PID). However $\mathbb{Z}_p$ is not divisible as a $\mathbb{Z}$-module since it is not divisible by $p \in \mathbb{Z}$.

Corollary 46. *Every R-module M has an injective resolution*

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{\psi_1} I^1 \xrightarrow{\psi^2} I^2 \longrightarrow \dots$$

Proof. Assume

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{\psi_1} I^1 \xrightarrow{\psi^2} I^2 \longrightarrow \dots \xrightarrow{\psi_{i-1}} I^i$$

is contructed. Choose I^{i+1} an injective R-module containing $I^i/\psi_{i-1}(I^{i-1})$. In this way

$$\begin{tikzcd} 0 \arrow[r] & I^i/\psi_{i-1}(I^{i-1}) \arrow[r] & I^{i+1} \\ & I^i \arrow[u] \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

so that $\ker \psi_i = \mathrm{im}\psi_{i-1}$. □

2.2 Essential extensions

Recall that, for M, N R-modules, $M \subseteq N$ is said to be essential if for all $K \subseteq N$, $K \neq 0$, then $K \cap M \neq 0$. Notice that to prove that $M \subseteq N$ is essential is enough to consider $K = kR$ cyclic submodules of N.

Examples. (1) If (R, m) is an Artinian local ring, then we have already proved that $\mathrm{Soc}(R) \subseteq R$ is essential.

(2) Let R be a domain. Then $R \subseteq Q(R)$ is essential.

Proof. Let $K = \left(\frac{a}{b}\right) R \subseteq Q(R)$, be a non-zero submodule.Then:

$$a \in \left(\frac{a}{b}\right) R \cap R \neq 0.$$

□

(3) $M \subseteq L \subseteq N$ and $M \subseteq N$ is essential if and only if $M \subseteq L$ and $L \subseteq N$ are essential.

(4) $M \subseteq N$ is essential if and only if for all $x \in N$, $x \neq 0$ there exists $r \in R$ such that $rx \in M$, $rx \neq 0$.

(5) $M \subseteq L_\alpha \subseteq N$ and $M \subseteq L_\alpha$ essential for all α, then $M \subseteq \bigcup_\alpha L_\alpha$ is essential.

Proof. Use (4). Notice that the inclusion of L_α in N is just to give sense to the union of the modules L_α. □

Proposition 47. *Let R be a ring and let E be a R-module. Then E is injective if and only if whenever $E \subseteq M$, then E splits out of M, i.e. there exists $N \subseteq M$ such that $M = E \oplus N$. We will denote $E|M$*

Proof. Assume E is injective and consider

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & E \arrow[u, "id"] \arrow[r, "i"'] & M \arrow[ul, dotted, "g"'] \end{tikzcd}$$

Then again $\ker(g) \cap E = 0$ because the inclusion ad the identity are injective maps, therefore $M = E \oplus \ker(g)$.

Conversely assume that $E \subseteq M$ implies E splits out of M. We know that there exists I injective, $E \subseteq I$. Then $I = E \oplus N$. Also:

$$\begin{tikzcd} & I \arrow[r, two heads] & E \\ & E \arrow[u] & \\ 0 \arrow[r] & K \arrow[u] \arrow[r] & L \arrow[ul, dotted, to path={}] \arrow[uul, dotted] \arrow[uu, dotted] \end{tikzcd}$$

i.e. E is injective. □

Proposition 48. *Let E be an R-module. Then E is injective if and only if E has not a proper essential extension, i.e. $E \subseteq M$ essential implies $E = M$.*

Proof. Assume E is injective and $E \subsetneq M$ a proper essential extension. Consider:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & E \arrow[u, "id"] \arrow[r, "i"'] & M \arrow[ul, dotted, "g"'] \end{tikzcd}$$

Then $\ker(g) \cap E = 0$ as in the proof of Proposition 47. This implies $\ker(g) = 0$ since $E \subseteq \ker(g) \subseteq M$ is an essential extension. But then M embeds into E and so $E \subseteq M \subseteq E$, which is $M = E$.

Conversely assume E has no proper essential extensions. Consider $E \subseteq I$ an injective R-module. If the extension is essential, then $E = I$ by assumption and E is injective. If the extension is not essential, then there exists $N \subseteq I$ such that $N \neq 0$ but $N \cap E = 0$. Using Zorn's Lemma define a maximal $M \subseteq I$, $M \neq 0$ and $M \cap E = 0$. Then

$$E \hookrightarrow I \twoheadrightarrow I/M$$

is an injection $E \hookrightarrow I/M$. By maximality of M, if $N/M \subseteq I/M$ then $N \cap E \neq 0$, i.e. I/M is essential over E. But E has no proper essential extensions, hence $E \simeq I/M$, i.e. $E + M = I$ and $M \cap E = 0$ by assumption. Therefore $I = E \oplus M$ and E is injective, as seen inside the proof of Proposition 47. $\square$

Proposition 49. *Let R be a ring and let $\{E_i\}$ be injective R-modules. Then*

(1) $\prod_i E_i$ is injective.

(2) If R is Noetherian, then $\bigoplus_i E_i$ is injective.

(3) If direct sums of injective modules are injective, then R is Noetherian.

Proof. (1) Consider

$$\begin{tikzcd} & E_i & \\ & \prod_i E_i \arrow[u, two heads] & \\ 0 \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[uul, dotted, "g_i"'] \end{tikzcd}$$

and take $\prod_i g_i : N \to \prod_i E_i$. It works:

$$\begin{tikzcd} & \prod_i E_i & \\ 0 \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[ul, "\prod_i g_i"'] \end{tikzcd}$$

(2) Use Baer's Criterian. It is enough to show

$$\begin{tikzcd} & \bigoplus_i E_i & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r] & R \arrow[ul, dotted, "g"'] \end{tikzcd}$$

R is Noetherian, then I is finitely generated and $f(I)$ has entries only in finitely many E_i, say $1 \le i \le n$. Consider $\prod_{i=1}^{n} g_i : R \to \oplus_i E_i$, with the g_i's as in (1). It works:

$$\begin{tikzcd} & \bigoplus_i E_i & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r] & R \arrow[ul, dotted, "\prod_{i=1}^{n} g_i"'] \end{tikzcd}$$

(3) Suppose we have $I_1 \subseteq I_2 \subseteq \dots$ an ascending chain of ideals in R and let $I := \bigcup_i I_i$. We need to show that there exists $j \in \mathbb{N}$ such that $I = I_j$. For all i there exists an injective module E_i such that

$$0 \longrightarrow I/I_i \xrightarrow{\pi_i} E_i$$

and by assumption $\bigoplus_i E_i$ is injective. Consider:

$$\begin{tikzcd} & \bigoplus_i E_i & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r] & R \arrow[ul, dotted, "g"'] \end{tikzcd}$$

where $f = \prod_i \pi_i$. The map f is well defined because for all $x \in I = \bigcup_i I_i$ there exists $j \in \mathbb{N}$ such that $x \in I_j$. This implies $x \in I_k$ for all $k \ge j$ and so $\pi_k(x) = 0$ for all $k \ge j$. Therefore $f(x) \in \bigoplus_{i=1}^{j-1} E_i \subseteq \bigoplus_i E_i$. The commutativity of the diagrams implies that for all $x \in I$ $f(x) = xg(1)$. Say $g(1) = (g_1, \dots, g_n, 0, \dots) \in \bigoplus_i E_i$, then $\pi_{n+1}(x) = 0$ for all $x \in I$ and hence $I = I_{n+1}$. $\square$

Theorem 50. *Let R be a ring, let $M \subseteq E$ be R-modules. The following statements are equivalent:*

(1) E is the maximal essential extension of M.

