A Proof Concerning the Number of Points where Two Curves of Arbitrary Order May Intersect*

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I. In the previous piece I reported without demonstration this proposition: that two algebraic curves, one of which is of order m and the other of order n, may intersect at mn points. The truth of this proposition is recognized by every Geometer, although one must admit, that nowhere does one find a sufficiently rigorous demonstration of it. There are general truths that we are ready to embrace as soon as the truth in some particular cases is recognized: and it is amongst this type of truth, that one can correctly place the proposition which I have just mentioned, since one finds it true not only in some, or many cases, but also in an infinite number of different cases. However, one will easily acknowledge that all these infinitely many proofs are not capable of protecting this proposition from every objection that an adversary might form, and a rigorous demonstration is absolutely necessary in order to silence him.

II. Before we undertake the demonstration of this proposition, it is necessary to establish its meaning. First, one may point out, that the number of intersections of two curves, one of which is of order m, the other of order n, does not necessarily $= mn$, but it can very often be smaller. Thus it may occur, that 2 straight lines do not intersect at all when they are parallel: and that a straight line intersects a parabola in only one point, and that 2 conic sections intersect each other at only 2 points, or at no point at all. Thus

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the meaning of our proposition is that the number of the intersections can never be greater than mn, although it is very often smaller; and thus we may consider either that some intersections extend towards infinity, or that they become imaginary. So that in counting the intersections to infinity, the imaginary ones as well as the real ones, one may say, that the number of intersections is always $= mn$.

III. Yet there may occur some cases, where the number of intersections is infinite, if one wishes to consider the coincidence of 2 equal and similar lines as an infinite number of intersections. This case will occur therefore, if the 2 equations, which describe the 2 lines, are the same, or if they have equal factors. But as the perfect coincidence may not properly be considered as an infinity of intersections, since this is rather a continual contact, the contents of the proposition faces no real exception on this score; and if the question depends on the number of intersections of 2 curves, one still supposes that they are neither coincident, nor do they have parts, of which the one falls perfectly on the other. Thus one will be able to state the proposition in question in this manner: that 2 curves, the one of order m, and the other of order n, the equations of which are neither the same nor have any common factor, can never intersect in more than mn points, although the number of intersections can very often be smaller.

IV. One will easily recognize the truth of this general proposition in an infinity of different cases, and this could even be demonstrated, if one was eager not to put forward anything in Geometry that was not justified by a rigorous proof. However, since these particular proofs contribute a lot to the greater understanding of this proposition, and in order to grasp their importance, I will start with the explanation of these proofs, before undertaking the general demonstration. To begin with, the truth of this proposition is recognized in the case, where one of 2 lines which intersect is straight, or of the first order, that is to say if $m = 1$, since then it is easy to demonstrate that the number of intersections of a curve of order n by a straight line is equal to n, or smaller. Since the general equation of curves of order n being:

$$\alpha y^n + (\beta + \gamma x)y^{n-1} + (\delta + \varepsilon x + \zeta x^2)y^{n-2} + \text{ etc. } = 0,$$

if for the equation of an arbitrary straight line

$$ax + by + c = 0$$

we substitute the value of

$$y = -\frac{bx + c}{a},$$

we will obtain an equation, in which the degree of the unknown x is no greater than n. Therefore, since each intersection corresponds to a root of x in this equation, it is clear that the number of intersections is equal to the number of roots of this equation, and that it consequently cannot be larger than n. Furthermore, we will see that the number of intersections is $= n$, if all the roots are real, and that it will be smaller, if some of these roots are imaginary. Now, if the highest powers of x cancel, and the equation after the elimination of y is reduced to a lower degree, this is a sign that some points of intersection extend towards infinity.

V. Let $m = 2$, and let the line of the second order be composed of 2 straight lines, which occurs when the equation is resolvable into 2 factors, as

$$(ay + bx + c)(dy + ex + f) = 0.$$

Now, the other line is an arbitrary curve of order n, the nature of which is expressed by the equation

$$\alpha y^n + (\beta + \gamma x)y^{n-1} + (\delta + \varepsilon x + \zeta x^2)y^{n-2} + \text{ etc. } = 0.$$

In this case it is clear, since this curve of order n can only intersect a straight line in n points, that 2 straight lines which are considered as a single curve of the second order can intersect at $2n$ points, when each intersects in n points: this conforms to the statement of the proposition, since mn in this case becomes $= 2n$.

