Figure 1: the sequent-calculus proof of Theorem 30

$$\dfrac{\dfrac{\dfrac{\text{property 11}}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m] \longrightarrow t(d) = [t(a), \langle \vec{b}, \vec{C} \rangle, m]} \qquad \dfrac{\text{definition of } \phi}{t(d) = [t(a), \langle \vec{b}, \vec{C} \rangle, m], \phi(t(d)) \longrightarrow \chi(t(a), \vec{b}, \vec{C})}}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m], \phi(t(d)) \longrightarrow \chi(t(a), \vec{b}, \vec{C})}}{(\mathbf{1})}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bigwedge_{s \in S_0} \psi(s, \vec{b}, \vec{c}_s) \longrightarrow \bigwedge_{s \in S_0} \psi(s, \vec{b}, \vec{c}_s) \qquad \dfrac{\text{induction hypothesis}}{a \notin S_0, \{\theta'(t(a), \vec{b})\}_{t \in f} \longrightarrow \theta'(a, \vec{b})}}{a \notin S_0, \bigwedge_{s \in S_0} \psi(s, \vec{b}, \vec{c}_s), \{(\bigwedge_{s \in S_0} \psi(s, \vec{b}, \vec{c}_s) \supset \theta'(t(a), \vec{b})\}_{t \in f} \longrightarrow \theta'(a, \vec{b})}}{a \notin S_0, \{(\bigwedge_{s \in S_0} \psi(s, \vec{b}, \vec{c}_s) \supset \theta'(t(a), \vec{b})\}_{t \in f} \longrightarrow \bigwedge_{s \in S_0} \psi(s, \vec{b}, \vec{c}_s) \supset \theta'(a, \vec{b})}}{a \notin S_0, \{\chi(t(a), \vec{b}, \vec{C})\}_{t \in f} \longrightarrow \chi(a, \vec{b}, \vec{C})} \qquad \dfrac{\text{definition of } \phi}{d = [a, \langle \vec{b}, \vec{C} \rangle, m], \chi(a, \vec{b}, \vec{C}) \longrightarrow \phi(d)}}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m], \{\chi(t(a), \vec{b}, \vec{C})\}_{t \in f} \longrightarrow \phi(d)}}{(\mathbf{2})}$$

Figure 1 continued

$$\dfrac{\dfrac{(\mathbf{1})}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m], \phi(t(d)) \longrightarrow \chi(t(a), \vec{b}, \vec{C})} \qquad \dfrac{(\mathbf{2})}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m], \{\chi(t(a), \vec{b}, \vec{C})\}_{t \in f} \longrightarrow \phi(d)}}{\dfrac{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m], \{\phi(t(d))\}_{t \in f} \longrightarrow \phi(d)}{(\mathbf{3})}}$$

$$\dfrac{\dfrac{\text{property 11}}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m] \longrightarrow t(d) = [t(a), \langle \vec{b}, \vec{C} \rangle, m]} \qquad \dfrac{\text{property 13}}{t(d) = [t(a), \langle \vec{b}, \vec{C} \rangle, m] \longrightarrow R(t(d))}}{\dfrac{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m] \longrightarrow R(t(d))}{(\mathbf{4})}}$$

$$\dfrac{\dfrac{(\mathbf{4})}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m] \longrightarrow R(t(d))} \qquad \dfrac{(\mathbf{3})}{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m], \{\phi(t(d))\}_{t \in f} \longrightarrow \phi(d)}}{\dfrac{a \notin S_0, d = [a, \langle \vec{b}, \vec{C} \rangle, m], \{R(t(d)) \supset \phi(t(d))\}_{t \in f} \longrightarrow \phi(d)}{(\mathbf{5})}}$$

Figure 1 continued

$$
\begin{array}{rcl}
\psi(a,\vec{b},\vec{c}_a) & \longrightarrow & \psi(a,\vec{b},\vec{c}_a) \\ \hline
a \in S_0, \{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0} & \longrightarrow & \psi(a,\vec{b},\vec{c}_a) \\ \hline
a \in S_0, \bigwedge_{s\in S_0} \psi(s,\vec{b},\vec{c}_s) & \longrightarrow & \psi(a,\vec{b},\vec{c}_a) \\ \hline
a \in S_0, \bigwedge_{s\in S_0} \psi(s,\vec{b},\vec{c}_s) & \longrightarrow & (\exists\vec{z})\psi(a,\vec{b},\vec{z}) \\ \hline
a \in S_0 & \longrightarrow & \bigwedge_{s\in S_0)} \psi(s,\vec{b},\vec{c}_s) \supset (\exists\vec{z})\psi(a,\vec{b},\vec{z}) \\ \hline
a \in S_0 & \longrightarrow & \bigwedge_{s\in S_0)} \psi(s,\vec{b},\vec{c}_s) \supset \theta'(a,\vec{b}) \\ \hline
a \in S_0 & \longrightarrow & \chi(a,\vec{b},\vec{C}) \\ \hline
\end{array}
$$

$$(\mathbf{6})$$

$$
\dfrac{\dfrac{\dfrac{(\mathbf{6})}{a \in S_0 \longrightarrow \chi(a,\vec{b},\vec{C})} \qquad \dfrac{\text{definition of } \phi}{d = [a,\langle\vec{b},\vec{C}\rangle,m], \chi(a,\vec{b},\vec{C}) \longrightarrow \phi(d)}}{a \in S_0, d = [a,\langle\vec{b},\vec{C}\rangle,m] \longrightarrow \phi(d)}}{(\mathbf{7})}
$$