(2) E is injective and $M \subseteq E$ is essential.

(3) E is injective and if $M \subseteq I \subseteq E$, with I injective, then $I = E$.

Furthermore, given M, such a module E exists and it is unique up to isomorphism. E is called the injective hull (or injective envelope) of M and it is denoted by $E_R(M)$.

Proof. (1) $\Rightarrow$ (2) $M \subseteq E$ is of course essential. Since E has no proper essential extensions, then E is injective by Proposition 48.
(2) $\Rightarrow$ (3) Suppose $M \subseteq I \subseteq E$ with I injective. By Proposition 48 there are no proper essential extensions of I and by assumption $M \subseteq E$ is essential, so $I \subseteq E$ is essential. So $I = E$.
(3) $\Rightarrow$ (1) Let $M \subseteq E$, with E injective. Choose I to be the maximal essential

extension of M inside E, which exists by Zorn's Lemma, since union of essential extensions is essential. Assume $I \subseteq N$ is an essential extension, then:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "i"] \arrow[r, "f"'] & N \arrow[ul, dotted, "g"'] \end{tikzcd}$$

with $i : I \to E$ the inclusion. $\ker(g) \cap I = 0$ because f and i are injective, then $\ker(g) = 0$ since $I \subseteq N$ is essential. But this implies $I \subseteq g(N) \subseteq E$ and so $I = N$ by maximality of I. So I has no proper essential extensions, therefore it is injective. By assumption, since $M \subseteq I \subseteq E$ and I is injective, we have $I = E$, i.e. E is the maximal essential extension of M.

This shows also the existence of such a module, since it is enough to take any injective module $E \supseteq M$ and find the maximal essential extension of M inside it. For uniqueness suppose $M \subseteq E$ and $M \subseteq E'$ satisfying the three equivalent conditions (1),(2),(3). Consider:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "i"] \arrow[r, "f"'] & E' \arrow[ul, dotted, "g"'] \end{tikzcd}$$

g is injective as above, so $M \subseteq g(E') \subseteq E$ and by (3) $E' \simeq g(E') = E$. □

Theorem 51 (Structure of injectives over Noetherian rings, part 1)**.** *Let R be a Noetherian ring.*

(1) An injective R-module E is indecomposable $\iff$ $E \simeq E_R(R/P)$ for some $P \in \operatorname{Spec}(R)$.

(2) Every injective R-module is isomorphic to a direct sum of indecomposable injective R-modules.

Proof. (1) Let $P \in \operatorname{Spec}(R)$, then $E_R(R/P)$ is indecomposable. If not $E_R(R/P) = M_1 \oplus M_2$. Let $I_1 = M_1 \cap R/P$ and $I_2 = M_2 \cap R/P$. $M_1 \neq 0$ and $M_2 \neq 0$, then $I_1 \neq 0$ and $I_2 \neq 0$ since $R/P \subseteq E_R(R/P)$ is essential by Theorem 50. Also $I_1 \cap I_2 = 0$ since $M_1 \cap M_2 = 0$. But R/P is a domain and two non-zero ideals must intersect since:

$$0 \neq I_1 I_2 \subseteq I_1 \cap I_2.$$

More generally every extension of a domain is essential.

Conversely let E be an injective indecomposable R-module. There exists $P \in \operatorname{Spec}(R)$ such that $P \in \operatorname{Ass}(E)$, i.e. $R/P \hookrightarrow E$. Then $R/P \subseteq E_R(R/P) \hookrightarrow E$ since by Theorem 50 the injective hull is the maximal essential extension of R/P inside any injective $E \supseteq R/P$. But $E_R(R/P)$ is injective, so $E_R(R/P)|E$, which implies $E \simeq E_R(R/P)$ because E is indecomposable.

(2) Let E be an injective R-module. Take $P \in \operatorname{Ass}(E)$, then $E_R(R/P)|E$ as above. Consider

$$\Lambda := \left\{ E_i : E_i \subseteq E, E_i \text{ indecomposable injective}, \sum E_i \simeq \bigoplus E_i \right\}.$$

Set

$$\mathcal{S} := \{\Lambda : \Lambda \text{ as above }\}.$$

Note that $\{E_R(R/P)\} \in \mathcal{S} \neq \varnothing$. Write $\Lambda \leq \Lambda'$ if for all $E_i \in \Lambda$, $E_i \in \Lambda'$. Use Zorn's Lemma to find Λ a maximal element in $\mathcal{S}$. If $\sum_{E_i \in \Lambda} E_i = E$ then $E \simeq \bigoplus E_i$ and the theorem is proved. If not $\sum E_i \simeq \oplus E_i$ is injective since R is Noetherian. Then $E \simeq \sum E_i \oplus N$ with $N \neq 0$ and N is injective since E is injective. Choose $Q \in \operatorname{Ass}(N)$, then $R/Q \hookrightarrow N$ and so $E_R(R/Q)|N$. Now $\Lambda < \Lambda \cup \{E_R(R/Q)\} \in \mathcal{S}$, which contradicts the maximality of Λ. ☐

Remark. Inside the proof of Theorem 51 we have actually proved that $\operatorname{Ass}(E_R(R/P)) = \{P\}$.

2.3 Structure of $E_R(k)$

Let us denote by (R, m, k, E) a local Noetherian ring with $E := E_R(k)$.

Proposition 52. *Let (R, m, k, E) be local Noetherian. Then*

(1) $\operatorname{Supp}(E) = \{m\}$.

(2) $\operatorname{Soc}(E) \simeq k$.

Proof. (1) We have $\operatorname{Ass}(E) = \{m\} \subseteq \operatorname{Supp}(E)$. Also, if there exists $P \in \operatorname{Supp}(E)$, $P \neq m$, then we would have a smaller associated prime of E, which cannot be.
(2) We know that $k \subseteq E$ is an essential extension. Let $x \in E$, then $mx = 0$, i.e. $k = Rx$.

Claim. $\operatorname{Soc}(E) = Rx$.

Proof of the Claim. If not choose $y \in \operatorname{Soc}(E)$, $y \notin Rx$. Since $Rx = k \subseteq E$ is essential $Ry \cap Rx \neq (0)$. But $mx = my = 0$, so there exist $a, b \notin m$ such that $ay = bx$. a and b are units, so $y = a^{-1}bx \in Rx$, which is a contradiction. ☐

Therefore $\operatorname{Soc}(E) = Rx \simeq k$. ☐

Notation. Denote $M^\vee := \operatorname{Hom}_R(M, E)$.

Theorem 53. *Let (R, m, k, E) be a 0-dimensional Noetherian local ring. Then*

(1) $\lambda(M) = \lambda(M^\vee)$ *for all* $M \in \operatorname{Mod}^{\text{fg}}(R)$.