VI. If one of the two proposed curves is of the third order, but is composed of 3 straight lines, the other remaining an arbitrary curve of order n, it is clear that the number of intersections will be $= 3n$, or less, as the proposition requires. And it will be the same for a line of arbitrary order m, if it consists of m straight lines, or if its equation is resolvable in as many simple equations of the form $ay + bx + c = 0$. For since each of these straight lines can intersect the other proposed curve of order n in n points, the number of all the intersections may amount to mn, in accordance with the statement of the proposition. And hence we already have an infinite number of cases, where the truth of this proposition is found to be solidly established. But in all these cases, one of the two proposed curves is not truly a curve, but rather a collection of many straight lines, according to the order to which it belongs.

VII. But there is also an infinity of curves where the truth stands out with great clarity. For let one of the two curves be a parabola expressed by the equation

$$y = axx + bx + c,$$

and therefore $m = 2$. Let the other curve be expressed by the general equation of order n

$$\alpha y^n + (\beta + \gamma x)y^{n-1} + (\delta + \varepsilon x + \zeta xx)y^{n-2} + \text{ etc. } = 0,$$

and it is evident that if one substitutes y everywhere with its value $axx + bx + c$, this equation will increase to the degree $2n$, and the variable x may have as many roots, of which all the intersections will be indicated: thus it will be possible that the curve of order n is intersected by the parabola in $2n$ points, and although the number of intersections can often be smaller, one sees nevertheless that it can never be greater than $2n$.

VIII. The same thing appears also, if one of the two curves is a parabolic curve of an arbitrary order

$$y = ax^m + bx^{m-1} + cx^{m-2} + \text{ etc.}$$

For if one substitutes this value for y in the equation for the other curve, of order n, we will see without difficulty, that the variable x will obtain an order of mn in the resulting equation, which denotes as many roots and hence as many intersections, all as the proposition states. From here one will also conclude, since the axis of the two curves is arbitrary, that even if one of the two curves is not expressed by an equation such as

$$y = ax^m + bx^{m-1} + cx^{m-2} + \text{ etc.},$$

as long as the equation may be reduced to this form by changing the axis, and even inclination between the coordinates, the number of intersections will equally well be $= mn$. The equation which designates the intersections always has this degree, or a lesser one, but never a higher one.

IX. These particular cases taken together lead us to a much more general case, where the truth of the proposition is found to be confirmed. Since every time that the equation of the first curve, which I suppose is of order m, may be resolved into factors expressing either straight lines, or parabolic curves, this equation being

$$(y-P)(y-Q)(y-R)(y-S) \text{ etc. } = 0,$$

where P, Q, R, S, etc., are rational functions of x and the first factor $y-P = 0$ denotes a curve of order p, the second, $y-Q=0$, one of order q, the third of order r, etc., so that $p+q+r+s+$ etc. $= m$, this curve will be composed of all these straight lines or curves together. The other curve, which I suppose to be of order n, may intersect the part of the first which is expressed by the factor $y-P=0$ in pn points, the part comprised in $y-Q=0$ in qn points, the part comprised in $y-R=0$ in rn points, etc. And hence the curve of order n may intersect every part of the first curve of order m in $pn+qn+rn+sn+$ etc. points, which is to say, in mn points since $p+q+r+s+$ etc. $= m$.

X. Although these cases continue to infinity, one will acknowledge nevertheless that much more is needed, in order that the truth of the proposition be demonstrated in all of its extent. And in order to reach such a demonstration it is necessary to prove that two equations of arbitrary order being proposed, such as

$$\begin{aligned} ay^m + (b+cx)y^{m-1} + (d+ex+fxx)y^{m-2} + \text{ etc.} &= 0, \\ \alpha y^n + (\beta+\gamma x)y^{n-1} + (\delta+\varepsilon x+\zeta xx)y^{n-2} + \text{ etc.} &= 0, \end{aligned}$$

if one eliminates from this one or the other of the 2 variables x and y, the other reaches only the power mn after the elimination. It is certainly true that it will be impossible in general to achieve this elimination in order to reveal to what order the other variable can rise, and even in most cases if one uses the ordinary methods of elimination, one will reach an equation of higher order than mn; so that employing this method, we would rather believe that the proposition is false. For although the equation which we derive by these means has factors, one has reason to doubt, whether one may ignore these factors, and whether they include roots which denote intersections.