Figure 1 continued

$$
\dfrac{
\dfrac{\text{property 12}}{R(d) \longrightarrow (\exists x,y,z)(d=[x,y,z])}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{(\mathbf{7})}{a \in S_0, d=[a,\langle \vec{b},\vec{c}\rangle,m] \;\longrightarrow\; \phi(d)}
}{d=[a,\langle \vec{b},\vec{c}\rangle,m] \;\longrightarrow\; \phi(d), a \notin S_0}
\quad
\dfrac{(\mathbf{5})}{a \notin S_0, d=[a,\langle \vec{b},\vec{C}\rangle,m], \{R(t(d)) \supset \phi(t(d))\}_{t\in f} \;\longrightarrow\; \phi(d)}
}{d=[a,\langle \vec{b},\vec{C}\rangle,m], \{R(t(d)) \supset \phi(t(d))\}_{t\in f} \;\longrightarrow\; \phi(d)}
}{d=[a,\langle \vec{b},\vec{C}\rangle,m], \bigwedge_{t\in f}(R(t(d)) \supset \phi(t(d))) \;\longrightarrow\; \phi(d)}
}{(\exists x,y,z)(d=[x,y,z], \bigwedge_{t\in f}(R(t(d)) \supset \phi(t(d))) \;\longrightarrow\; \phi(d)}
}{
\dfrac{
\dfrac{R(d), \bigwedge_{t\in f}(R(t(d)) \supset \phi(t(d)) \;\longrightarrow\; \phi(d)}{\bigwedge_{t\in f}(R(t(d)) \supset \phi(t(d)) \;\longrightarrow\; R(d) \supset \phi(d)}
}{d \notin S_0, \bigwedge_{t\in f}(R(t(d)) \supset \phi(t(d)) \;\longrightarrow\; R(d) \supset \phi(d)}
}
$$

$$(\mathbf{8})$$

Figure 1 continued

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{properties 4 and 12}}{R(s) \longrightarrow (\exists x,\vec{y},\vec{z},w)(s=[x,\langle\vec{y},\vec{z}\rangle,w])} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{property 10}}{s \in S_0, s=[a,\langle\vec{b},\vec{C}\rangle,m] \longrightarrow a \in S_0} \quad \dfrac{(\mathbf{6})}{a \in S_0 \longrightarrow \chi(a,\vec{b},\vec{C})}}{s \in S_0, s=[a,\langle\vec{b},\vec{C}\rangle,m] \longrightarrow \chi(a,\vec{b},\vec{C})} \quad \dfrac{\text{definition of } \phi}{s=[a,\langle\vec{b},\vec{C}\rangle,m], \chi(a,\vec{b},\vec{C}) \longrightarrow \phi(s)}}{s \in S_0, s=[a,\langle\vec{b},\vec{C}\rangle,m] \quad \longrightarrow \quad \phi(s)}}{s \in S_0, (\exists x,\vec{y},\vec{z},w)(s=[x,\langle\vec{y},\vec{z}\rangle,w]) \quad \longrightarrow \quad \phi(s)}}{s \in S_0, R(s) \quad \longrightarrow \quad \phi(s)}}{s \in S_0 \quad \longrightarrow \quad R(s) \supset \phi(s)}}{\longrightarrow \quad \bigwedge_{s \in S_0}(R(s) \supset \phi(s))}}{(\mathbf{9})}$$

$$\dfrac{\dfrac{\dfrac{(\mathbf{9})}{\longrightarrow \bigwedge_{s \in S_0}(R(s) \supset \phi(s))} \qquad \dfrac{\dfrac{(\mathbf{8})}{d \notin S_0, \bigwedge_{t \in f}(R(t(d)) \supset \phi(t(d)) \longrightarrow R(d) \supset \phi(d)}}{\bigwedge_{s \in S_0}(R(s) \supset \phi(s)) \longrightarrow R(E(u,\langle\vec{b},\vec{C}\rangle)) \supset \phi(E(u,\langle\vec{b},\vec{C}\rangle))}\ \text{pfIND}}{\longrightarrow \quad R(E(u,\langle\vec{b},\vec{C}\rangle)) \supset \phi(E(u,\langle\vec{b},\vec{C}\rangle))}}{(\mathbf{10})}$$