(2) $M = 0$ *if and only if* $M^\vee = 0$.

(3) $M \simeq M^{\vee\vee}$ *for all* $M \in \operatorname{Mod}^{\text{fg}}(R)$.

(4) $\lambda(E) = \lambda(R)$.

Proof. (1) By induction on $\lambda(M)$:
If $\lambda(M) = 1$, then $M \simeq k$. In this case $M^\vee = k^\vee = \mathrm{Hom}_R(k, E) = \mathrm{Hom}_R(R/m, E) = 0 :_E m = \mathrm{Soc}(E) \simeq k$ by (2) of Proposition 52.
Assume now $\lambda(M) > 1$ and start a composition series:

$$0 \longrightarrow k \longrightarrow M \longrightarrow M' \longrightarrow 0.$$

Length is additive, so $\lambda(M) = \lambda(M') + 1$. Also E is injective, so $Hom_R(\cdot, E) = (\ \cdot\ )^\vee$ preserves short exact sequences:

$$0 \longrightarrow (M')^\vee \longrightarrow M^\vee \longrightarrow k^\vee \longrightarrow 0 \quad \text{is exact.}$$

By induction $\lambda(M^\vee) = \lambda((M')^\vee) + 1 = \lambda(M') + 1 = \lambda(M)$.
(2) Follows immediately from (1).
(3) For all M, N R-modules there exists a natural map:

$$\begin{aligned} &M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), N) \\ &m \mapsto (f \mapsto f(m)) \end{aligned}$$

Therefore there exists a natural map $\theta : M \to M^{\vee\vee}$. By (1) used twice $\lambda(M) = \lambda(M^{\vee\vee})$, so to show that θ is an isomorphism is enough to show that it is one-to-one. If not there exists $u \in M$ such that for all $f \in M^\vee$ $f(u) = 0$. But consider the short exact sequence:

$$0 \longrightarrow Ru \longrightarrow M \longrightarrow M/Ru \longrightarrow 0$$

and apply $(\ \cdot\ )^\vee$:

$$0 \longrightarrow (M/Ru)^\vee \longrightarrow M^\vee \xrightarrow{\varphi} (Ru)^\vee \longrightarrow 0$$

where φ is just the restriction to Ru, i.e. if $f \in M^\vee$, $f : M \to E$, then $\varphi(f) = f|_{Ru} \in (Ru)^\vee$. But $f(u) = 0$ for all $f \in M^\vee$ and φ surjective means $(Ru)^\vee = 0$, if and only if $Ru = 0$ by (2), i.e. $u = 0$ and θ is one-to-one. Hence $M \simeq M^{\vee\vee}$.
(4) Note that $R^\vee = Hom_R(R, E) \simeq E$, therefore $\lambda(E) = \lambda(R^\vee) = \lambda(R)$ follows from (1). □

Remark. (1) in Theorem 53 does not imply $M \simeq M^\vee$.

Remark. Let $(\ \cdot\ )^* := \mathrm{Hom}_R(\cdot, R)$. Then we cannot replace $(\ \cdot\ )^\vee$ by $(\ \cdot\ )^*$: let $V := \mathrm{Soc}(R) \simeq k^{\oplus t}$, then

$$V^* = \mathrm{Hom}_R(V, R) = \mathrm{Hom}_R(k^{\oplus t}, R) \simeq \mathrm{Hom}_R(k, R)^{\oplus t} \simeq V^{\oplus t} = k^{\oplus t^2}.$$

So $\lambda(V) = t$ and $\lambda(V^*) = t^2$, so it works (since $t \neq 0$) if and only if $t = 1$. But in this case R is (zero dimensional ??) Gorenstein and next theorem shows that R is injective and $R \simeq E$, so $V^* = V^\vee$ in this case.

Theorem 54. *Let (R, m, k, E) be a 0-dimensional Noetherian local ring. The following conditions are equivalent:*

(1) R is Gorenstein.

(2) $R \simeq E$.

(3) R is injective as an R-module.

Proof. (1) $\Rightarrow$ (2) R is essential over $\mathrm{Soc}(R) \simeq k$. But E is the maximal essential extension of k, so $k \subseteq R \subseteq E$. But $\lambda(R) = \lambda(E)$ by Proposition 53 (4), therefore $R \simeq E$.
(2) $\Rightarrow$ (3) Clear since E is an injective R-module.
(3) $\Rightarrow$ (1) R is injective, then R is a direct sum of $E_R(R/P)$, with $P \in \mathrm{Spec}(R)$. But R is 0-dimensional, so m is the only prime and $R \simeq \bigoplus E$. Now, R is local, so it is indecomposable, i.e. $R \simeq E$ and R is Gorenstein since $\mathrm{Soc}(E) \simeq k$ by Proposition 52 (2). □

Remark. If $\alpha \in \mathrm{Ext}^1_R(k, R)$, $\alpha \neq 0$, this means that there exists M a R-module, $M \neq R \oplus k$, such that

$$0 \longrightarrow R \longrightarrow M \longrightarrow k \longrightarrow 0 \quad \text{is exact.}$$

Since $k \simeq \mathrm{Soc}(R) \subseteq R$ is essential, and $R \hookrightarrow M$, then $k \subseteq M$ is essential unless $M = R \oplus k$. If R is Gorenstein $R \simeq E$ and k cannot have essential extensions properly containing R. So it has to be $M = R \oplus k$, in accordance with the fact that $\mathrm{Ext}^1_R(k, R) = 0$ since R is Gorenstein, therefore injective.

Corollary 55. *Let (R, m, k, E) be a 0-dimensional Noetherian local ring. Then the natural map*

$$\begin{aligned} R &\to \mathrm{Hom}_R(E, E) = E^{\vee} \\ r &\mapsto (e \mapsto er) \end{aligned}$$

is an isomorphism.

Proof. $R^{\vee} \simeq E$ and $E^{\vee} \simeq R^{\vee\vee} \simeq R$ by the map above. □

Theorem 56. *Let (R, m, k, E) be a Noetherian local ring. Then*

$$\widehat{R} \simeq \mathrm{Hom}_R(E, E) = E^{\vee}$$

Proof. Recall that $\widehat{R} = \lim_{\leftarrow} R/m^n$. Set $E_n = \mathrm{Hom}_R(R/m^n, E) \simeq \{e \in E : m^n E = 0\} \subseteq E$.
Claim (1st Key Claim). $E_n = E_{R/m^n}(k)$

Proof of the 1st *Key Claim.* Note that

- E_n is injective as an R/m^n module by Proposition 43.

- $E_1 = \mathrm{Soc}(E) \simeq k \subseteq E_n \subseteq E_{R/m^n}(k)$ is essential, hence $k \subseteq E_n$ is essential.

- E_n injective and essential implies $E_n = E_{R/m^n}(k)$ (also because $\lambda(E_n) = \lambda((R/m^n)^\vee) = \lambda(R/m^n) = \lambda(E_{R/m^n}(k))$ and $E_n \subseteq E_{R/m^n}(k)$).

$\square$

Claim (2^{nd} Key Claim). Let $f \in \mathrm{Hom}_R(E, E)$, then clearly $f_n := f|_{E_n} \in \mathrm{Hom}_R(E_n, E)$. We claim that $f_n \in \mathrm{Hom}_R(E_n, E_n)$.