XI. In order to perceive this difficulty more clearly, I will eliminate, in the usual way, the quantity y from these two equations:

$$\text{(I.)} \quad Py^3 + Qy^2 + Ry + S = 0,$$

$$\text{(II.)} \quad py^3 + qy^2 + ry + s = 0,$$

where P, Q, R, S, p, q, r, s, are arbitrary functions of the other variable quantity x. Let us multiply the first by s, and the second by S, and the difference divided by y will give

$$\text{(III.)} \qquad (Ps - pS)y^2 + (Qs - qS)y + Rs - rS = 0.$$

Then let us multiply the first by p, and the second by P, and the difference will give

$$\text{(IV.)} \qquad (Qp - qP)y^2 + (Rp - rP)y + Sp - sP = 0.$$

In the same manner, from these two equations of the second degree we will draw two of the first degree in y:

$$\begin{aligned} \text{(V.)} \qquad & ((Ps - pS)(Sp - sP) - (Qp - qP)(Rs - rS))\, y \\ + \quad & (Qs - qS)(Sp - sP) - (Rp - rP)(Rs - rS) = 0. \\ \text{(VI.)} \qquad & ((Qs - qS)(Qp - qP) - (Rp - rP)(Ps - pS))\, y \\ + \quad & (Rs - rS)(Qp - qP) - (Sp - sP)(Ps - pS) = 0. \end{aligned}$$

And from here we will draw this equation, in which the quantity y is no longer present:

$$\begin{aligned} \text{(VII.)} \qquad & (Ps - pS)(Sp - sP)(Rs - rS)(Qp - qP) \\ - \quad & (Ps - pS)^2(Sp - sP)^2 - (Qp - qP)^2(Rs - rS)^2 \\ + \quad & (Qp - qP)(Rs - rS)(Sp - sP)(Ps - pS) \\ = \quad & (Qs - qS)^2(Qp - qP)(Sp - sP) \\ - \quad & (Qs - qS)(Qp - qP)(Rp - rP)(Rs - rS) \\ - \quad & (Rp - rP)(Ps - pS)(Qs - qS)(Sp - sP) \\ + \quad & (Rp - rP)^2(Ps - pS)(Rs - rS) \end{aligned}$$

which changes into this:

$$\begin{aligned} 0 \quad = \quad & (Ps - pS)^4 \\ + \quad & 2(Qp - qP)(Rs - rS)(Ps - pS)^2 + (Rp - rP)(Qs - qS)(Ps - pS)^2 \\ - \quad & (Qp - qP)(Qs - qS)^2(Ps - pS) + (Rs - rS)(Rp - rP)^2(Ps - pS) \\ + \quad & (Qp - qP)^2(Rs - rS)^2 - (Qp - qP)(Qs - qS)(Rp - rP)(Rs - rS). \end{aligned}$$

However, the last terms, which do not contain the factor $(Ps - pS)$ reduce to:

$$(Qp - qP)(Rs - rS)\left((Qp - qP)(Rs - rS) - (Qs - qS)(Rp - rP)\right),$$

which is

$$(Qp - qP)(Rs - rS)(Ps - pS)(Qr - qR).$$

Hence the whole equation will be divisible by $Ps - pS$, giving:

$$\begin{aligned} 0 &= (Ps - pS)^4 \\ &+ [2(Qp - qP)(Rs - rS) + (Rp - rP)(Qs - qS)](Ps - pS)^2 \\ &+ [-(Qp - qP)(Qs - qS)^2 + (Rs - rS)(Rp - rP)^2](Ps - sP) \\ &+ (Qp - qP)(Qr - qR)(Rs - rS)(Ps - pS). \end{aligned}$$

XII. It is clear enough that in this case the factor $Ps - pS$, being set $= 0$, can not denote an intersection, and that by consequence the intersections of the two proposed curves will be contained in this equation:

$$\begin{aligned} (Ps - pS)^3 + 2(Qp - qP)(Rs - rS)(Ps - sP) \\ -(Qp - qP)(Qs - qS)^2 + (Rp - rP)(Qs - qS)(Ps - pS) \\ +(Rs - rS)(Rp - rP)^2 + (Qp - qP)(Qr - qR)(Rs - rS) &= 0. \end{aligned}$$