Figure 1 continued

$$
\dfrac{
\dfrac{\text{property 5}}{\longrightarrow (\exists v)(E(u,\langle\vec{b},\vec{C}\rangle) = [u,\langle\vec{b},\vec{C}\rangle,v])}
\qquad
\dfrac{\dfrac{\text{property 11}}{E(u,\langle\vec{b},\vec{C}\rangle) = [u,\langle\vec{b},\vec{C}\rangle,v] \quad\longrightarrow\quad R(E(u,\langle\vec{b},\vec{C}\rangle))}}{(\exists v)(E(u,\langle\vec{b},\vec{C}\rangle) = [u,\langle\vec{b},\vec{C}\rangle,v]) \quad\longrightarrow\quad R(E(u,\langle\vec{b},\vec{C}\rangle))}
}{
\dfrac{\longrightarrow \quad R(E(u,\langle\vec{b},\vec{C}\rangle))}{(\mathbf{11})}
}
$$

$$
\dfrac{
\dfrac{(\mathbf{10})}{\longrightarrow \quad R(E(u,\langle\vec{b},\vec{C}\rangle)) \supset \phi(E(u,\langle\vec{b},\vec{C}\rangle))}
\qquad
\dfrac{\dfrac{(\mathbf{11})}{\longrightarrow \quad R(E(u,\langle\vec{b},\vec{C}\rangle))} \qquad \phi(E(u,\langle\vec{b},\vec{C}\rangle)) \longrightarrow \phi(E(u,\langle\vec{b},\vec{C}\rangle))}{R(E(u,\langle\vec{b},\vec{C}\rangle)) \supset \phi(E(u,\langle\vec{b},\vec{C}\rangle)) \longrightarrow \phi(E(u,\langle\vec{b},\vec{C}\rangle))}
}{
\dfrac{\longrightarrow \quad \phi(E(u,\langle\vec{b},\vec{C}\rangle))}{(\mathbf{12})}
}
$$

Figure 1 continued

$$
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\dfrac{
\dfrac{
\dfrac{
\dfrac{\text{properties 5 \& 6}}{E(u,\langle\vec{b},\vec{C}\rangle)=[d,\langle\vec{e},\vec{F}\rangle,g]\ \longrightarrow\ u=d\wedge\vec{b}=\vec{e}\wedge\vec{C}=\vec{F}}
\quad \text{equality axioms}
}{E(u,\langle\vec{b},\vec{C}\rangle)=[d,\langle\vec{e},\vec{F}\rangle,g],\chi(d,\vec{e},\vec{F})\longrightarrow\chi(u,\vec{b},\vec{C})}
\qquad
\dfrac{\text{definition of }\chi}{\chi(u,\vec{b},\vec{C}),\{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0}\longrightarrow\theta'(u,\vec{b})}
}{E(u,\langle\vec{b},\vec{C}\rangle)=[d,\langle\vec{e},\vec{F}\rangle,g],\chi(d,\vec{e},\vec{F}),\{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0}\ \longrightarrow\ \theta'(u,\vec{b})}
}{E(u,\langle\vec{b},\vec{C}\rangle)=[d,\langle\vec{e},\vec{F}\rangle,g]\wedge\chi(d,\vec{e},\vec{F}),\{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0}\ \longrightarrow\ \theta'(u,\vec{b})}
}{(\exists x,\vec{y},\vec{z},w)(E(u,\langle\vec{b},\vec{C}\rangle)=[x,\langle\vec{y},\vec{z}\rangle,w]\wedge\chi(x,\vec{y},\vec{z})),\{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0}\ \longrightarrow\ \theta'(u,\vec{b})}
}{\phi(E(u,\langle\vec{b},\vec{C}\rangle)),\{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0}\ \longrightarrow\ \theta'(u,\vec{b})}
}{}
}{}
\;
$$

$$(\mathbf{13})$$

$$
\dfrac{
\Gamma(\vec{b}),\theta(a,\vec{b})\longrightarrow\theta'(a,\vec{b})
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{(\mathbf{12})}{\longrightarrow\phi(E(u,\langle\vec{b},\vec{C}\rangle))}
\qquad
\dfrac{(\mathbf{13})}{\phi(E(u,\langle\vec{b},\vec{C}\rangle)),\{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0}\longrightarrow\theta'(u,\vec{b})}
}{\{\psi(s,\vec{b},\vec{c}_s)\}_{s\in S_0}\ \longrightarrow\ \theta'(u,\vec{b})}
}{\{(\exists\vec{z})\psi(s,\vec{b},\vec{z})\}_{s\in S_0}\ \longrightarrow\ \theta'(u,\vec{b})}
}{\{\theta'(s,\vec{b})\}_{s\in S_0}\ \longrightarrow\ \theta'(u,\vec{b})}
\qquad
\theta'(u,\vec{b})\longrightarrow\theta(u,\vec{b}),\Delta(\vec{b})
}{\Gamma(\vec{b}),\{\theta(s,\vec{b})\}_{s\in S_0}\longrightarrow\theta(u,\vec{b}),\Delta(\vec{b})}
$$