Proof of the 2^{nd} Key Claim. Let $u \in E_n$, then $m^n u = 0$. So $0 = f(m^n u) = m^n f(u)$, therefore $f(u) = f_n(u) \in E_n$. $\square$

Claim (Final Claim). $\mathrm{Hom}_R(E, E) \simeq \lim_{\leftarrow} \mathrm{Hom}_{R/m^n}(E_n, E_n)$

Proof of the Final Claim. First of all note that

$$\mathrm{Hom}_{R/m^n}(E_n, E_n) = \lim_{\leftarrow} \mathrm{Hom}_R(E_n, E_n)$$

since m^n kills E_n and the maps are restrictions. Consider the map:

$$\varphi : f \in \mathrm{Hom}_R(E, E) \mapsto (f_1, f_2, \dots) \in \lim_{\leftarrow} \mathrm{Hom}_R(E_n, E_n)$$

It is well defined since $f_{n+1}|_{E_n} = f_n$ for all $n \geq 1$. Also φ is a homomorphism and it is one-to-one: $\mathrm{Ass}(E) = \{m\}$, therefore every element in E is killed by some power of the maximal ideal. Therefore $\bigcup_n E_n = E$ and $f_n = 0$ for all $n \geq 1$ implies $f = 0$.

Conversely take $(g_1, g_2, \dots) \in \lim_{\leftarrow} \mathrm{Hom}_R(E_n, E_n)$, i.e. $g_{n+1}|_{E_n} = g_n$. Take $u \in E$, then $u \in E_n$ for some n since $\bigcup_n E_n = E$. Define $g : E \to E$, $g(u) := g_n(u)$. It is a homomorphism, so $g \in \mathrm{Hom}_R(E, E)$ and $\varphi(g) = (g_1, g_2, \dots)$. Therefore φ is an isomorphism. $\square$

Consider now the following diagram:

$$
\begin{tikzcd}
{} & {} \\
R/m^n \arrow[u, two heads, dotted] \arrow[r] & \mathrm{Hom}_R(E_{R/m^n}(k), E_{R/m^n}(k)) = \mathrm{Hom}_R(E_n, E_n) \arrow[u, two heads, dotted] \\
R/m^{n+1} \arrow[u, two heads] \arrow[r] & \mathrm{Hom}_R(E_{R/m^{n+1}}(k), E_{R/m^{n+1}}(k)) = \mathrm{Hom}_R(E_{n+1}, E_{n+1}) \arrow[u, two heads] \\
R/m^{n+2} \arrow[u, two heads] \arrow[r] & \mathrm{Hom}_R(E_{R/m^{n+2}}(k), E_{R/m^{n+2}}(k)) = \mathrm{Hom}_R(E_{n+2}, E_{n+2}) \arrow[u, two heads] \\
\widehat{R} \simeq \lim\limits_{\leftarrow} R/m^n \arrow[u, two heads, dotted] & \lim\limits_{\leftarrow} \mathrm{Hom}_R(E_n, E_n) \simeq \mathrm{Hom}_R(E, E) \arrow[u, two heads, dotted]
\end{tikzcd}
$$

The maps $R/m^n \to \mathrm{Hom}_R(E_{R/m^n}(k), E_{R/m^n}(k))$ are all isomorphism by Corollary 55, because they act as multiplication and R/m^n is 0-dimensional. The equalities $\mathrm{Hom}_R(E_{R/m^n}(k), E_{R/m^n}(k)) = \mathrm{Hom}_R(E_n, E_n)$ follow by 1^{st} Key Claim and finally $\lim_{\leftarrow} \mathrm{Hom}_R(E_n, E_n) \simeq \mathrm{Hom}_R(E, E)$ by the Final Claim. The diagram commutes since

$$R/m^{n+1} \twoheadrightarrow R/m^n \simeq \mathrm{Hom}_R(E_n, E_n)$$

is restriction and then multiplication, and this commutes with

$$R/m^{n+1} \simeq \mathrm{Hom}_R(E_{n+1}, E_{n+1}) \twoheadrightarrow \mathrm{Hom}_R(E_n, E_n)$$

which is multiplication and then restriction of the map. This proves the theorem since

$$\widehat{R} \simeq \lim_{\leftarrow} R/m^n \simeq \lim_{\leftarrow} \mathrm{Hom}_R(E_n, E_n) \simeq \mathrm{Hom}_R(E, E) = E^\vee.$$

$\square$

Theorem 57 (Matlis Duality)**.** *Let (R, m, k, E) be a local Noetherian ring. Then $(\ \cdot\ )^\vee$ gives a one-to-one arrow reversing correspondence between*

$$\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}) \qquad \overset{\vee}{\longleftrightarrow} \qquad \textit{Artinian } R-\textit{modules}$$

If M is a module on either side, then $M \simeq M^{\vee\vee}$.

Before proving this theorem we need some discussion on Artinian R-modules.

Remark. $(\ \cdot\ )^\vee$ is exact but sends injections in surjections and viceversa. For instance if $0 \to A \to B$ is an injection, then checking (i.e. applying $(\ \cdot\ )^\vee$) the sequence becomes $B^\vee \to A^\vee \to 0$. Similarly starting with a surjection.

If N is an Artinian R-module, then so it is every $K \subseteq N$ submodule. In particular, every finitely generated submodule $K \subseteq N$ has to have finite length, because it is finitely generated and Artinian. This means that for all $x \in N$ there exists $n >> 0$ such that $m^n x = 0$, since Rx is finitely generated and $\mathrm{Supp}(Rx) = \{m\}$.

Remark. An Artinian R-module N is essential over $\mathrm{Soc}(N) = \{x \in N : mx = 0\}$.

Proof of the Remark. We have already proved this result for 0-dimensional rings. It is in fact true in general for any dimension. As already noticed it is enough to prove essentiality for principal modules. Let $Rx \subseteq N$, $x \neq 0$. Choose $n \in \mathbb{N}$ least such that $m^n x = 0$. Then $m^{n-1}x \neq 0$ is inside $Rx \cap \mathrm{Soc}(N) \neq 0$, i.e. $\mathrm{Soc}(N) \subseteq N$ is essential. □

$\mathrm{Soc}(N) \subseteq N$ is a submodule of an Artinian module and it is also a k-vector space. So it has to be $\dim_k \mathrm{Soc}(N) < \infty$, otherwise it cannot be Artinian. Write $\mathrm{Soc}(N) = k^t$, then $k^t \subseteq N$ is essential, hence $k^t \subseteq E_R(k^t) = E_R(k)^t = E^t$, since $E_R(\ \cdot\ )$ commutes with direct sums.

Lemma 58. *E is Artinian.*

Proof. Take a descending chain of submodules of E:

$$\cdots \subseteq E_{n+1} \subseteq E_n \subseteq \cdots \subseteq E_1 \subseteq E.$$

Checking we get

$$E^\vee \twoheadrightarrow {E_1}^\vee \twoheadrightarrow \ldots$$

But $E^\vee = \widehat{R}$ and also all the maps are surjections, hence ${E_n}^\vee \simeq \widehat{R}/I_n$, for some ideal I_n. Since ${E_n}^\vee \twoheadrightarrow {E_{n+1}}^\vee$, we have $\widehat{R}/I_n \twoheadrightarrow \widehat{R}/I_{n+1}$, i.e. we can consider the ascending chain of ideals in $\widehat{R}$:

$$I_n \subseteq I_{n+1} \subseteq \ldots$$

But $\widehat{R}$ is Noetherian, therefore the chain stabilizes and so ${E_n}^\vee = {E_{n+1}}^\vee$ for some $n \in \mathbb{N}$.