In the case of two curves of the third order, the coefficients P and p are constant; Q and q functions of x of the first degree such as $\alpha + \beta x$, R and r functions of x of the second degree such as $\alpha + \beta x + \gamma x^2$, and S and s functions of x of the third degree such as $\alpha + \beta x + \gamma x^2 + \delta x^3$. As a consequence, the factors which are found in this equation will be functions of x

$Ps - pS$ of degree 3	$Qs - qS$ of degree 4
$Qp - qP$ of degree 1	$Qr - qR$ of degree 3
$Rs - rS$ of degree 5	$Rp - rP$ of degree 2

from which it is evident that the equation, which indicates the intersections, will be of order 9, and consequently that, in general, two lines of the third order may intersect in 9 points.

XIII. These same equations

$$\begin{aligned} Py^3 + Qy^2 + Ry + S &= 0, \\ py^3 + qy^2 + ry + s &= 0, \end{aligned}$$

can also illustrate the number of intersections in an infinity of other cases. Since the first equation expresses a curve of order m, and the second equation a curve of order n, which will be the case, if the coefficients are entirely rational functions [i.e. polynomials *ed.*] of x with

P of degree $m-3$	p of degree $n-3$
Q of degree $m-2$	q of degree $n-2$
R of degree $m-1$	r of degree $n-1$
S of degree m	s of degree n

Thus the factors, which constitute the equation and no longer contain the variable y, will be functions of x:

$Ps - pS$ of degree $m+n-3$	$Rp - rP$ of degree $m+n-4$
$Qp - qP$ of degree $m+n-5$	$Qr - qR$ of degree $m+n-3$
$Rs - rS$ of degree $m+n-1$	$Qs - qS$ of degree $m+n-2$

And hence the number of intersections of these two curves will be $= 3m + 3n - 9$, which is always smaller than mn if m and n are larger than 3. For let $m = 3 + \alpha$ and $n = 3 + \beta$, the number of intersections will be $= 9 + 3(\alpha + \beta)$ instead of $mn = 9 + 3(\alpha + \beta) + \alpha\beta$. But one clearly sees that this decrease in the number of intersections comes from the fact that the chosen equations do not describe the general curves of orders m and n, but only special cases of these orders, from which it is not surprising that the number of intersections was found to be smaller than the proposition demands.

XIV. The elimination of the unknown y from two cubic equations, for which I have done the calculation, led to an equation whose degree was too great, and could not be reduced to the correct degree except through division by a factor, which one may clearly see did not include any intersections. Thus in equations where y has a higher degree, we will derive, through the elimination of y, an equation of even higher degree, which in fact will have a divisor. But this method, besides being impractical in equations, will not guarantee us that we will always find such a divisor which does not contain intersection. And still less, if after the division the equation will really be of the same dimension as the general proposition denotes, that is to say if the degree is never greater than mn, where the two given equations are of the order m and n. This situation proves all the more so the necessity of demonstrating the general proposition to its greatest extent, since without this one could well have reason to doubt its truth.

XV. Therefore it is principally on the details of the elimination that the demonstration of our general proposition depends. It is necessary to take care, that through the elimination one does not achieve an equation which includes spurious roots. Since two equations being given, each of which contain the same variable y, which must be eliminated, one easily sees that the elimination can be made from an infinity of different methods, depedning on the arbitrary quantities by which we multiply one or the other equation. It is a matter then of fixing a method of elimination, and following this operation, so that the equation at which one arrives contains no other roots than those which denote intersections, and that one can be assured, that it does not include superfluous factors, of which one may doubt whether or not they indicate intersections.

XVI. Therefore let two arbitrary equations be given:

$$\begin{aligned} y^m - Py^{m-1} + Qy^{m-2} - Ry^{m-3} + Sy^{m-4} - \text{ etc.} &= 0, \\ y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + sy^{n-4} - \text{ etc.} &= 0, \end{aligned}$$

which must be combined so that the equation which results no longer contains the variable y. Now at first we see that the value of y, which results from one of these equations, must be equal to the value of y, which results from the other. Therefore if both equations gives several values of y, the two given equations may be satisfied simultaneously, if an arbitrary value of y from the one will be equal to an arbitrary value of y from the other. Let us suppose that all the roots from the first equation are:

$$A, B, C, D, E, F, G \text{ etc.}$$

and the roots of the other equation are:

$$a, b, c, d, e, f, g \text{ etc.}$$

Given this, it is clear that each of the two given equations will be satisfied in every case where each of the roots of the first equation is equal to one of the roots of the other.