Claim. $E_n = E_{n+1}$

Proof of the Claim. If $0 \to K \to L \to L/K \to 0$ is an exact sequence, then checking we get an exact sequence $0 \to (L/M)^\vee \to L^\vee \to K^\vee \to 0$. So if $L^\vee = K^\vee$, then $(L/K)^\vee = 0$. So we need to prove that if T is any module, then $T^\vee = 0$ implies $T = 0$. Note that we have already proved this result if we assume R 0-dimensional and $T \in \mathrm{Mod}^{\mathrm{fg}}(R)$. By way of contradiction choose $x \in T$, $x \neq 0$. Since $Rx \subseteq T$ we have $0 = T^\vee \twoheadrightarrow (Rx)^\vee$, i.e. $(Rx)^\vee = 0$. In this way we have reduced the problem to a finitely generated module, since we can now assume $T = Rx$. There exists a non zero surjective map $T \twoheadrightarrow k$ (just kill the maximal ideal and project onto one copy of k, hence checking $k^\vee \subseteq T^\vee$. But we know that $k^\vee = k$ and by assumption $T^\vee = 0$, and this is a contradiction. Therefore $T = 0$.

Now apply this result to $K = E_{n+1}$ and $L = E_n$ to prove $E_n/E_{n+1} = 0$, i.e. $E_n = E_{n+1}$. $\square$

This claim completes the prove because $E_n = E_{n+1}$ and the initial ascending chain stabilizes. $\square$

This Lemma gives a characterization of Artinian modules over a Noetherian local ring R.

Corollary 59. *An R-module N is Artinian if and only if $N \subseteq E^t$ for some $t \in \mathbb{N}$.*

Proof. We have already seen that if N is Artinian, then $N \subseteq E^t$ for some $t \in \mathbb{N}$ $(t = \dim_k \operatorname{Soc}(N))$. Conversely E is Artinian, therefore E^t is Artinian, and $N \subseteq E^t$ is a submodule of an Artinian module, hence Artinian. $\square$

Remark. Any Artinian R-module is naturally an $\widehat{R}$-module. In fact let $\widehat{r} \in \widehat{R}$, then $\widehat{r} = \lim r_n$, with $r_n \in R$ and $r - r_n \in m^n$. If x is an element of N, then there exists $n >> 0$ such that $m^n x = 0$. Then $\widehat{r}x = r_t x$ for all $t \geq n$ is forced and well defined.

Remark. With a similar argument, if N is an Artinian R-module and M is an $\widehat{R}$-module, then $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_{\widehat{R}}(M, N)$.

Exercise 1. This is of course not true in general. For instance

$$\operatorname{Hom}_{k[\![t]\!]}(k[\![t]\!], k[\![t]\!]) \subsetneq \operatorname{Hom}_{k[t]}(k[\![t]\!], k[\![t]\!]).$$

Find a $k[t]$-automorphism of $k[\![t]\!]$ which is not a $k[\![t]\!]$- automorphism.

Proof of Theorem 57, Matlis Duality. Let N be an Artinian R-module, by Corollary 59 $N \subseteq E^t$. Checking:

$$(E^t)^\vee \twoheadrightarrow N^\vee \quad \text{and} \quad (E^t)^\vee = (E^\vee)^t \simeq \widehat{R}^t.$$

So $N^\vee$ is a finitely generated $\widehat{R}$-module. Conversely let $M \in \operatorname{Mod}^{\mathrm{fg}}(\widehat{R})$, then we have a presentation $(\widehat{R})^t \to M \to 0$, and checking we get $M^\vee \subseteq ((\widehat{R})^\vee)^t$. By Corollary 59 we just have to show that $(\widehat{R})^\vee \simeq E$. By the previous discussion on Artinian modules it turns out that

$$(\widehat{R})^\vee = \operatorname{Hom}_R(\widehat{R}, E) = \operatorname{Hom}_{\widehat{R}}(\widehat{R}, E) \simeq E.$$

So far we have shown that the correspondence is well defined and also that $\widehat{R} \overset{\vee}{\longleftrightarrow} E$.

Suppose now $M \in \mathrm{Mod}^{\mathrm{fg}}(\widehat{R})$, take a presentation and double-check it:

$$\begin{tikzcd}
(\widehat{R})^s \arrow[r] \arrow[d, "\simeq"'] & (\widehat{R})^t \arrow[r] \arrow[d, "\simeq"'] & M \arrow[r] \arrow[d] & 0 \\
((\widehat{R})^{\vee\vee})^s \arrow[r] & ((\widehat{R})^{\vee\vee})^t \arrow[r] & M^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

where the up-to-down arrows are the natural maps of a module in the double dual. They are isomorphisms since $(\widehat{R})^{\vee} \simeq E$ and so $(\widehat{R})^{\vee\vee} \simeq E^{\vee} \simeq \widehat{R}$. So five lemma implies $M \simeq M^{\vee\vee}$.

On the other way let N be an Artinian R-module, then $0 \to N \to E^t \to N_1 \to 0$, where N_1 is the cokernel. Since E^t is Artinian also N_1 is Artinian, so $N_1 \subseteq E^s$. Double-checking:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & E^t \arrow[r] \arrow[d, "\simeq"'] & E^s \arrow[d, "\simeq"'] \\
0 \arrow[r] & N^{\vee\vee} \arrow[r] & (E^{\vee\vee})^t \arrow[r] & (E^{\vee\vee})^s
\end{tikzcd}$$

Again the maps are the natural ones of a module in its double dual and

$$E^{\vee\vee} \simeq (\widehat{R})^{\vee} \simeq E,$$

so five lemma implies $N \simeq N^{\vee\vee}$. $\square$

Remark. What happens if M is just a finitely generated R-module and we check it?

$$\begin{aligned}
M^{\vee} &= \mathrm{Hom}_R(M, E) \\
&\simeq \mathrm{Hom}_R(M, \mathrm{Hom}_{\widehat{R}}(\widehat{R}, E)) \\
&\simeq \mathrm{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, E) \\
&= \mathrm{Hom}_{\widehat{R}}(\widehat{M}, E) \\
&= \mathrm{Hom}_R(\widehat{M}, E) \\
&= (\widehat{M})^{\vee}.
\end{aligned}$$

Therefore $M^{\vee} \simeq (\widehat{M})^{\vee}$ if M is a finitely generated R-module. Also $M^{\vee\vee} \simeq (\widehat{M})^{\vee\vee} \simeq \widehat{M}$ since $\widehat{M} \in \mathrm{Mod}^{\mathrm{fg}}(\widehat{R})$.

Remark. If M is both an Artinian R-module and a finitely generated R-module (hence a $\widehat{R}$-module since it is an Artinian R-module), then M belongs to both the sides of the correspondence. In this case $M^{\vee\vee} = M$, and $(\,\cdot\,)^{\vee}$ is an involution.