XVII. The number of roots A, B, C, D, etc. of the first equation will be $= m$, and the number of roots of the other equation will be $= n$. Therefore the proposed equation may be represented in the following form:

$$\begin{aligned} (y-A)(y-B)(y-C)(y-D)(y-E) \text{ etc.} &= 0, \\ (y-a)(y-b)(y-c)(y-d)(y-e) \text{ etc.} &= 0. \end{aligned}$$

Now it is clear that if $A = a$, the value $y = A = a$ will satisfy both equations; the same thing occurs if $A = b$, or $A = c$, or $A = d$, or $A = e$, etc. Furthermore the value $y = B$ will satisfy both equations if $B = a$, or $B = b$, or $B = c$, or $B = d$, or $B = e$, etc. And the value $y = C$ will satisfy both equations if $C = a$, or $C = b$, or $C = c$, or $C = d$, or $C = e$, etc., and similarly for the others. And it is evident that all these combinations together represent all the possible cases where the two given equations may be simultaneously satisfied.

XVIII. Therefore since the equation that we seek through elimination must contain all the possible cases where the same value taken for y satisfies both equations at the same time, it is clear that it must contain all the cases under consideration and hence it will be composed of all these factors

$$\left.\begin{array}{c}(A-a)(A-b)(A-c)(A-d)(A-e)\text{ etc.}\\(B-a)(B-b)(B-c)(B-d)(B-e)\text{ etc.}\\(C-a)(C-b)(C-c)(C-d)(C-e)\text{ etc.}\\(D-a)(D-b)(D-c)(D-d)(D-e)\text{ etc.}\\(E-a)(E-b)(E-c)(E-d)(E-e)\text{ etc.}\\\text{etc.}\end{array}\right\}=0.$$

Therefore, since the quantity y is no longer found in this equation, this is what we seek by the elimination, and which exhibits every case, in which the two given equations may have the same root. But as the roots A, B, C, D etc., a, b, c, d, etc., are often impossible to determine, it is a matter of explaining this equation by the coefficients P, Q, R, S, etc., p, q, r, s, etc., whose relationship to the roots is known.

XIX. Because, as we have seen, the product of all the factors

$$(y-a)(y-b)(y-c)(y-d)(y-e)\text{ etc.}$$

is equal to the expression

$$y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + sy^{n-4} - \text{ etc.},$$

if we successively substitute for y the values A, B, C, D, etc., the equation, which must result from the elimination, will be composed of the following factors:

$$\left.\begin{array}{c}(A^n - pA^{n-1} + qA^{n-2} - rA^{n-3} + sA^{n-4} - \text{etc.})\\(B^n - pB^{n-1} + qB^{n-2} - rB^{n-3} + sB^{n-4} - \text{etc.})\\(C^n - pC^{n-1} + qC^{n-2} - rC^{n-3} + sC^{n-4} - \text{etc.})\\(D^n - pD^{n-1} + qD^{n-2} - rD^{n-3} + sD^{n-4} - \text{etc.})\\(E^n - pE^{n-1} + qE^{n-2} - rE^{n-3} + sE^{n-4} - \text{etc.})\\\text{etc.}\end{array}\right\} = 0,$$

where the number of these factors is $= m$, according to the number of roots of the first equation. Here it is also obvious that in changing the equations, the equation that results from the elimination may also be represented in the form:

$$\left.\begin{array}{c}(a^m - Pa^{m-1} + Qa^{m-2} - Ra^{m-3} + Sa^{m-4} - \text{etc.})\\(b^m - Pb^{m-1} + Qb^{m-2} - Rb^{m-3} + Sb^{m-4} - \text{etc.})\\(c^m - Pc^{m-1} + Qc^{m-2} - Rc^{m-3} + Sc^{m-4} - \text{etc.})\\(d^m - Pd^{m-1} + Qd^{m-2} - Rd^{m-3} + Sd^{m-4} - \text{etc.})\\(e^m - Pe^{m-1} + Qe^{m-2} - Re^{m-3} + Se^{m-4} - \text{etc.})\\\text{etc.}\end{array}\right\} = 0,$$

where the number of factors is $= n$.

XX. Although the expressions of the roots A, B, C, D, etc., and a, b, c, d, etc., are for the most part very irrational, and often such that one can not determine them, one nevertheless knows that the sum

$$\begin{array}{rcl}\text{of all the roots } A,\ B,\ C,\ D, \text{ etc., is} & = & P,\\\text{the sum of the products taken 2 at a time} & = & Q,\\\text{the sum of the products taken 3 at a time} & = & R,\\\text{the sum of the products taken 4 at a time} & = & S,\\\text{etc.} & & \end{array}$$

And from these values P, Q, R, S, etc., we are able to express all the expressions, into which all the roots enter equally, by rational formulas composed of P, Q, R, S, etc. Now we easily see, that if one multiplies the factors previously mentioned, one always derives similar expressions, which include all the roots equally, and in the place of which one may put the rational functions of the coefficients P, Q, R, S, etc., and p, q, r, s, etc. This is also clear from the double form of this equation from the previous section. For if some irrationality remained in the first form, this would be an irrationality in the first equation, but from the second form we see, that there may not be an irrationality from the first equation. From this it follows, that either form must lead us to the same rational expression, which includes only the coefficients P, Q, R, S, etc., and p, q, r, s.

XXI. If we reflect now, that in the given equations

$$\begin{aligned} y^m - Py^{m-1} + Qy^{m-2} - Ry^{m-3} + Sy^{m-4} - \text{ etc.} &= 0, \\ y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + sy^{n-4} - \text{ etc.} &= 0, \end{aligned}$$

given that they express curves of the orders m and n, the coefficients P and p denote functions of the first degree in x such as $\alpha + \beta x$, the coefficients Q and q functions of the second degree $\alpha + \beta x + \gamma xx$, the coefficients R and r functions of the third degree $\alpha+\beta x+\gamma x^2+\delta x^3$, etc., such that the sum of the roots A, B, C, D, etc., or a, b, c, d, etc., will be expressed by a function of x of degree one, the sum of the products taken 2 at a time of these roots by a function of the second degree, the sum of the products taken 3 at a time by a function of the third degree, and so forth. This is why in the composition of all the roots in the first form (Paragraph XVIII) one may see each root as a function of x of degree 1, and hence this form being composed of mn simple factors, it is of degree mn in x and as a consequence it designates mn intersections of the two given curves.

XXII. If there is in this demonstration still some obscurity, this comes from its great universality, and all the doubts that one may have about this disappear entirely, as soon as one makes the application to some particular cases, in which one first recognizes that all this that I have given regarding the degrees of each part must hold not only in these cases, but also in general. I will start with two equations of the second order, which are

$$\begin{array}{rcl|c} & & & \text{Roots} \\ yy - Py + Q & = & 0 & A, B \\ yy - py + q & = & 0 & a, b \end{array}$$

Then, since $m = 2$ and $n = 2$, the equation where the elimination must lead, will be

$$(A^2 - pA + q)(B^2 - pB + q) = 0,$$

which, being expanded, will give

$$A^2B^2 - pAB(A+B) + q(A^2+B^2) + ppAB - pa(A+B) + qq = 0.$$

Now letting $AB = Q$ and $A + B = P$, we will have $AA + BB = PP - 2Q$. By consequence the desired equation will be

$$Q^2 - pPQ + aPP - 2Qq + ppQ - pqP + qq = 0,$$

of which each term will be of degree 4 in x, given that P and p are of degree 1 in x, and Q and q of degree 2.