Remark. If R is (zero dimensional ??) Gorenstein, then

$$(\ \cdot\ )^\vee = \mathrm{Hom}_R(\cdot, E) = \mathrm{Hom}_R(\cdot, R) = (\ \cdot\ )^*.$$

Question. *Which modules M satisfy $M^{\vee\vee} \simeq M$? Assume $R = \widehat{R}$ to simplify the problem. In this case there is a theorem which states that all the modules of this type are M such that $0 \to A \to M \to N \to 0$ with A Artinian and N finitely generated. In this case in fact:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\simeq"'] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & A^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & N^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

and $M \simeq M^{\vee\vee}$ by five lemma.

Lemma 60. *Let (R, m, k, E) be a 0-dimensional Noetherian local ring. The following facts are equivalent:*

(1) *R is Gorenstein.*

(2) $\mathrm{id}_R\, R \leq \infty$

Proof. (1) $\Rightarrow$ (2) R is Gorenstein, therefore $R \simeq E$ is injective and $\mathrm{id}_R\, R = 0$.
(2) $\Rightarrow$ (1) Suppose $\mathrm{id}_R\, R = n \leq \infty$, then we have a finite injective resolution. Since every injective is a direct sum of indecomposable injective modules, and the only prime in R is m, the injective modules are all direct sums of E, i.e.:

$$0 \longrightarrow R \longrightarrow E^{b_0} \longrightarrow E^{b_1} \longrightarrow \ldots\ldots \longrightarrow E^{b_n} \longrightarrow 0.$$

Since R is 0-dimensional $E^\vee \simeq \widehat{R} \simeq R$, so checking:

$$0 \longrightarrow R^{b_n} \longrightarrow R^{b_{n-1}} \longrightarrow \ldots\ldots \longrightarrow R^{b_0} \longrightarrow E \longrightarrow 0.$$

So $\mathrm{pdim}R \leq \infty$ and by Auslander-Buchsbaum formula $\mathrm{pdim}E + \mathrm{depth}E = \mathrm{depth}R$. But $\mathrm{depth}E = \mathrm{depth}R = 0$ since all is Artinian, so E is free and $E \simeq R^t$. Finally $\lambda(R) = \lambda(E) = \lambda(R^t) = t\lambda(R)$, i.e. $t = 1$ and $E \simeq R$ is Gorenstein. $\square$

2.4 Structure of $E_R(R/P)$

Theorem 61. *Let R be a Noetherian ring, $P \in \mathrm{Spec}(R)$. Then $E_R(R/P) \simeq E_{R_P}(k(P))$, where $k(P) = R_P/PR_P$.*

Proof. Let us prove the theorem in three steps.

(1) $E_R(R/P) \simeq E_R(k(P))$ as R-modules:
Note that $k(P) = R_P/PR_P = (R/P)_P$ so it is the fraction field of R/P. For this reason $R/P \subseteq k(P)$ is essential and also

$$\begin{tikzcd} 0 \arrow[r] & R/P \arrow[r] \arrow[d, hook] & k(P) \arrow[dl, hook, dotted] \\ & E_R(R/P) & \end{tikzcd}$$

there exists an embedding $k(P) \hookrightarrow E_R(R/P)$, which is essential itself. But $E_R(R/P)$ is injective, so $E_R(R/P) = E_R(k(P))$ by definition of injective hull (any injective between a module and its injective hull is the injective hull).

(2) $E_R(R/P)$ is an R_P-module:
This is equivalent to say that every $x \notin P$ acts on $E_R(R/P)$ as a unit, i.e.

- x is a NZD on $E_R(R/P)$, in fact suppose $u \in E_R(R/P)$, $u \neq 0$ and $xu = 0$. Then $Ru \cap R/P \neq 0$ as $R/P \subseteq E_R(R/P)$ is essential. Therefore there exists $r \in R$ such that $ru \in R/P$, $ru \neq 0$, which implies $xru \neq 0$ since $x \notin P$, and this is a contradiction. So x is a NZD.
- $xE_R(R/P) = E_R(R/P)$, consider in fact

$$\begin{array}{ccc} k(P) & \subseteq & E_R(R/P) \\ \| & & \cup| \\ xk(P) & \subseteq & xE_R(R/P) \end{array}$$

where the equality holds since $k(P)$ is a field and x is a unit in $k(P)$. Then $k(P) \subseteq xE_R(R/P) \subseteq E_R(R/P)$ and since x is a NZD $xE_R(R/P) \simeq E_R(R/P)$, so $xE_R(R/P)$ is injective. But any injective contained in the injective hull is the injective hull, so in fact $xE_R(R/P) = E_R(R/P)$.

(3) The proof, i.e. $E_R(R/P) \simeq E_{R_P}(k(P))$ as R_P-modules:
$k(P) \subseteq E_R(R/P)$ is essential and $E_R(R/P)$ is also a R_P-module. We want to show that $E_R(R/P)$ is an injective R_P-module. Consider:

$$\begin{tikzcd} & E_R(R/P) & \\ 0 \arrow[r] & M \arrow[r] \arrow[u] & N \end{tikzcd}$$

where modules and maps are all over R_P. Then, since all modules and maps can be considered over R and since $E_R(R/P)$ is an injective R-module there exists $g : N \to E_R(R/P)$ which makes the diagram commute. It is enough to prove that g is a R_P homomorphism.

Fact. If $g : K \to L$ is a R-homomorphism and K, L are R_P-modules, then g is a R_P-homomorphism.

Proof of the Fact. Take $x \in P$ and $u \in K$. Then

$$g(u) = g\left(\frac{1}{x} \cdot x \cdot u\right) = xg\left(\frac{u}{x}\right)$$

which implies $g\left(\frac{u}{x}\right) = \frac{g(u)}{x}$ since L is a R_P-module. $\square$

Now, this completes the proof since the only essential injective extension of a module is the injective hull, i.e. $E_R(R/P) \simeq E_{R_P}(k(P))$ as R_P-modules. $\square$

Remark. $E_{R_P}(k(P)) = E_{\widehat{R_P}}(k(P))$.

Corollary 62. *Let R be a Noetherian ring, $P, Q \in \operatorname{Spec}(R)$. Then*

$$E_R(R/P)_Q = \begin{cases} 0 & P \not\subseteq Q \\ E_R(R/P) & P \subseteq Q \end{cases}$$

Proof. If $P \subseteq Q$, then by Theorem 61 we have both $E_{R_P}(k(P)) = E_R(R/P)$ and $E_{R_P}(k(P)) = E_{R_Q}((R/P)_Q)$, first localizing at Q and then at P. If instead $P \not\subseteq Q$ choose $x \in P \setminus Q$ and take $u \in E_R(R/P)$. Since $\operatorname{Ass}(Ru) = \{P\}$ there exists $l >> 0$ such that $P^l u = 0$, in particular $x^l u = 0$. Localizing at Q x becomes invertible and so $\frac{u}{1} = 0$, i.e. $E_R(R/P)_Q = 0$. $\square$

Corollary 63. *Let R be a Noetherian ring and E be an injective R-module. Then E_Q is injective for all $Q \in \operatorname{Spec}(R)$.*

Proof. E is a sum of indecomposable injective modules, write $E = \bigoplus_i E_R(R/P_i)$. Then $E_Q = \bigoplus_i E_{R_Q}(R/P_i)_Q$ is a sum of either zero modules or $E_R(R/P_i)$, and so it is injective. $\square$

3 Minimal Injective Resolutions

Remark. If $I^{\cdot}$ is an injective resolution over R, then $I^{\cdot}_P$ is an injective resolution over R_P for all $P \in \operatorname{Spec}(R)$.

Lemma 64. *Let R be a Noetherian ring, then*

$$\operatorname{Hom}_R(R/xR, E_R(R/P)) = \begin{cases} 0 & x \notin P \\ E_{R/xR}(R/P) & x \in P \end{cases}$$

Proof. Recall that $\operatorname{Hom}_R(R/xR, E_R(R/P)) = \{u \in E_R(R/P) : ux = 0\}$. If $x \notin P$ then x acts as a unit on $E_R(R/P)$ (as seen in the proof of Theorem 61, so it cannot kill anything and $\operatorname{Hom}_R(R/xR, E_R(R/P)) = \{u \in E_R(R/P) : ux = 0\} = 0$.

If $x \in P$ then note that $\operatorname{Hom}_R(R/xR, E_R(R/P))$ is an R/xR injective module, so it is enough to show that it is essential over R/P. But $0 \to R/P \to E_R(R/P)$ is essential and $R/P \subseteq \operatorname{Hom}_R(R/xR, E_R(R/P)) = \{u \in E_R(R/P) : ux = 0\} \subseteq E_R(R/P)$. Hence $R/P \subseteq \operatorname{Hom}_R(R/xR, E_R(R/P))$ is essential and therefore $\operatorname{Hom}_R(R/xR, E_R(R/P)) = E_{R/xR}(R/P)$. $\square$

Definition. Let R be a ring and let $M \in \operatorname{Mod}^{\mathrm{fg}}(R)$. A *minimal injective resolution* $I^{\cdot}$ of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \ldots\ldots$$

such that I^j is an injective R-module and $I^{j+1} \simeq E_R(I^j/Im(d^{j-1}))$.

Fact. Up to isomorphism of complexes the minimal injective resolution of a module is unique.

Proposition 65 (Criterion for Injective Hulls)**.** *Let R be a Noetherian ring and $M \subseteq I$ be R-modules, with I injective. Then $I = E_R(M)$ if and only if for all $P \in \operatorname{Spec}(R)$*

$$\varphi_P : \operatorname{Hom}_{R_P}(k(P), M_P) \longrightarrow \operatorname{Hom}_{R_P}(k(P), I_P)$$

is an isomorphism.

Proof. Note that $M \subseteq I$ implies $M_P \subseteq I_P$ and by left-exactness of Hom

$$\operatorname{Hom}_{R_P}(k(P), M_P) \hookrightarrow \operatorname{Hom}_{R_P}(k(P), I_P)$$

So $I = E_R(M)$ if and only if for all $P \in \operatorname{Spec}(R)$ the map φ_P is surjective.

Suppose φ_P is surjective for all $P \in \operatorname{Spec}(R)$ and let $u \in I$, $u \neq 0$. It is enough to show that $Ru \cap M \neq 0$. Let $P \in \operatorname{Ass}(Ru)$, then $R/P \hookrightarrow Ru$. Set $v =$ the image of 1 under this map. We have

$$\begin{tikzcd}
 & R/P \arrow[d, phantom, "\cap\vert"] & \simeq & Rv & \subseteq & I \\
0 \arrow[r] & k(P) \arrow[rr, "f"'] & & I_P & &
\end{tikzcd}$$

and by assumption there exists $f : k(P) \to M_P$ such that

$$\begin{tikzcd}
k(P) \arrow[r, "g"] \arrow[d, equal] & M_P \arrow[d, phantom, "\cap\vert"] \\
k(P) \arrow[r, "f"'] & I_P \\
1 \arrow[r, mapsto] & \frac{v}{1}
\end{tikzcd}$$

Then $g(1) = \frac{v}{1} \in M_P$, i.e. there exists $s \notin P$ such that $sv \in M$, $sv \neq 0$ because $\frac{v}{1} \neq 0$ and $s \notin P$. So $0 \neq Rv \cap M \subseteq Ru \cap M$, i.e. $M \subseteq I$ is essential and $I = E_R(M)$ because it is injective.

Conversely assume $I = E_R(M)$ and take $f : k(P) \to I_P$ a R_P-homomorphism. We know that $\mathrm{Hom}_{R_P}(k(P), I_P) = (\mathrm{Hom}_R(R/P, I))_P$ because Hom commutes with localization. Therefore there exists $g : R/P \to I$ R-homomorphism such that $\frac{g}{s} = f$ for some $s \notin P$. Suppose $g(1) = u$, then $Ru \cap M \neq 0$ since $M \subseteq I = E_R(M)$ is essential. So there exists $r \notin P$ such that $0 \neq ru \in M$, and r is not in P because otherwise $ru = rg(1) = g(r) = g(0) = 0$ since $g : R/P \to I$. We have

$$\begin{array}{ccc} k(P) & \longrightarrow & M_P \\ \| & & \cap\,| \\ k(P) & \xrightarrow[f]{} & I_P \\ 1 & \longmapsto & f(1) \end{array}$$

So $f(1) = \frac{g(1)}{s} = \frac{u}{s} = \frac{ru}{su} \in M_P$, i.e. φ_P is surjective since we can consider $f : k(P) \to M_P$ restricting the target space. $\square$

Theorem 66. *Let R be a Noetherian ring and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$ and take*

$$I^{\cdot} : \qquad 0 \longrightarrow M \longrightarrow I^0 \xrightarrow{d} I^1 \xrightarrow{d} \cdots$$

an injective resolution of M. Then $I^{\cdot}$ is minimal if and only if for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and for all $j \geq 0$

$$\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^j_\mathfrak{p}) \longrightarrow \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^{j+1}_\mathfrak{p})$$

is the zero map.

Proof. Fix j and define $N = \mathrm{Coker}(I^{j-2} \to I^{j-1})$. Therefore we have the exact sequence

$$N \longrightarrow I^j \longrightarrow I^{j+1} \longrightarrow 0.$$

Seeing that localization is exact and $\mathrm{Hom}(\kappa(\mathfrak{p}), \cdot)$ is left exact, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), N_\mathfrak{p}) \xrightarrow{\alpha} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^j_\mathfrak{p}) \xrightarrow{\beta} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^{j+1}_\mathfrak{p}).$$

Now, for all $\mathfrak{p}$ and all j, we have $\beta = 0$ if and only if α is an isomorphism. By Proposition 65, this is the same as $I^j_\mathfrak{p}$ being the injective hull of $N_\mathfrak{p}$ for all j, that is, $I^{\cdot}$ is minimal. $\square$

Remark. For projective resolutions minimality was given by the condition

$$P_{j+1} \otimes R/m \to P_j \otimes R/m$$

is the zero map.