XXIII. Let the two given equations be of the third order:

$$\begin{array}{rcl|c} & & & \text{the roots being} \\ y^3 - Py^2 + Qy - R & = & 0 & A, B, C \text{ and } m = 3 \\ y^3 - py^2 + qy - r & = & 0 & a, b, c \text{ and } n = 3. \end{array}$$

Thus the desired equation by elimination of y will be:

$$(A^3 - pA^2 + qA - r)(B^3 - pB^2 + qB - r)(C^3 - pC^2 + qC - r) = 0,$$

which by the expansion will become

$$\begin{aligned}
&A^3B^3C^3 - pA^2B^2C^2(AB + AC + BC) + qABC(A^2B^2 + A^2C^2 + B^2C^2) \\
&-r(A^3B^3 + A^3C^3 + B^3C^3)_{+}p^2A^2B^2C^2(A+B+C) \\
&-pqABC(A^2B + AB^2 + A^2C + AC^2 + B^2C + BC^2) - p^3A^2B^2C^2 \\
&+q^2ABC(A^2 + B^2 + C^2) + pr(A^3B^2 + A^2B^3 + A^3C^2 + A^2C^3 + B^3C^2 + B^2C^3) \\
&+q^3ABC + r^2(A^3 + B^3 + C^3) - qr(A^3B + AB^3 + A^3C + AC^3 + B^3C + BC^3) \\
&-r^3 + p^2qABC(AB + AC + BC) \\
&+pqr(A^2B + AB^2 + A^2C + AC^2 + B^2C + BC^2) \\
&-p^2r(A^2B^2 + A^2C^2 + B^2C^2) - q^2r(AB + AC + BC) \\
&-pq^2ABC(A + B + C) + qr^2(A + B + C) \\
&-pr^2(A^2 + B^2 + C^2) = 0,
\end{aligned}$$

where it is necessary to note that

$$\begin{array}{rcll} A + B + C & = & P, & \text{of degree 1 in } x \\ AB + AC + BC & = & Q, & \text{of degree 2} \\ ABC & = & R, & \text{of degree 3} \end{array}$$

XXIV. As for the other expressions, one finds them formed from the coefficients P, Q, R such that

$$\begin{array}{rcll} & & & \text{degree} \\ A^2+B^2+C^2 & = & P^2-2Q & 2 \\ A^2B+AB^2+A^2C+AC^2+B^2C+BC^2 & = & PQ-3R & 3 \\ A^3+B^3+C^3 & = & P^3-3PQ+3R & 3 \\ A^3B+AB^3+A^3C+AC^3+B^3C+BC^3 & = & P^2Q-PR-2Q^2 & 4 \\ A^2B^2+A^2C^2+B^2C^2 & = & Q^2-2PR & 4 \\ A^3B^2+A^2B^3+A^3C^2+A^2C^3+B^3C^2+B^2C^3 & = & PQ^2-2P^2R-QR & 5 \\ A^3B^3+A^3C^3+B^3C^3 & = & Q^3-3PQR+3RR & 6 \end{array}$$

from which one clearly sees, since p, q, r are functions of the first, second, and third degree in x, that all the terms are of the same power of x, and that this power is $= 9$, as the enunciation of the proposition states. Now this substitution will give the following equation by the elimination of the variable y:

$$\begin{aligned} & R^3-pQR^2+qQ^2R-2qPR^2-rQ^3+3rPQR-3rR^2 \\ & -r^3+qr^2P-q^2rQ+2pr^2Q+q^3R-3pqrR+3r^2R \\ & +p^2PR^2-pqPQR+3pqRR+prPQ^2-2prP^2R-prQR+q^2P^2R \\ & -pr^2P^2+pqrPQ-3r^2PQ-pq^2PR+2p^2rPr+qrPR-p^2rQ^2 \\ & +rrP^3-2qqQR-qrP^2Q-p^3R^2+2qrQQ+ppqQR=0. \end{aligned}$$

XXV. This example will serve to convince us in general that, if the two proposed equations are:

$$\begin{aligned} y^m-Py^{m-1}+Qy^{m-2}-Ry^{m-3}+Sy^{m-4}-\ldots\pm V &= 0, \\ y^n-py^{n-1}+qy^{n-2}-ry^{n-3}+sy^{n-4}-\ldots\pm v &= 0, \end{aligned}$$

where P and p are functions of degree 1 in x, Q and q of degree 2, R and r of degree 3, etc., and the last terms V of degree m and v of degree n, that the first term, which is given by the equation of paragraph XIX, resulting from the elimination, will be $A^nB^nC^nD^n$ etc. $=V$, and hence is of degree mn in x. Since one also sees clearly that all the other terms, being expressible by the letters P, Q, R, etc., and p, q, r, etc., must be of the same degree in x, it is incontestably proven that the equation, which one reaches by the elimination of the letter y, will be of degree mn in x, just as the general proposition states.