Theorem 67. *Let (R, m, k) be a Noetherian local ring and let $M \in \mathrm{Mod}^{\mathrm{fg}}(R)$. Then*

$$\mathrm{id}_R\, M = \sup\{i : \mathrm{Ext}^i_R(k, M) \neq 0\}$$

Proof. INSERT PROOF!!!!!!!!!!!!!!!!!! □

Theorem 68. *Let (R, m, k) be a d-dimensional Cohen-Macaulay local ring. The following facts are equivalent:*

(1) R is Gorenstein.

(2) $\mathrm{id}_R\, R < \infty$.

Proof. Lemma 60 proves the Theorem if the ring is 0-dimensional. We will use induction on $\dim R$. Without loss of generality assume $\dim R > 0$. R is Cohen-Macaulay, so there exists $x \in m$ a NZD. Set $\overline{R} := R/xR$. We want to calculate $\mathrm{Ext}^1_R(R/xR, R)$ in two different ways. A free resolution of $\overline{R}$ is:

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow \overline{R} \longrightarrow 0$$

Applying $\mathrm{Hom}_R(\cdot, R)$:

$$0 \longrightarrow \mathrm{Hom}_R(\overline{R}, R) \longrightarrow \mathrm{Hom}_R(R, R) \xrightarrow{x} \mathrm{Hom}_R(R, R) \longrightarrow Ext^1_R(\overline{R}, R) \longrightarrow Ext^1_R(R, R) \longrightarrow \ldots\ldots$$

Since x is a NZD on R we have $\mathrm{Hom}_R(\overline{R}, R) = 0 :_R x = 0$ and $\mathrm{Ext}^i_R(R, R) = 0$ for all $i \geq 1$ since R is free. Hence:

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow \mathrm{Ext}^1_R(\overline{R}, R) \longrightarrow 0$$

which means:

$$\mathrm{Ext}^i_R(\overline{R}, R) = \begin{cases} 0 & i \neq 1 \\ \overline{R} & i = 1 \end{cases}$$

Claim. $E_R(R/P) \not| E_R(R)$ if $P \notin \mathrm{Ass}(R)$.

Proof of the Claim. If $P \notin \mathrm{Ass}R$ then $P_P \notin \mathrm{Ass}(R_P)$, therefore

$$\mathrm{Hom}_{R_P}(k(P), R_P) = 0.$$

But $0 = \mathrm{Hom}_{R_P}(k(P), R_P) \simeq \mathrm{Hom}_{R_P}(k(P), (E_R(R))_P)$, hence

$$E_R(R/P) \not| E_R(R)$$

because otherwise $E_R(R) = E_R(R/P) \oplus L$ which would imply $(E_R(R))_P \neq 0$ and so $\mathrm{Hom}_{R_P}(k(P), (E_R(R))_P) \neq 0$ since

$$\mathrm{Hom}_{R_P}(k(P), (E_R(R/P))_P) = \mathrm{Hom}_{R_P}(k(P), E_{R_P}(k(P))) = k(P)^\vee \simeq k(P).$$

□

Now take the minimal injective resolution of R:

$$0 \longrightarrow R \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \dots$$

Then we can compute Ext applying $\operatorname{Hom}_R(\overline{R}, \cdot)$:

$$\operatorname{Ext}^i_R(\overline{R}, R) = H^i(0 \longrightarrow R \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \dots)$$

But

$$\operatorname{Hom}_R(\overline{R}, E_R(R/P)) = \begin{cases} 0 & x \notin P \\ \\ E_{\overline{R}}(R/P) & x \in P \end{cases}$$

and $I^j = \bigoplus E_R(R/P)$ for some $P \in \operatorname{Spec}(R)$, therefore $\overline{I}^j := \operatorname{Hom}_R(\overline{R}, I^j)$ are all injective $\overline{R}$-modules. By the Claim $\overline{I}^0 = 0$ since $I^0 = E_R(R)$, also $E_R(R/P) \nmid E_R(R)$ if $P \notin \operatorname{Ass}(R)$ and $x \notin \operatorname{Ass}(R)$. Then:

$$\operatorname{Ext}^i_R(\overline{R}, R) = H^i(0 \longrightarrow R \longrightarrow 0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots)$$

Since $\operatorname{Ext}^i_R(\overline{R}, R) = 0$ if $i \neq 1$ and $\operatorname{Ext}^1_R(\overline{R}, R) = \overline{R}$ there is no cohomology in degree $j \geq 2$ and $\overline{R} = \ker(\overline{I}^1 \to \overline{I}^2)$, therefore

$$0 \longrightarrow \overline{R} \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \dots \quad \text{is exact.}$$

So this is a resolution of $\overline{R}$, and this shows $\operatorname{id}_{\overline{R}} \overline{R} < \operatorname{id}_R R$.
(2) $\Rightarrow$ (1) Assume $\operatorname{id}_R R < \infty$, then $\operatorname{id}_{\overline{R}} \overline{R} < \operatorname{id}_R R < \infty$ and hence, by induction, $\overline{R}$ is Gorenstein. But $x \in m$ is a NZD, therefore R is Gorenstein.
(1) $\Rightarrow$ (2) We claim that the resolution

$$0 \longrightarrow \overline{R} \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \dots \quad \text{is exact.}$$

is in fact minimal. Write $I^j = E^j \oplus L^j$, where $E_R(R/P) | I^j$ with $x \in P$ are all inside E^j, and all others are inside L^j. Note that $\overline{I}^j = \overline{E}^j$. Consider:

$$\begin{tikzcd}
\operatorname{Hom}_{R_P}(k(P), \overline{E}^j_P) \arrow[r] \arrow[d, "\simeq"'] & \operatorname{Hom}_{R_P}(k(P), \overline{E}^{j+1}_P) \arrow[d, "\simeq"'] \\
(\operatorname{Hom}_R(R/P, \operatorname{Hom}_R(\overline{R}, E^j)))_P \arrow[r] \arrow[d, "\simeq"'] & (\operatorname{Hom}_R(R/P, \operatorname{Hom}_R(\overline{R}, E^{j+1})))_P \arrow[d, "\simeq"'] \\
\operatorname{Hom}_{R_P}(k(P), E^j_P) \arrow[r] \arrow[d, hook] & \operatorname{Hom}_{R_P}(k(P), E^{j+1}_P) \arrow[d, hook] \\
\operatorname{Hom}_{R_P}(k(P), I^j_P) \arrow[r, "0"] & \operatorname{Hom}_{R_P}(k(P), I^{j+1}_P)
\end{tikzcd}$$

where the last map is zero since we started from a minimal injective resolution. Therefore $\operatorname{id}_{\overline{R}} \overline{R} \leq \operatorname{id}_R R - 1$ (INSERT PROOF THAT IT IS IN FACT EQUAL). Assuming R is Gorenstein, we have that $\overline{R}$ is Gorenstein because x is a NZD, and therefore $\operatorname{id}_{\overline{R}} \overline{R} < \infty$ by induction. But then $\operatorname{id}_R R = \operatorname{id}_{\overline{R}} \overline{R} + 1 < \infty$. □

Remark. Actually (2) implies that the ring R is Cohen-Macaulay , so the Theorem can be restated as follows:

$$R \text{ is Gorenstein } \iff \operatorname{id}_R R < \infty.$$

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